Mixed-phase coupled vibrators

Donald T. Easley

ABSTRACT

A simple mechanical model for a vibrator is given. The equations governing the dynamic evolution of such a model vibrator are derived. A set of these model vibrators may be coupled through an elastic half-space with the vibrators upon the free surface. The equations describing the coupling of the vibrators through the half-space are derived. The final coupled system of equations provides a means to more realistically model the wavefield generated by a set of interacting vertical vibrators.

INTRODUCTION

Since the field experiments of Edelmann (1981) there have been other field experiments with VSPs and some physical modeling experiments which indicate the existence of significant shear-wave energy propagaing vertically downwards generated by two vertically polarized surface sources in counterphase. Theoretical analysis of the situation based on the assumption of stresses in counterphase (Dankbaar, 1983) or interacting line sources without horizontal coupling (Tan, 1985) fails to explain the existence of the normally incident shear-wave energy; however, if the simple assumption of displacements in counterphase is made (Easley, 1992) shear-wave generation is a direct consequence. Both the stresses and displacements in counterphase are rather unrealistic impositions on vibrators and serve only to indicate the possible range of phenomena to be expected. Tan's (1985) approach is far more general. He uses a simple line-vibrator model. A set of these simple vibrator models are placed parallel to each other on a frictionless surface of an elastic half-space. The radiation patterns calculated by Tan also fail to show the existence of normal incident shear-waves. I believe the lack of shear waves propagating in this direction is in part due to the frictionless-surface assumption. I have used a methodology similar to Tan's and derived the coupled equations for a simple vibrator model over an elastic half-space allowing interactions in all three perpendicular spatial directions. I have made the assumption that plate rotation is negligible and the vibrators are in welded contact with the surface. With these assumptions a set of coupled equations is derived which allows the modeling of the wavefield within the elastic half-space along with the dynamics of the vibrators.

THEORETICAL DEVELOPMENT

Mechanical model of vertical vibrator

The equation of motion of a spring-mass-dashpot system which approximates the mechanics of a vertical vibrator will be developed. Figure 1 represents the system we will be using, with displacements and forces only in the vertical direction shown. Properties in this direction will be subscripted by the numeral 3.

