Solving physics pde's using Gabor multipliers

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Marmousi - 2D model of heterogeneous media

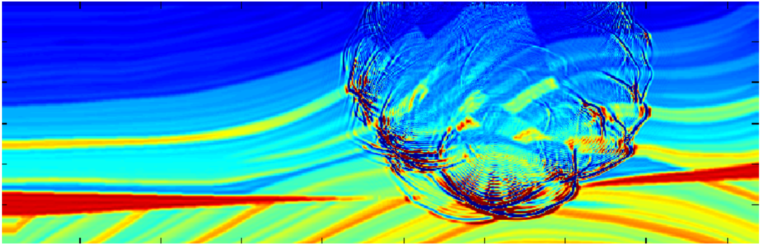

Figure: Seismic waves propagating through regions of different velocities. Simulate by solving the wave equation with non-constant coefficients.

Better mathematics to control accuracy, reduce artifacts.

Chunking the wave equation

The wave equation

$$\frac{1}{c^2}\partial_{tt}u - \nabla^2 u = g,$$

velocity $c = c(\mathbf{x})^2$ represents an inhomogeneous medium. Approximate locally as

$$\frac{1}{c_m^2}\partial_{tt}u - \nabla^2 u = g,$$

$c_m = c(\mathbf{x}_m)$ is a constant, locally fixed velocity for the medium. We solve locally as some function u_m, then reassemble into a full solution, by summing up the u_m.

Windowing the functions

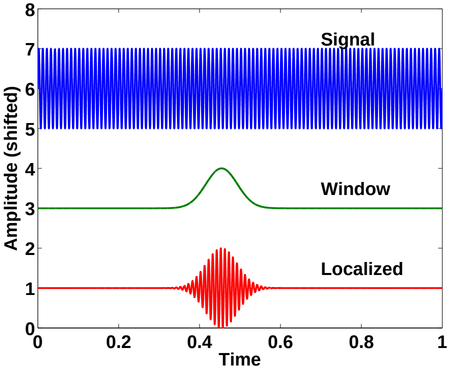

Figure: We localize signals such as the solution u by multiplying with a window. This gives a localized version of the signal.

Many windows

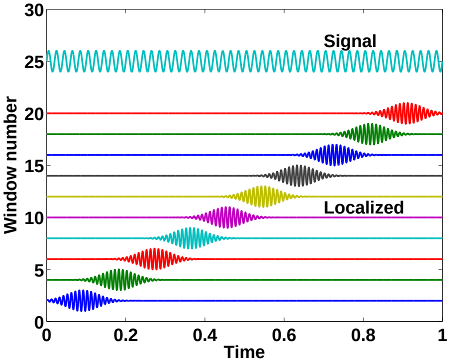

Figure: Many windows are used, to localize the signal at many different positions. The corresponding PDEs are solved with these local signals.

Another motivation - nonstationary filtering

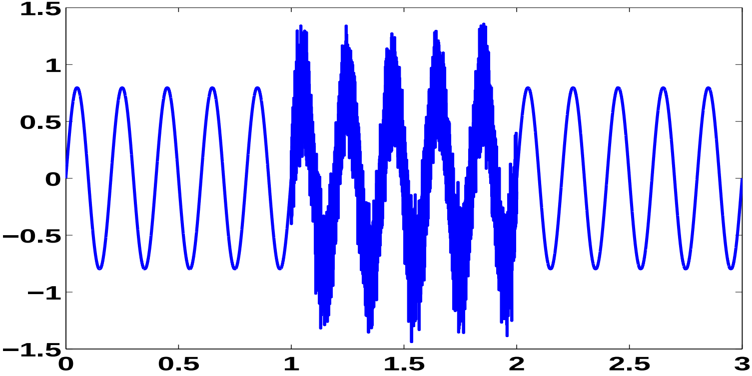

Figure: A nonstationary signal. We might want to filter the middle part differently.

