Defining an explosive point source in a transversely isotropic medium

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ABSTRACT

An explosive point-source in a transversely isotropic medium is defined here using the boundary conditions at the level of the source satisfied by the displacement-stress vector. These boundary conditions are equivalent to a pressure pulse in an isotropic medium. In a transversely isotropic medium the defined source generates both quasi P- and quasi S-waves.

INTRODUCTION

One of the basic elements in calculating the response of any structure to a seismic disturbance is the way the disturbance is represented as a source of elastic waves. The starting point is usually the physical picture related to our intuition and leading to a convenient mathematical formulation. Thus, when defining an explosive point-source in an isotropic medium, it is obvious physically that a pressure field is generated at the source, acting equally in all directions. Hence, the generated motion must be spherically symmetrical, the actual mechanism by which the motion is created consisting of bringing into the system, at the source, additional matter that pushes in all directions the matter already in the system; each particle will move adjacent particles located further away from the source so that a wave is generated, the particle motion at every point being normal to the spherical wavefront and starting when the particle is reached by this wave.

We find therefore that in an isotropic medium, an explosive point-source generates a P-wave which is spherically symmetric around the source. This physical picture is translated into a mathematical form by looking for solutions of the momentum equations that have spherical symmetry and are P-waves. As it is known (Love, 1944; Ewing et al., 1957), the displacement associated with such motion S "derives" from a potential Φ,

$$\vec{S} = \nabla \Phi \tag{1}$$

the potential being a solution of the wave equation in one space dimension. The mathematical form of the potential and all the derived quantities like displacement or stress components is a direct consequence of the desired physical picture, the main aspect of it being the spherical symmetry.

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For a non-isotropic medium, however, there cannot be a spherical symmetric solution and therefore it seems that it does not make much sense to talk about an explosion source in such a medium! However, as it has been shown (Abramovici, 1987), an explosive source in medium that is inhomogeneous in the vertical direction, is equivalent to a set of four boundary conditions at the level of the source, satisfied by the components of the displacement-stress vector: two of the components are continuous and two have jumps that depend upon the strength of the source and its time-variation. It is not difficult to understand these conditions on an intuitive basis and they can be taken as defining an explosion source. This definition works also for a transversely isotropic medium, since in such a medium which is vertically inhomogeneous, the cylindrical symmetry is preserved.

In the following sections we show how this formalism works and what are the components of the displacement-stress vector expressed as double integrals over wavenumber and frequency. The integrands are combinations of exponentials in the vertical variable z with coefficients that lead in the limit to those in an isotropic medium and satisfy a compatibility condition.

In the last section we show some seismograms based on the double integral representation for some transversely isotropic materials, illustrating the behavior of an explosive source in such media.

THE DOUBLE INTEGRAL SOLUTION

As mentioned, the potential Φ solves the P-wave equation,

$$\nabla^2\Phi - \frac{1}{\alpha^2}\frac{\partial^2 \Phi}{\partial t^2} = 0 \tag{2}$$

∇^2 being the Laplacian and α the velocity of the P-waves:

$$\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}} \ . \tag{3}$$

It is known that the solutions of the above wave equation that depend only upon the distance R to the source, located at the origin are of the form

$$\Phi = \frac{f\,(t - R\,/\,\alpha)}{R} \tag{4}$$

with $R = \sqrt{x^2 + y^2 + z^2}$. The function f must satisfy some regularity conditions in order to allow Φ to satisfy the wave equation, e.g. must be continuous and have first and second order derivatives. Moreover, this function must represent the source defined above so that we must connect it with the above physical definition.

Considering a small sphere of radius R centered at the source, the additional matter brought by the source to this sphere will be a spherical shell of volume $4\pi R^2 S_R$ with S_R the displacement in the radial direction. Therefore, denoting by V the increase in volume of matter at the source, due to the activity of this source, we find the following relation:

$$V = \lim_{R \to 0} \left(4\pi R^2 S_R \right) \tag{5}$$

and since $S_R = \frac{\partial \Phi}{\partial R}$,

$$V = \lim_{R \to 0} \left(4\pi R^2 \frac{\partial \Phi}{\partial R} \right) . \tag{6}$$