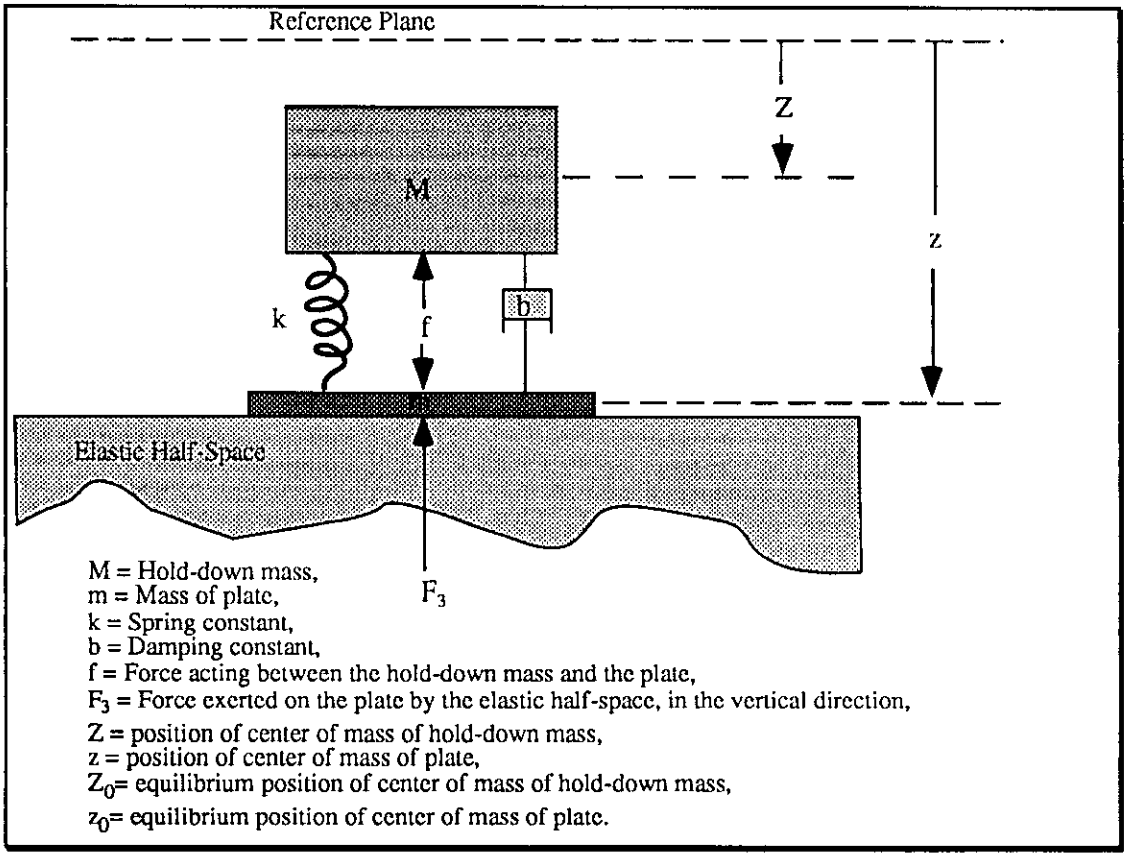

FIG. 1 Coupled spring, dashpot and mass model of a vertical vibrator.

When the system is in equilibrium the centers of mass of the hold-down mass and the plate are in positions Z^0 and z^0 respectively. I will assume the system has reached the equilibrium state prior to the time $t = 0$. Since in equilibrium there is no net force acting on the system, the gravitational force due to the hold-down mass and plate must be equal to but opposite in direction to the vertical reaction force of the elastic half-space. Let the vertical force supplied by the half-space be F_3^0 when the system is in equilibrium. This means:

$$(M + m)g = -F_3^0 \tag{1a}$$

which is a restatement of the equilibrium condition of this system. The forces in any other two perpendicular directions are assumed to be zero at equilibrium. Symbolically this can be written as:

$$F_1^0 = F_2^0 = 0, \tag{1b}$$

where we have taken two arbitrary mutually perpendicular directions on the horizontal plane to be represented by subscripts 1 and 2. The 1,2, and 3 axis are assumed to form a right-handed rectangular coordinate system. Another relation of the equilibrium state comes from considering the hold-down mass being solely supported by the spring; by using Hooke's law this can be expressed by:

$$k\,(z^0 - Z^0 - L) = -Mg\ , \tag{2}$$

where L is the length of the unstressed spring. The equilibrium equations provide a means of testing the dynamic equations that will be developed. When a dynamic force $f(t)$ is applied between the hold-down mass and plate, the two masses will in general move from their equilibrium position. Let the displacements of the center of mass of the hold-down mass and plate in the vertical direction from equilibrium be respectively:

$$U_3 = Z - Z^0, \tag{3a}$$

and

$$u_3 = z - z^0. \tag{3b}$$

As the masses are displaced, the distance between them can change. The change in this distance can be written as:

$$z - Z = u_3 - U_3 + (z^0 - Z^0). \tag{4}$$

Substitution of equation (2) into equation (4) results in:

$$z - Z = u_3 - U_3 - \frac{Mg}{k} + L. \tag{5}$$

As a first step towards developing the coupled system of equations of motion for the two masses we will consider all the forces acting on each mass individually. The forces acting on the plate are:

$$mg = \text{gravitational force,} \tag{6a}$$

$$-k\,(z - Z - L) = \text{force due to the spring,} \tag{6b'}$$

$$-b\,(\dot{z} - \dot{Z}) = \text{frictional damping force from the dashpot,} \tag{6c'}$$

$$f(t) = \text{applied force between the masses,} \tag{6d}$$

and

$$F_3(t) = \text{force due to the elastic half space.} \tag{6e}$$

Forces (6b') and (6c') can be put in terms of displacement by using equation (5) as:

$$-k\,(u_3 - U_3) + Mg\ , \tag{6b}$$

and

$$-b\,(\dot{u}_3 - \dot{U}_3). \tag{6c}$$

The corresponding forces acting on the hold-down mass are

$$Mg = \text{gravitational force,} \tag{7a}$$

$$-f(t) = \text{applied force acting between the masses.} \tag{7b}$$

$$k\,(u_3 - U_3) - Mg \ = \text{force due to the spring,} \tag{7c}$$

and

$$b\,(\dot{u}_3 - \dot{U}_3) = \text{frictional damping force from the dashpot.} \tag{7d}$$