Three signals, Gabor transforms (STFT)

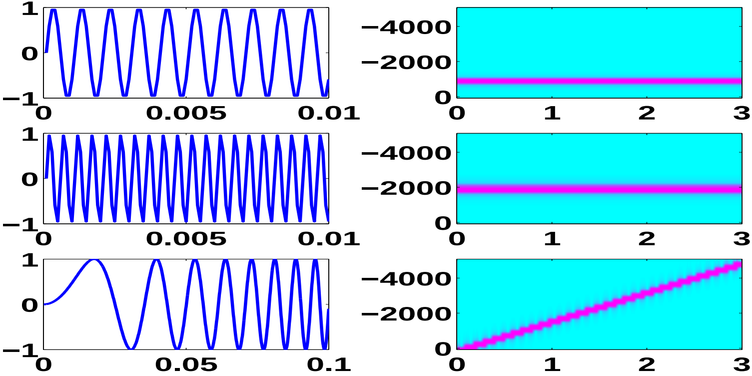

Figure: A **1kHz** signal, a **2kHz** signal, and a **linear chirp**. With their STFTs on 0 to 3 secs, 0 to 5kHz.

Frequency content, at 3 localizations

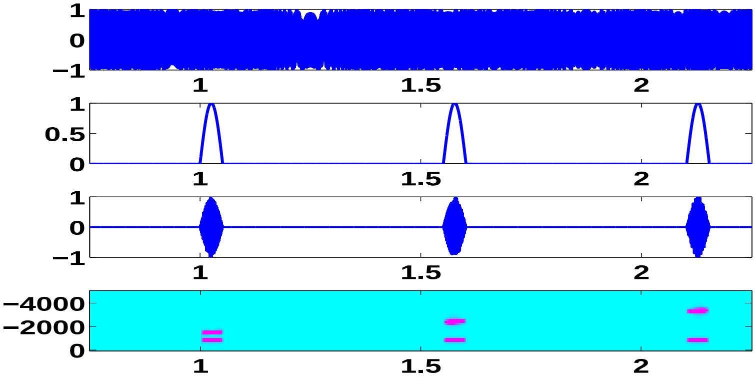

Figure: 1) Signal. 2) Three windows. 3) **Windowed signal.** 4) FFTs.

A beep and a chirp, filtered

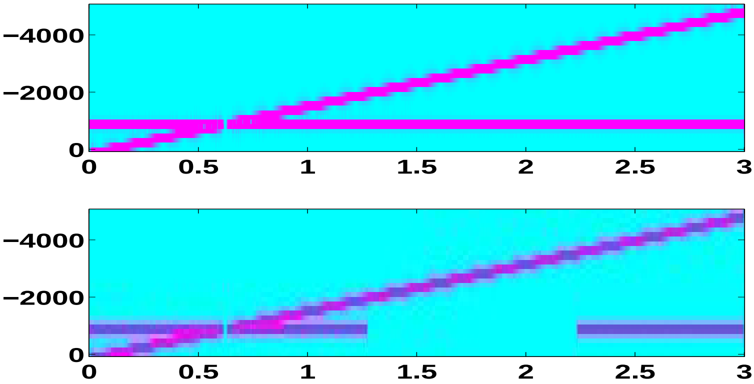

Figure: A **beep+chirp** signal, and a **filtered** version.

Idea: zero out entries in the t-f representation, to selectively filter.

The Gabor multiplier

- Modify a signal f by changing in the time-frequency domain.
- Signal $f(t)$ Gabor-transforms to $F(m, \omega)$:
  m the localization index, ω the frequency index.
- Multiply with fixed function $\alpha(m, \omega)$, obtain $\alpha(m, \omega)F(m, \omega)$.
- Invert the Gabor transform, obtain the modified signal,
  $\tilde{f} = G_\alpha f$.
- Gabor multiplier is the operator

$$G_\alpha = \mathcal{G}^{-1} M_\alpha \mathcal{G}.$$

Problems with Gabor multipliers

However:

- What multplier α is used to represent a PDE?.
- Even if α is well-behaved, the operator G_α might not be. (eg. unpredictable operator bounds)
- Combining operators may not behave in the way we expect.
- The Gabor approximation to a PDE may not be very good.
- Inverting the Gabor approximation may not be easy.