Using the above expression for Φ, we have

$$\frac{\partial \Phi}{\partial R} = - \frac{f'(t - R/\alpha)}{\alpha R} - \frac{f(t - R/\alpha)}{R^2} \tag{7}$$

where f' stands for the derivative of f with respect to its whole argument. On the other hand, V is a given function of time, e.g. of the form

$$V = V_o \, g(t) \tag{8}$$

where V_o is a dimensional constant, e.g. the volume of matter generated at the source in one second. Thus, we have

$$V_o \, g(t) = \lim_{R \to 0} 4\pi \left\{ - \frac{R f'(t - R/\alpha)}{\alpha} - f(t - R/\alpha) \right\} . \tag{9}$$

As f' must be continuous due to the assumption that f'' exists, we find from this relation an expression for $f(t)$:

$$f(t) = - \frac{V_o}{4\pi} g(t) \tag{10}$$

with $g(t)$ representing the time variation of the source. Using this expression, we find the corresponding potential Φ :

$$\Phi = -\frac{V_o}{4\pi R}\, g\,(t - R/\alpha) \quad . \tag{11}$$

Representing the shifted time function $g(t - R/\alpha)$ as a Fourier integral

$$g(t - R/\alpha) = \frac{1}{\pi}\,\mathrm{Re}\int_0^\infty G(\omega)\, e^{i\,\omega(t - R/\alpha)}\, d\,\omega \tag{12}$$

where $G(\omega)$ is the Fourier transform of $g(t)$ assumed causal,

$$G(\omega) = \int_0^\infty g\,(t)\, e^{-i\,\omega t}\, d\,t \tag{13}$$

we find the following representation for the potential Φ :

$$\Phi = -\frac{V_o}{4\pi^2}\,\mathrm{Re}\int_0^\infty G(\omega)\,\frac{e^{i\,\omega(t - R/\alpha)}}{R}\, d\,\omega \quad . \tag{14}$$

Using now the well known Sommerfeld integral (Ewing et al., 1957)

$$\frac{e^{-i\frac{\omega}{\alpha}R}}{R} = \int_0^\infty \frac{e^{-K_p|z|}}{K_p}\, k\, J_0(k\, r)\, d\, k \tag{15}$$

where

$$K_p = \sqrt{k^2 - \frac{\omega^2}{\alpha^2}} \tag{16}$$

we find finally the representation of Φ as a double integral:

$$\Phi = -\frac{V_o}{4\pi^2}\,\mathrm{Re}\int_0^\infty G(\omega)\, e^{i\,\omega t}\, d\,\omega \int_0^\infty \frac{e^{-K_p|z|}}{K_p}\, J_o(kr)\, k\, d\, k \quad . \tag{17}$$

THE BOUNDARY CONDITIONS AT THE SOURCE LEVEL

All the components of the displacement-stress vector are obtained from the potential Φ through well known relations in cylindrical coordinates (Love, 1944) :

$$s_z = \frac{\partial \Phi}{\partial z}, \qquad s_r = \frac{\partial \Phi}{\partial r}, \tag{18}$$

$$\tau_{zr} = \mu \left(\frac{\partial s_r}{\partial z} + \frac{\partial s_z}{\partial r} \right), \qquad \tau_{zz} = 2\,\mu \frac{\partial s_z}{\partial r} + \lambda \Delta \tag{19}$$

with

$$\Delta = \frac{\partial s_r}{\partial r} + \frac{\partial s_z}{\partial z} + \frac{s_r}{r} \tag{20}$$

where s_z, s_r are the vertical and radial displacements and τ_{zz}, τ_{zr}, the vertical and radial stresses on a surface having the normal in the vertical direction. Applying these relations to the double integral representing Φ and interchanging the derivatives with respect to z and r with the integral signs, we find double integral representations for all the components of the displacement-stress vector, of the form:

$$s_z = \mathrm{Re} \int_0^{\infty} G(\omega)\, e^{i\omega t}\, d\omega \int_0^{\infty} k\, w\, J_o(kr)\, dk \tag{21}$$

$$s_r = \mathrm{Re} \int_0^{\infty} G(\omega)\, e^{i\omega t}\, d\omega \int_0^{\infty} q\, J_o'(kr)\, dk \tag{22}$$

$$\tau_{zz} = \mathrm{Re} \int_0^{\infty} G(\omega)\, e^{i\omega t}\, d\omega \int_0^{\infty} k\, T_w\, J_o(kr)\, dk \tag{23}$$

$$\tau_{zr} = \mathrm{Re} \int_0^{\infty} G(\omega)\, e^{i\omega t}\, d\omega \int_0^{\infty} T_q\, J_o'(kr)\, dk \tag{24}$$

where w, q, T_w, T_q are certain functions of z obtained by the procedure described above.