Forces (6a) to (6e) and (7a) to (7d), along with Newton's second, law can be used to obtain the equations of motion in the vertical direction for the plate and hold-down mass respectively as:

$$\ddot{u}_3 m = (m + M)g \ + f \ + F_3 \ - k(u_3 - U_3) - b(\dot{u}_3 - \dot{U}_3), \tag{8a}$$

and

$$\dot{U}_3 M = -f \ + k(u_3 - U_3) + b(\dot{u}_3 - \dot{U}_3). \tag{8b}$$

These dynamic equations reduce to equation (1a) in the static situation. The only forces acting in the other two orthogonal directions are assumed to arise solely from the elastic half space. Let the forces in the 1 and 2 directions from the elastic half space be respectively F_1 and F_2. By assuming that the vibrator acts as a solid unit to forces in these directions, which means $u_1 = U_1$ and $u_2 = U_2$, then Newton's second law in these directions takes on the form:

$$(M + m)\,\ddot{u}_1 = F_1, \tag{8c}$$

and

$$(M + m)\,\ddot{u}_2 = F_2. \tag{8d}$$

We have assumed that the vibrator remains vertical throughout the experiment. Multiflying equation (8a) and (8b) by m and M, respectively, taking the difference and substitute a few variables, we arrive at the following equation:

$$\ddot{v}\mu + \dot{v}b + vk = \Pi, \tag{9}$$

where

$$v = u_3 - U_3,$$

$$\mu = \frac{mM}{m+M},$$

and

$$\Pi = f + \frac{\mu}{m}F_3 + Mg .$$

Equation (9) represents a damped harmonic oscillator with displacement v and forcing term Π . The first step we take in solving equation (9) is to consider the solution of homogeneous form ($\Pi = 0$) of equation (9) (Symon, 1971), that is:

$$v = A e^{-\gamma t} \cos(\omega_1 t + \theta), \tag{10}$$

where A and θ are constants of integration which are determined by boundary conditions and $\omega_1 = \sqrt{\omega_0^2 - \gamma^2}$ such that $\omega_0 = \sqrt{\frac{k}{\mu}}$ and $\gamma = \frac{b}{2\mu}$. As a second step towards the solution of equation (9), we will find the Green's function v_G associated with it. The causal Green's function is defined to be the solution of the following problem:

$$\ddot{v}_G \mu + \dot{v}_G b + v_G k = \delta(t - \tau), \tag{11a}$$

such that:

$$-\infty < (t,\tau) < \infty .$$

By causal, I mean there will be no response until the impulse at $t = \tau$. The initial value problem given by equation (11a) can be cast in another form by incorporating a jump condition as:

$$\ddot{v}_G \mu + \dot{v}_G b + v_G k = 0. \tag{11b}$$

such that:

$$t \in \{Reals: t \neq \tau\},$$

with the following additional conditions:

v_G continuous at $t = \tau$,

and

$$\lim_{\varepsilon \to 0} \dot{v}_G(\tau + \varepsilon) - \dot{v}_G(\tau - \varepsilon) = \frac{1}{\mu}.$$

The causality condition is still enforced. The Green's function for equations (11a) and (11b) (Stackgold, 1979) is:

$$v_G(t,\tau) = \mathrm{H}(t - \tau)\,\frac{e^{-\gamma t}}{\omega_1 \mu}\,\sin(\omega_1\,[t - \tau]) = v_G(t - \tau), \tag{12}$$

where the Heaviside function H(t) is defined to be:

$$\mathrm{H}(t) = \left\{ \begin{array}{ll} 1 & : t < 0 \\ 1 & : t > 0 \end{array} \right\}.$$