Problem 1: Representing a derivative

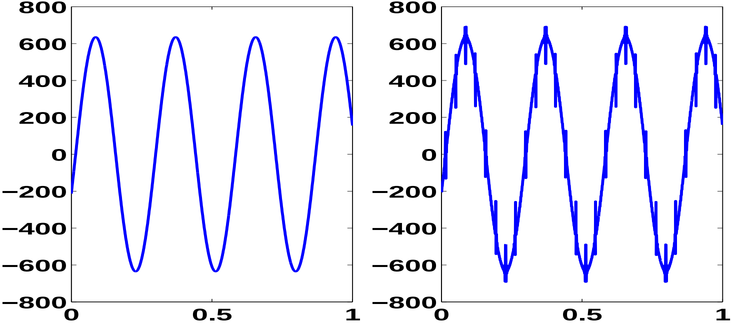

Figure: We expect a 2nd derivative operator to have multipler $\alpha = -\omega^2$. FT on left, good. Gabor multiplier (right) shows obvious errors.

Problem 2: Unpredictable operator bounds

Multiplier representing a series of phase shifts,

$$\alpha(m,\omega) = \exp(2\pi i t_m \cdot \omega),$$

a poor choice of windows can give a large operator norm

$$||G_\alpha|| \approx \sqrt{M}$$

where M is the number of windows.
Each phase shift represents an energy-preserving operator, but the corresponding Gabor multiplier can boost the energy of the input signal. Grows with the number of windows.

Problem 3: Combining operators may not behave well.

Add and subtract Gabor multipliers in the obvious way:

$$G_\alpha + G_\beta = G_{\alpha+\beta}$$

$$G_\alpha - G_\beta = G_{\alpha-\beta}$$

Products and inverses are not quite right:

$$G_\alpha G_\beta \neq G_{\alpha\cdot\beta}$$

$$(G_\alpha)^{-1} \neq G_{\alpha^{-1}}.$$

Functions of an operator not quite right:

$$\exp(G_\alpha) \neq G_{\exp(\alpha)}.$$

Very different than the functional calculus for Fourier multipliers.

Problem 4: Approximating a PDE may not be good.

A second order ODE of the form

$$a(t)\partial_{tt}u + b(t)\partial_t u + c(t)u = f(t)$$

should be represented, naively, with a Gabor multiplier like

$$\alpha(m,\omega) = -a(t_m)\omega^2 + ib(t_m)\omega + c(t_m).$$

The second order derivative already has problems (see Problem 1). There is also the question of how well the sampled functions $a(t_m), b(t_m), c(t_m)$ can represent the continuous functions $a(t), b(t), c(t)$.

Problem 5: Inverting the Gabor approximation may not be easy.

- The functional calculus for inverses does not work.
- The local inverses might not combine into a proper global inverse.

0. The convolution theorem

Theorem: The Gabor multiplier G_α is a sum of localized convolutions operators,

$$G_\alpha = \sum_{m=1}^{M} M_{v_m} C_{g_m} M_{w_m}.$$

M_{w_m}, M_{v_m} are multiplication by the windows.
C_{g_m} is convolution by g_m, where $\mathcal{FT}(g_m) = \alpha(m, \cdot)$.

The windows w_m, v_m come from the forward and inverse transform.
Partition of unity condition,

$$\sum_{m=1}^{M} v_m(t) w_m(t) = 1 \qquad \text{for all } t.$$

The Partition of Unity Condition

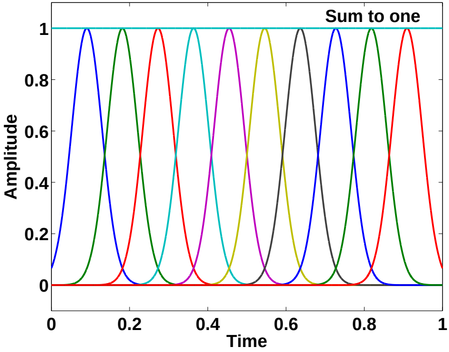

Figure: The windows must sum up to one.

1. Correcting the multiplier

For one-sided windows, the Gabor multiplier = Fourier multiplier, for constant coefficient differential operators.