Taking the difference between the expression for $z > 0$, i.e. above the source and $z < 0$, i.e. below the source, and making z tend to zero, we find the boundary conditions at the source level, characterizing this type of source:

$$\lim_{z \to 0} \left[(s_z)_{z>0} - (s_z)_{z<0} \right] = \frac{V_o}{2\pi^2} \operatorname{Re} \int_0^\infty G(\omega)\, e^{i\omega t}\, d\omega \int_0^\infty k\, J_o(kr)\, dk \tag{25}$$

$$\lim_{z \to 0} \left[(s_r)_{z>0} - (s_r)_{z<0} \right] = 0 \tag{26}$$

$$\lim_{z \to 0} \left[(\tau_{zz})_{z>0} - (\tau_{zz})_{z<0} \right] = 0 \tag{27}$$

$$\lim_{z \to 0} \left[(\tau_{zr})_{z>0} - (\tau_{zr})_{z<0} \right] = \frac{V_o \mu}{\pi^2} \operatorname{Re} \int_0^\infty G(\omega)\, e^{i\omega t}\, d\omega \int_0^\infty k\, J_o'(kr)\, dk \ . \tag{28}$$

These conditions are usually taken into account not in the "integral" form, as they are presented here, but in their "local" form, i.e. as conditions for the integrands w, q, T_w and T_q:

$$\lim_{z \to 0} \left[w_{z>0} - w_{z<0} \right] = \frac{V_o}{2\pi^2} \tag{29}$$

$$\lim_{z \to 0} \left[q_{z>0} - q_{z<0} \right] = 0 \tag{30}$$

$$\lim_{z \to 0} \left[(T_w)_{z>0} - (T_w)_{z<0} \right] = 0 \tag{31}$$

$$\lim_{z \to 0} \left[(T_q)_{z>0} - (T_q)_{z<0} \right] = \frac{V_o \mu}{\pi^2} k \ . \tag{32}$$

AN EXPLOSIVE SOURCE IN A TRANSVERSELY ISOTROPIC MEDIUM

For a transversely isotropic medium, the momentum equations are, in cylindrical coordinates, in terms of the "local" z-dependent functions, w, q, T_w *and* T_q :

$$\frac{d}{dz}\begin{bmatrix} w \\ q \\ T_w \\ T_q \end{bmatrix} = \begin{bmatrix} 0 & \frac{Fk}{C} & \frac{1}{C} & 0 \\ -k & 0 & 0 & \frac{1}{L} \\ -\rho\,\omega^2 & 0 & 0 & k \\ 0 & \tilde{X} & -\frac{Fk}{C} & 0 \end{bmatrix} \begin{bmatrix} w \\ q \\ T_w \\ T_q \end{bmatrix} \tag{33}$$

where

$$\tilde{X} = -\rho\,\omega^2 + k^2\left(A - \frac{F^2}{C}\right) \tag{34}$$

and A, C, F and L are, together with another quantity denoted usually N, the five elastic parameters characterizing the transversely isotropic medium.

In a homogeneous medium there are two upgoing and two downgoing exponential solutions of the form $e^{\eta z}$, where η satisfies the compatibility condition:

$$C\,L\,\eta^4 - \{-\rho\,\omega^2(C+L) + k^2[\,C\,A - F\,(F+2L)]\}\,\eta^2$$
$$+ (k^2 L - \rho\,\omega^2)(k^2 A - \rho\,\omega^2) = 0\;. \tag{35}$$

The roots of this equation $\pm\,\eta_P$ and $\pm\,\eta_S$ tend, respectively, to

$$\sqrt{k^2 - \frac{\omega^2}{\alpha^2}}\quad,\quad \sqrt{k^2 - \frac{\omega^2}{\beta^2}} \tag{36}$$

when the medium tends to an isotropic one.