Before proceeding to writing down the solution for equation (9) it is necessary to state the following. Equation (9) can be broken into two equations, namely:

$$\ddot{v}_1 \mu + \dot{v}_1 b + v_1 k = \pi_1(t), \tag{13a}$$

and

$$\ddot{v}_2 \mu + \dot{v}_2 b + v_2 k = \pi_2, \tag{13b}$$

with:

$$\pi_1(t) = \frac{\mu}{m}\,\Delta F_3(t) + f(t),$$

and

$$\pi_2 = \frac{\mu}{m}\,F_3^0 + Mg\ ,$$

where the force due to the elastic half-space, $F_3(t)$, is partitioned into constant equilibrium restoring force term F_3^0 and a term $\Delta F_3(t)$ representing a perturbation from equilibrium. This can be represented by:

$$F_3(t) = F_3^0 + \Delta F_3(t)\ . \tag{13c}$$

Note that F_3^0 was defined in equation (1a). Due to the linearity of equation (9) its solution can be written as:

$$v = v_1 + v_2. \tag{14}$$

By direct substitution of equation (1a) into the definition of π_2 in equation (13b) we can see that π_2 is identically zero. Since the unique solution of the homogeneous problem with homogeneous initial conditions is the trivial solution, we have:

$$v_2 = 0. \tag{15}$$

We will use the Green's function of equation (12) to obtain a particular solution of equation (13a). The form of this solution is:

$$v_1(t) = \int_{-\infty}^{\infty} v_G(t - \tau)\, \pi_1(\tau)\, d\tau . \tag{16}$$

If we assume, with no loss of generality, that π_1 is zero before t = 0, we can rewrite equation (16) in the following manner:

$$v_1(t) = \int_0^t \frac{e^{-\gamma[t - \tau]}}{\omega_1 \mu} \sin\!\left(\omega_1 [t - \tau]\right) \pi_1(\tau)\, d\tau . \tag{17}$$

Now that we have particular solutions to equations (13a) and (13b), we can use equation (14) to obtain a particular solution to equation (9) then add the solution of the homogeneous equation given by equation (10) to get the general solution of equation (9).The general solution would then be given by:

$$v(t) = \int_0^t \frac{e^{-\gamma[t - \tau]}}{\omega_1 \mu} \sin(\omega_1 [t - \tau])\, \pi_1(\tau)\, d\tau \; + A\, e^{-\gamma t} \cos(\omega_1 t \; +\theta), \tag{18}$$

in which A and θ are arbitrary constants to be determined from boundary conditions. Following the technique above, we shall solve another intermediate problem. By adding equations (8a) and (8b) and defining a few new variables we get:

$$\ddot{w} = \lambda(t), \tag{19}$$

where

$$w = u_3 m + U_3 M,$$

and

$$\lambda(t) = (m + M)\, g + F_3(t).$$

As in the previous differential equation we begin by solving two related problems; first we solve the associated homogeneous problem given by:

$$\ddot{w}_0 = 0;; \tag{20}$$

secondly we solve the associated causal Green's problem below:

$$\ddot{w}_G = \delta(t - \tau), \tag{21}$$

such that

$$-\infty < t,\tau < \infty.$$

The general solutions of differential equations (20) and (21) are respectively:

$$w_0 = At + B\ , \tag{22}$$

and:

$$w_G = \mathrm{H}(t - \tau)\ (t - \tau), \tag{23}$$

where $\mathrm{H}(t)$ is the Heaviside function previously defined with A and B being constants to be determined by boundary conditions. As before we will now split the linear equation (19) into two separate problems. The sum of the solutions of the two related problems is the solution of equation (19). The two problems are:

$$\ddot{w}_1 = \frac{\lambda_1 t^2}{2}, \tag{24a}$$

and

$$\ddot{w}_2 = \lambda_2(t), \tag{24b}$$

where

$$w = w_1 + w_2 \text{ and } \lambda(t) = \lambda_1 + \lambda_2(t),$$

such that

$$\lambda_1 = (m + M)\ g + F_3^0 \text{ and } \lambda_2(t) = \Delta F_3(t).$$

Noting that λ_1 is identically zero, by equation (1a), particular solutions of equations (24a) and (24b) are respectively:

$$w_1(t) = 0, \tag{25a}$$

and

$$w_2(t) = \int_0^t (t - \tau)\ \lambda_2(\tau)\ \mathrm{d}\tau. \tag{25b}$$

Equation (25a) can be obtained by inspection of differential equation (24a) while equation (25b) was obtained by using the Green's function of equation (23) in equation (24b). Since $w = w_1 + w_2$, we can construct the general solution for equation (19) by adding the general solution to the homogeneous equation $w_0(t)$ to the particular solutions $w_1(t)$ and $w_2(t)$, giving:

$$w(t) = w_2(t) + w_1(t) + w_0(t),$$

or

$$w(t) = \int_0^t (t - \tau)\, \lambda_2(\tau)\, d\tau + At + B\,, \tag{26}$$

where A and B are again constant to be determined from boundary conditions. By manipulating the definitions of w and v in equations (9) and (19) we can obtain:

$$u_3(t) = \frac{\mu}{m}\left[\frac{w(t)}{M} + v(t)\right], \tag{27a}$$

and

$$U_3(t) = \frac{\mu}{M}\left[\frac{w(t)}{m} + v(t)\right]. \tag{27b}$$

Assuming the relatively reasonable initial conditions of $U_3(0) = u_3(0) = 0$ and $\dot{U}_3(0) = \dot{u}_3(0) = 0$ then using equations (18) and (26) we can expand equations (27a) and (27b) to its final form:

$$u_3(t) = \frac{\mu}{m}\int_0^t \frac{(t - \tau)\, \lambda_2(\tau)}{M} + \frac{e^{-\gamma[t - \tau]}}{\omega_1 \mu} \sin(\omega_1 [t - \tau])\, \pi_1(\tau)\, d\tau, \tag{28a}$$

and

$$U_3(t) = \frac{\mu}{M}\int_0^t \frac{(t - \tau)\, \lambda_2(\tau)}{m} + \frac{e^{-\gamma[t - \tau]}}{\omega_1 \mu} \sin(\omega_1 [t - \tau])\, \pi_1(\tau)\, d\tau, \tag{28b}$$

where

$$\pi_1(t) = \frac{\mu}{m}\, \Delta F_3(t) + f(t),$$

$$\lambda_2(t) = \Delta F_3(t),$$

$$\omega_1 = \sqrt{\omega_0^2 - \gamma^2},$$

such that

$$\omega_0 = \sqrt{\frac{k}{\mu}}\,,\ \gamma = \frac{b}{2\mu}\,,\ \mu = \frac{mM}{m+M}\ \text{ and }\ F_3(t) = F_3^0 + \Delta F_3(t)\,.$$

By direct comparison of equations (8c) and (8d) to equation (19), we find that the forms are identical so we can adapt the solution to equation (19), given by equation (26), for equations (8c) and (8d). The solutions are respectively:

$$u_1\,(t) = \int_0^t (t-\tau)\,\frac{F_1(\tau)}{(M\ +m)}\,\mathrm{d}\tau, \tag{29a}$$

and

$$u_2\,(t) = \int_0^t (t-\tau)\,\frac{F_2(\tau)}{(M\ +m)}\,\mathrm{d}\tau, \tag{29b}$$

where I have assumed homogeneous initial conditions. We now have a full set of equations for the motion of the vibrator as summarized below:

$$u_1\,(t) = \int_0^t (t-\tau)\,\frac{F_1(\tau)}{(M\ +m)}\,\mathrm{d}\tau\ , \tag{30a}$$

$$u_2\,(t) = \int_0^t (t-\tau)\,\frac{F_2(\tau)}{(M\ +m)}\,\mathrm{d}\tau, \tag{30b}$$

and

$$u_3(t) = \frac{\mu}{m}\int_0^t \frac{(t-\tau)\,\Delta F_3(\tau)}{M} + \frac{\mathrm{e}^{-\gamma[t-\tau]}}{\omega_1\mu}\,\sin\!\left(\omega_1\,[t-\tau]\right)\left[\frac{\mu}{m}\,\Delta F_3(\tau) + f(\tau)\right]\mathrm{d}\tau\,, \tag{30c}$$