For split windows, there is a correction term. Eg: the second order derivative gives

$$\partial_{tt} = G_\alpha + M_d$$

where $d = \sum_k w_k w_k''$ is a correction term.

For an n-th order differential operator, there will be a (n-2)-order differential operator as a correction term.

2. The boundedness theorem

For split windows, positive, matched: $v_m = w_m \geq 0$. Then

Theorem: The norm of the Gabor multiplier G_α satisfies

$$||G_\alpha|| \leq \max_{m,\omega} |\alpha(m,\omega)|.$$

Eg: a combination of phase shifts will always be energy preserving. The sup norm on the multiplier function α determines a bound on the norm of G_α, as with Fourier multipliers.

3. Combining operators

We have an approximation for the composition of Gabor multipliers

$$G_\alpha G_\beta \approx G_{\alpha\beta}.$$

For boxcar windows, the error term is

$$\Delta = \sum_k M_{w_k}[C_{\alpha_k}, M_{w_k}]C_{\beta_k} M_{w_k}.$$

A Fourier multiplier composes with a one-way windowed Gabor multiplier,

$$F_\alpha G_\beta = G_{\alpha\beta}$$

and this equality is exact.

4. Approximating PDEs

The Gabor multiplier (with symmetric windows) cannot approximate a 2nd order PDE. However, by adding a correction term of 0-th order, we get an exact representation of a constant coefficient PDE.
For non-constant coefficient, the Gabor representation, with correction term, is approximate to the order of the sup-norm distance between the variable coefficients $a(t), b(t), c(t)$ and their sampled versions $a(t_m), b(t_m), c(t_m)$. (We believe...)

Instead of using many small windows...

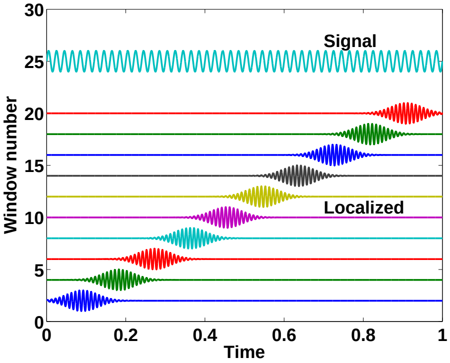

Figure: Many windows are used, to localize the signal at many different positions.

Use a few chunky windows

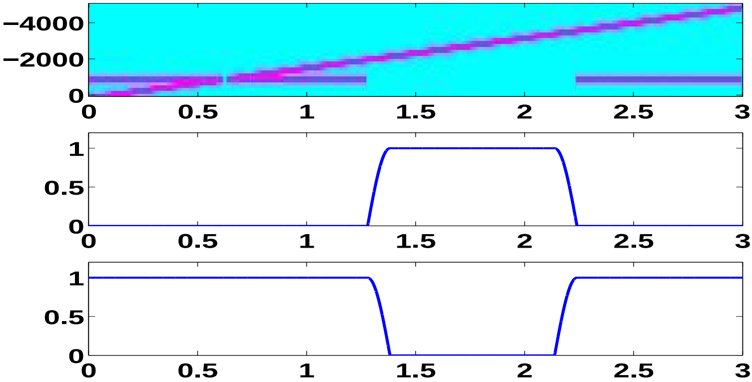

Figure: For this simple n.s. filter, two big windows are enough.

Velocity profile

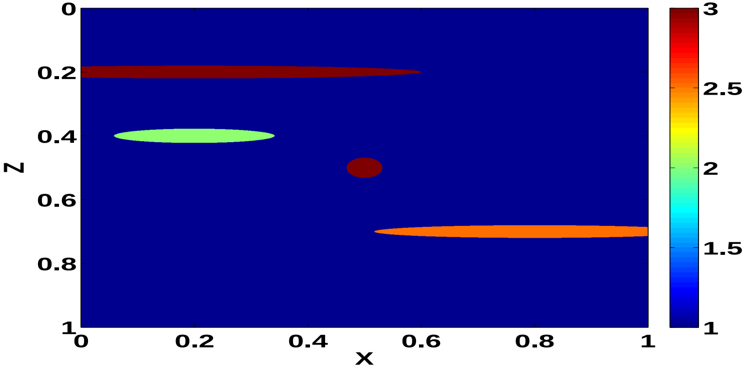

Figure: A blocky velocity profile.