The corresponding displacement-stress vector is given by:

$$\begin{bmatrix} w \\ q \\ T_w \\ T_q \end{bmatrix} = \frac{1}{F+L}\begin{bmatrix} \eta \\ \rho\,\omega^2 + C\,\eta^2 - L\,k^2 \\ F(-\rho\,\omega^2 + L\,k^2) + L\,C\,\eta^2 \\ L(\rho\,\omega^2 + F\,k^2 + C\,\eta^2)\,\eta \end{bmatrix} e^{\eta z}\;. \tag{37}$$

Taking linear combinations of such solutions above and below the source, we get a fourth-order inhomogeneous system of equations, corresponding to the boundary conditions at the level of the source shown in the previous section. We find solving this algebraic system that both coefficients of the solutions involving exponentials $e^{\pm \eta_P z}$ and $e^{\pm \eta_S z}$ are different from zero, so that the considered explosive source generates both quasi P-wave and quasi S-wave.

SYNTHETIC SEISMOGRAMS

Using the described double integrals, we calculated seismograms for several transversely isotropic structures, the characteristics of which are given below. These materials have been used previously for a vertical point force with similar results (White, 1982), the calculations being based on the use of potentials.

Table 1 : The elastic parameters of four transversely isotropic materials are in units of dynes/cm^3, multiplied by 10^{10} and the density is in gm/cm^3 (taken from White, 1982).

parameters / models	A	C	F	L	ρ
Pierre-shale	10.00	9.20	6.80	1.30	2.00
Austin-chalk	22.00	14.00	12.00	2.40	2.20
Gypsum-soil	28.40	8.50	4.30	1.50	2.35
Plexiglas-aluminum	51.80	21.40	13.00	3.65	1.95

Figure 1 shows the source-receiver locations while Figure 2 shows the seismograms calculated for a homogeneous space made of Pierre shale, for a receiver located at a distance of 500m from the source, at different angles. The top seismograms show the vertical and horizontal components, whereas the bottom ones the radial and transverse components of motion. All we can see are a strong quasi P-wave arriving earlier and a weak quasi S-wave arriving later. On the seismograms for the radial component, the S arrival is almost inexistent. We can find the explanation of the relatively strong amplitude of the P-arrival versus the S-arrival by analyzing the reflectivity for this type of structure which is not very far from an isotropic one, for which the S-wave will be inexistent as it is not generated by the source.

On the next set of seismograms shown on Figure 3, the picture is similar, but the S-arrival is more pronounced, as the structure is less similar to an isotropic one. However, the picture is completely different on Figures 4 and 5 corresponding to structures presenting a much stronger anisotropy. Here there is an additional arrival at a later time. The explanation for this new arrival as well as the theoretical determination of the arrival times of the quasi-P and quasi-S phases was given by White (1982) based on the asymptotic approximation of the double integrals giving the seismic field. The essential ingredient of this calculation is the compatibility equation described previously, that can be used to obtain the phase velocity as a function of the phase angle. The magnitude of the phase velocities is, as usual, the ratio between frequency and wavenumber, ω/k. As it is known, the energy of the wavefield travels at a velocity equal not to the phase velocity, but to the corresponding group velocity, the direction of travel for a group being different from that dictated by the phase velocity (Berryman, 1979).

On Figure 6 we show the phase velocities (dashed curves) and the group velocities (solid curves) for the quasi-*P* and quasi-*S* as function of phase or group angle, calculated for the materials in Table 1. As we can see, for the weakly anisotropic materials, there are no cusps on the group velocity curves like those for the strongly anisotropic materials. Moreover, even for a not so strongly isotropic material like Austin-chalk, the group velocity curve does not coincide with that of the phase velocity. Plotting the travel time curves by broken lines based on the group velocities, we find agreement with the arrival times on the numerical seismograms represented in Figures 2-5. Moreover, since the curves presenting cusps correspond to the quasi-*S* waves, we conclude that the late arrivals are of quasi-*S* type, in particular the third arrival on the seismograms for the strongly anisotropic structures shown on Figures 4 and 5.