where

$$\omega_1 = \sqrt{\omega_0^2 - \gamma^2},$$

such that

$$\omega_0 = \sqrt{\frac{k}{\mu}}\,,\ \gamma = \frac{b}{2\mu}\,,$$

$$\mu = \frac{mM}{m+M}\,,$$

and

$$F_3(t) = F_3^0 + \Delta F_3(t)\,.$$

As can be seen from equations (30a), (30b) and (30c) the displacement of the vibrator plate is inextricably linked to the forces of the elastic half-space $F_1(\tau), F_2(\tau)$, and $F_3(\tau)$. In order to complete the characterization we must have a description of the deformation and stresses within the elastic half-space due to the displacements $u_1(t)$, $u_2(t)$, and $u_3(t)$ of the plate. To accomplish this we turn to the representation theorem (Aki and Richards, 1980) which can be use to relate the displacement field throughout the elastic half-space due to sources within a given volume of the half-space as well as on the surface of that volume.

Elastic half space portion of vibrator problem:

Consider a group of N vibrators the i'th element of which has base-plate displacement characterized by:

$$u_j^i(t): \ i = 1, 2, \cdots, N \text{ and } j = 1, 2, 3, \tag{31a}$$

where the subscript j indicates three orthogonal components. These base-plates are situated on surface elements:

$$S_i(t): i = 1, 2, \cdots, N\ . \tag{31b}$$

It is assumed that the three displacements given by equation (31a) completely describe the motion of the base-plate and the half-space under the base-plate to which it is in welded contact. This rather severe restriction can be partially justified in the following manner.

By assuming the base-plate to be rigid, we have reduced its possible deformations to just a rotation and a displacement. In two dimensions a rotation and translation can be written in matrix form as:

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} u_x \\ u_y \end{bmatrix} \tag{32a}$$

where x and y are the coordinates of a point in the undeformed reference frame while X and Y are the coordinates of the same point after a rotation by angle θ and displacements in the x and y direction is given by u_x and u_y respectively. If θ is small compared to unity, then we can use small-angle approximations to rewrite equation (32a) as:

$$\begin{bmatrix} X \\ Y \end{bmatrix} \approx \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

$$= \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} u_x \\ u_y \end{bmatrix} \quad , \qquad (32b)$$

where only the translational/displacement term has survived the approximation. This restriction, however, is not crucial to our development. The forces F_1^i, F_2^i and F_3^i due to the half-space acting on the ith base-plate results from the stresses set up in the elastic half-space, either by base-plate self interaction, interaction with the other vibrators, stresses due to other external or internal sources, or any combination of these. If we represent the stress field in the elastic half space by σ_{ij}, the forces acting on the individual base-plates can be represented as:

$$F_j^i(t) = \int_{S_i} \sigma_{jk}\, n_k \, dS \qquad (33)$$

where n_k is the outward normal of the surface S of which S_i is a part. The surface S enclosing volume V together encompasses all sources of the half-space. Figure 2 shows this in two dimensions. The surface S and the volume V can be extended to cover all of the half-space.