Many little windows

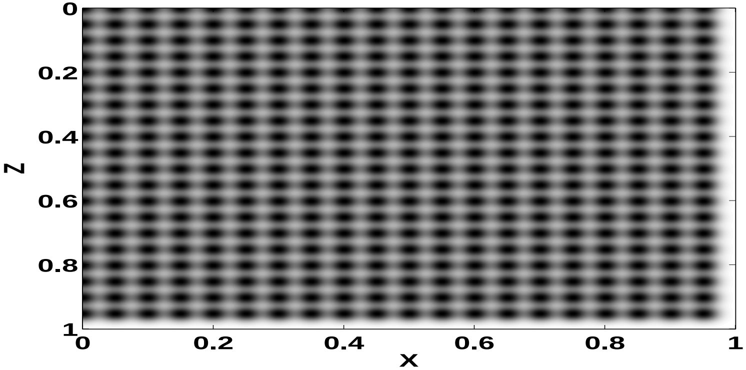

Figure: Cover velocity profile with many small, identical windows.

Many little windows - lots of computation

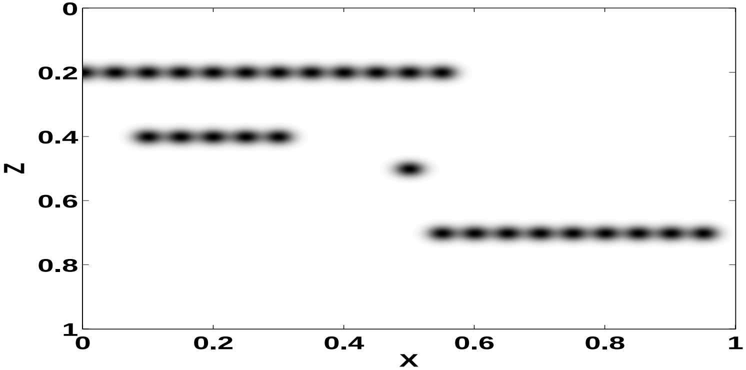

Figure: Small windows can cover the geology, but need many of them.

A few big windows

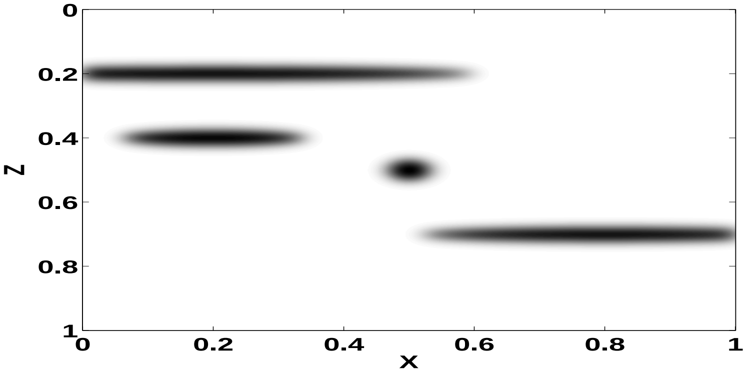

Figure: Cover with four windows of different shapes, one for each sample velocity. Background covered with one big window, with holes.

Lebesgue window results

Windowed approximation to the velocity field is accurate to an error no bigger than

$$\frac{v_{max} - v_{min}}{\# \text{ windows}}.$$

Sets the bound on how well the (corrected) Gabor multiplier approximates the PDE.

windows sets the speed of computation.

Conclusions

- Gabor multipliers can be used to model non-constant coefficient linear PDEs.
- We are developing an accurate mathematical theory to determine:
  - multipliers as product-convolution operators;
  - correction terms for higher order diff'l operators;
  - correction terms for functional calculus;
  - accuracy of approximate operators;
  - accuracy of solutions and simulations of PDEs.

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