Finally, we calculated seismograms for a vertical array of receivers located at 20m separation as shown in Figure 7, at an offset of 400m. The vertical and horizontal components are shown in Figure 8. In addition, we show on Figure 9 seismograms for a horizontal array of receivers at 30m interval, located at a level 400m below the source. In both the VSP and the horizontal array, we notice the appearance of the second quasi-*S* wave, at a minimal distance or, equivalently, a minimal group angle.

CONCLUSIONS

We presented here a formalism by which an explosive point-source is defined in a transversely isotropic medium. Such a source generates both quasi-*P* and quasi-*S* waves. For strongly anisotropic solid, multiple quasi *S*-wave is generated by the source, which means that in a layered medium the physical picture of rays bouncing up and down in various layers is much more complicated. The formalism presented here can handle any layered, sectionally continuous structure with results that can be obtained within a given numerical tolerance.

ACKNOWLEDGMENTS

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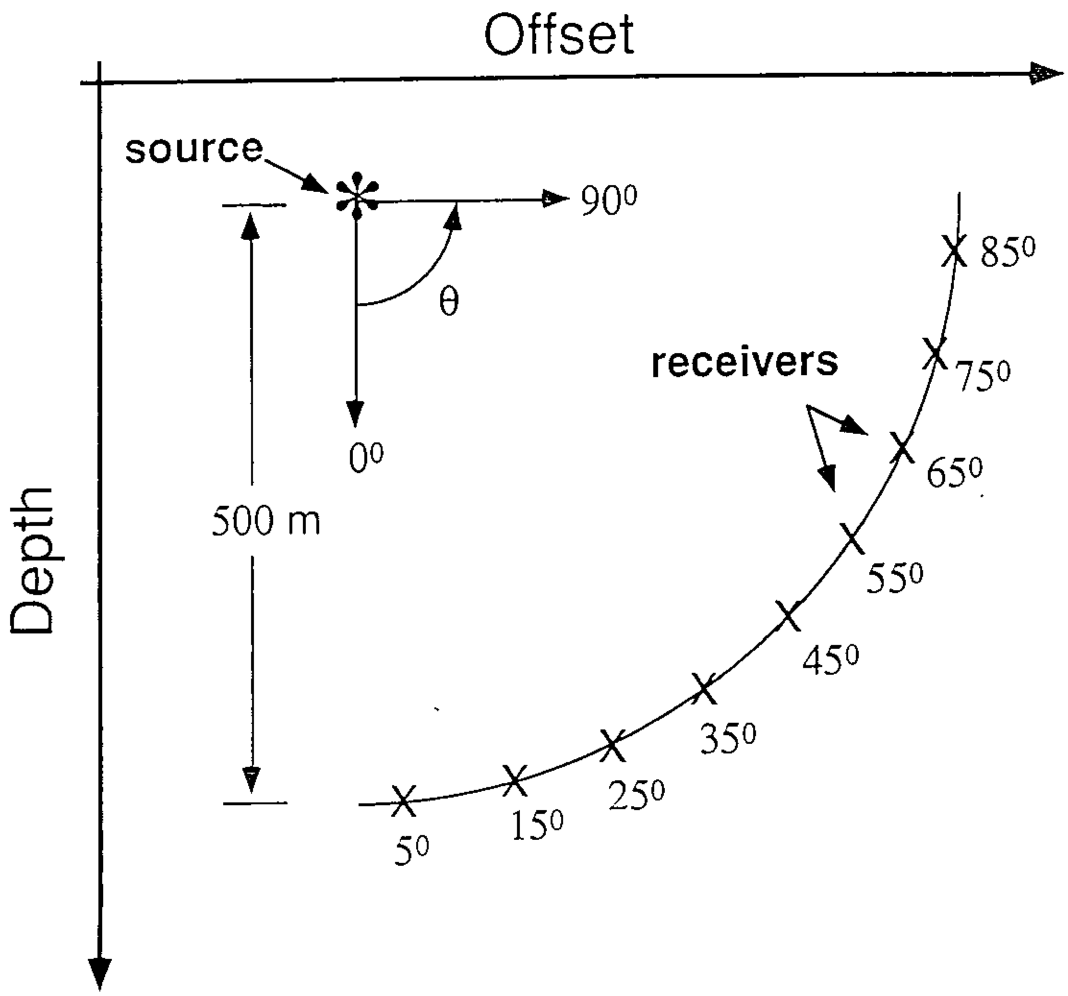

Figure 1: The location of source and receivers for the computed seismograms.