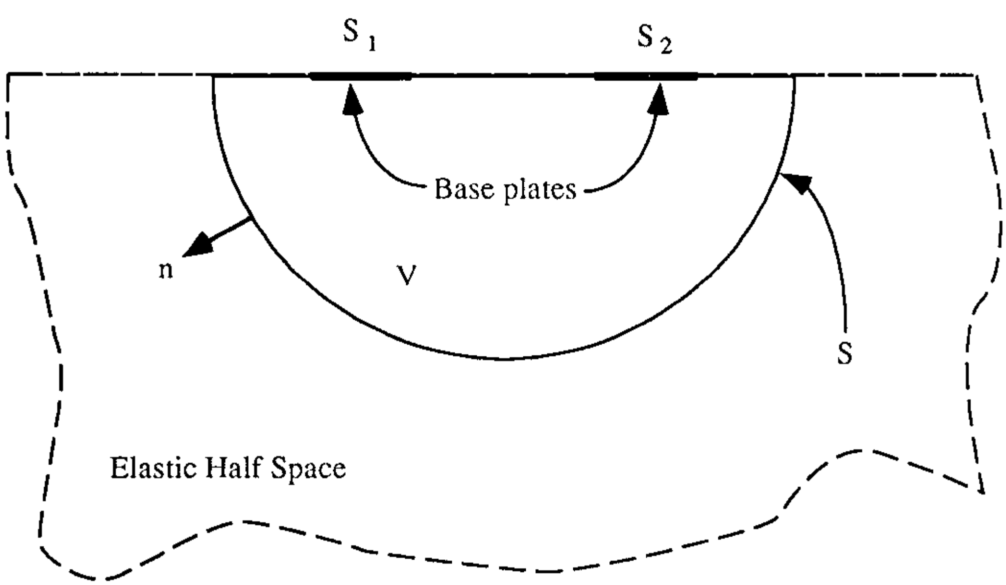

FIG. 2 Source volume in elastic half space

We will use the elastic representation theorem (Aki and Richards, 1980) to relate the displacements of the base-plates to the displacements observable throughout the elastic half-space. Since the only sources are postulated to be due to the displacements of the base plates the appropriate form of the representation theorem would be:

$$u_n(\mathbf{x},t) = \int_{-\infty}^{\infty} d\tau \int_S u_m(\xi,\tau)\, c_{mjkl}(\xi)\, n_j\, G_{nk,l}(\mathbf{x},t-\tau;\xi,0)\, dS(\xi). \tag{34}$$

where	$\mathbf{u}$	$\equiv$	displacement vector,
	$\mathbf{x}$	$\equiv$	observation point vector,
	ξ	$\equiv$	source point vector,
	$G_{nm}(\mathbf{x},t-\tau;\xi,0)$	$\equiv$	n'th component of the elastodynamic Green's function with unit impulse applied at ξ and time τ, subject to homogeneous boundary conditions,
	$G_{nk,l}$	$\equiv$	partial derivative of the Green's function with respect to source coordinate ξ_l ,
	V	$\equiv$	volume of integration containing source mechanisms,
	S	$\equiv$	closed orientable surface containing V,
	$\mathbf{n}$	$\equiv$	unit outward normal of surface S,
and	$c_{mjkl}(\xi)$	$\equiv$	elastic tensor at source location ξ.

We will be considering only homogeneous isotropic half spaces so the elastic tensor takes the special form:

$$c_{ijkl} = \lambda\, \delta_{ij}\delta_{kl} + \mu\left[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right]. \tag{35}$$

The final connection between the vibrators and the elastic half-space is made through the constitutive relation:

$$\sigma_{ij} = c_{ijkl}\, \varepsilon_{kl}, \tag{36}$$

where

$$\varepsilon_{kl} = \frac{1}{2}\left(u_{k,l} + u_{l,k}\right).$$

Due to the symmetries of the elastic tensor equation (36) can be written in the following more useful form:

$$\sigma_{ij} = c_{ijkl}\, u_{k,l}. \tag{37}$$

Conveniently we have ready made solutions for a point force on the free surface for both vertical and horizontal directions. The solutions come form the works of Miller and Pursey (1953) for vertical point forces and Cherry (1962) for horizontal forces. They were calculating the response of an elastic half space to a vibrating circular disk where the traction under the plate is assumed to be known. They then proceeded to derive the asymptotic expansions to their integral representations. As paraphrased from Cherry (1962), if the amplitude multiplied by the square of the radius of the disk remains finite as the radius of the disk approaches zero, then we have the response due to a point force on the free surface. It is just such a response which is needed in equation (34) for the Green's function. This completes the definitions needed to solve the complete set of coupled equations.

Solving the entire problem

So far we have developed a complete set of equations describing the motion of the vibrators, as given by equations (30a) through (30c); however, these equations are dependent upon the forces that the half-space exerts on the vibrators. The equation describing the motion of the half-space is given by equation (34), which in turn depends on the motion of the vibrators. These two sets of equations are related to each other by equations (33) and (37). If we start with the following initial conditions:

$$u_1 = u_2 = u_3 = 0, \tag{38a}$$

for displacements, and

$$F_1^i = F_2^i = 0 \text{ and } F_3^i = F_3^0, \tag{38b}$$

for forces, we can use the coupled equations to solve for all future displacements within the half-space. The actual procedure is most easily implemented by numerical integration and further simplification can be sought by transforming the equations into the frequency domain. Flowchart of figure 3 shows how the variables are related to each other and indicates how an algorithm may be implemented.

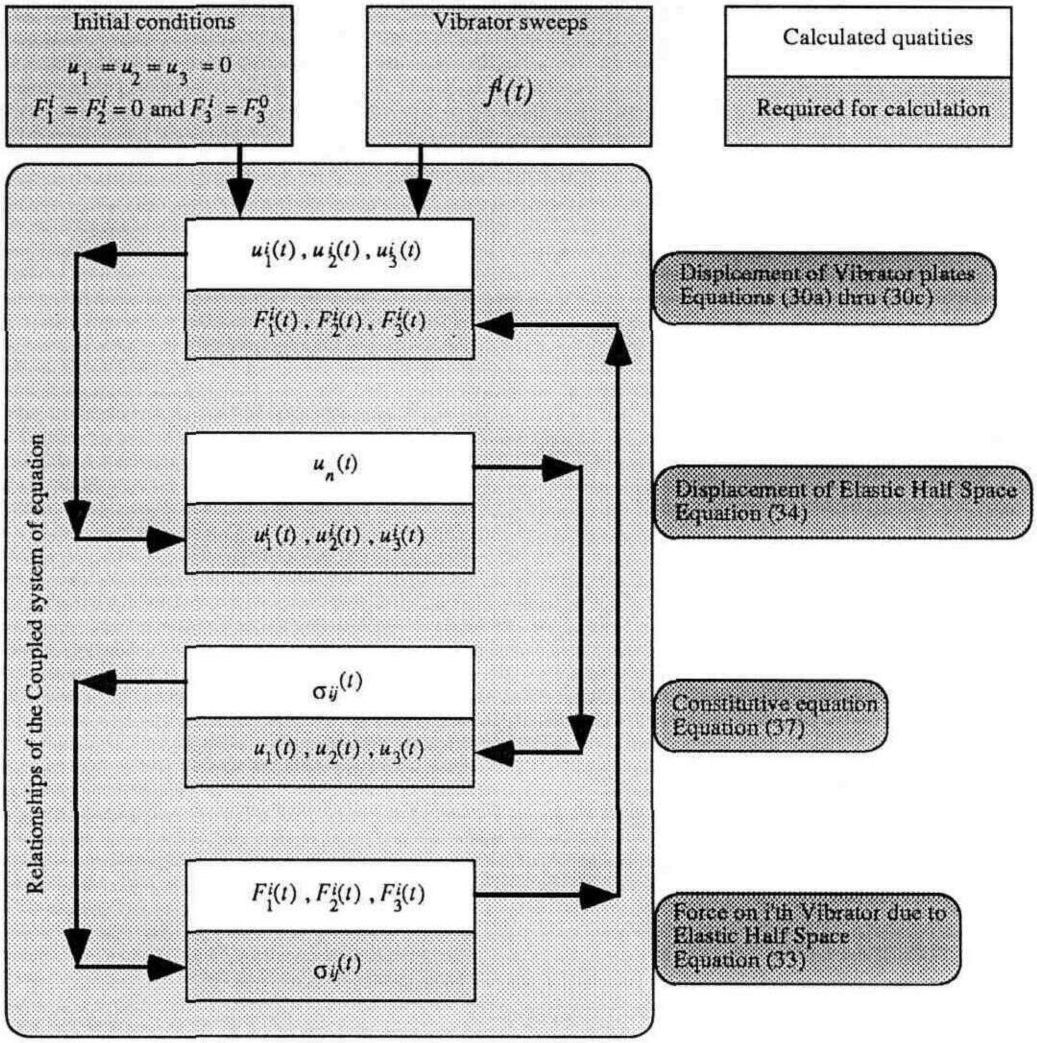

FIG. 3: Flowchart Coupled equations relationships

CONCLUSION

The equations governing the dynamic evolution of a simple vibrator model consisting of a hold-down mass a base-plate joined in series by a spring and dashpot were derived. A set of these vibrators was then allowed to interact upon a free surface over an elastic half-space. The equations controlling the dynamics of the elastic half-space were developed. These coupled equations were shown to completely describe the wavefield generated by a set of these simple vibrators over an elastic half-space. The next step would be to solve the coupled set of equations numerically.

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