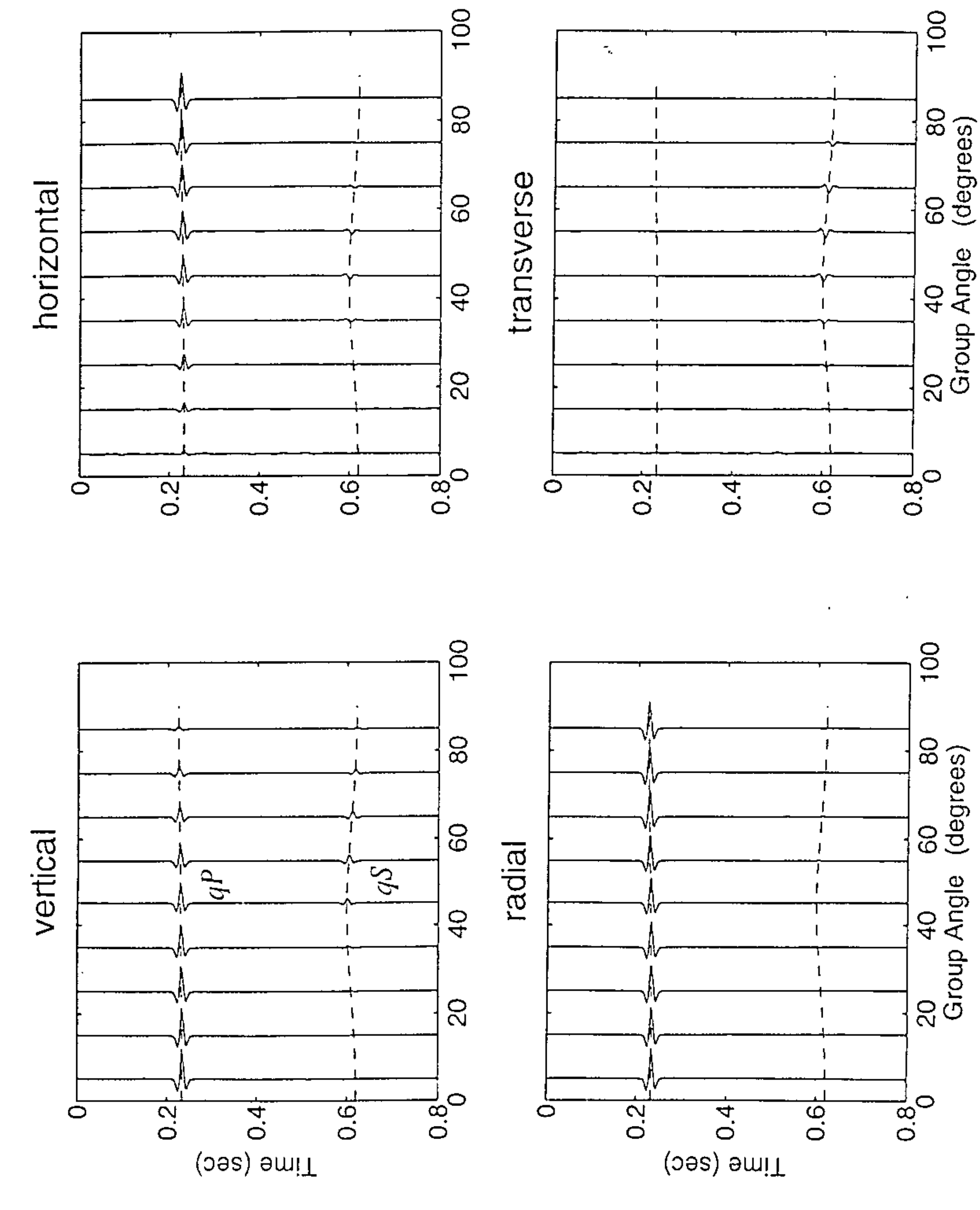

Figure 2: A profile of synthetic seismograms for a homogenous space, consisting of Pierre shale material. The receivers are located at a distance of 500m from the source, at different group angles.

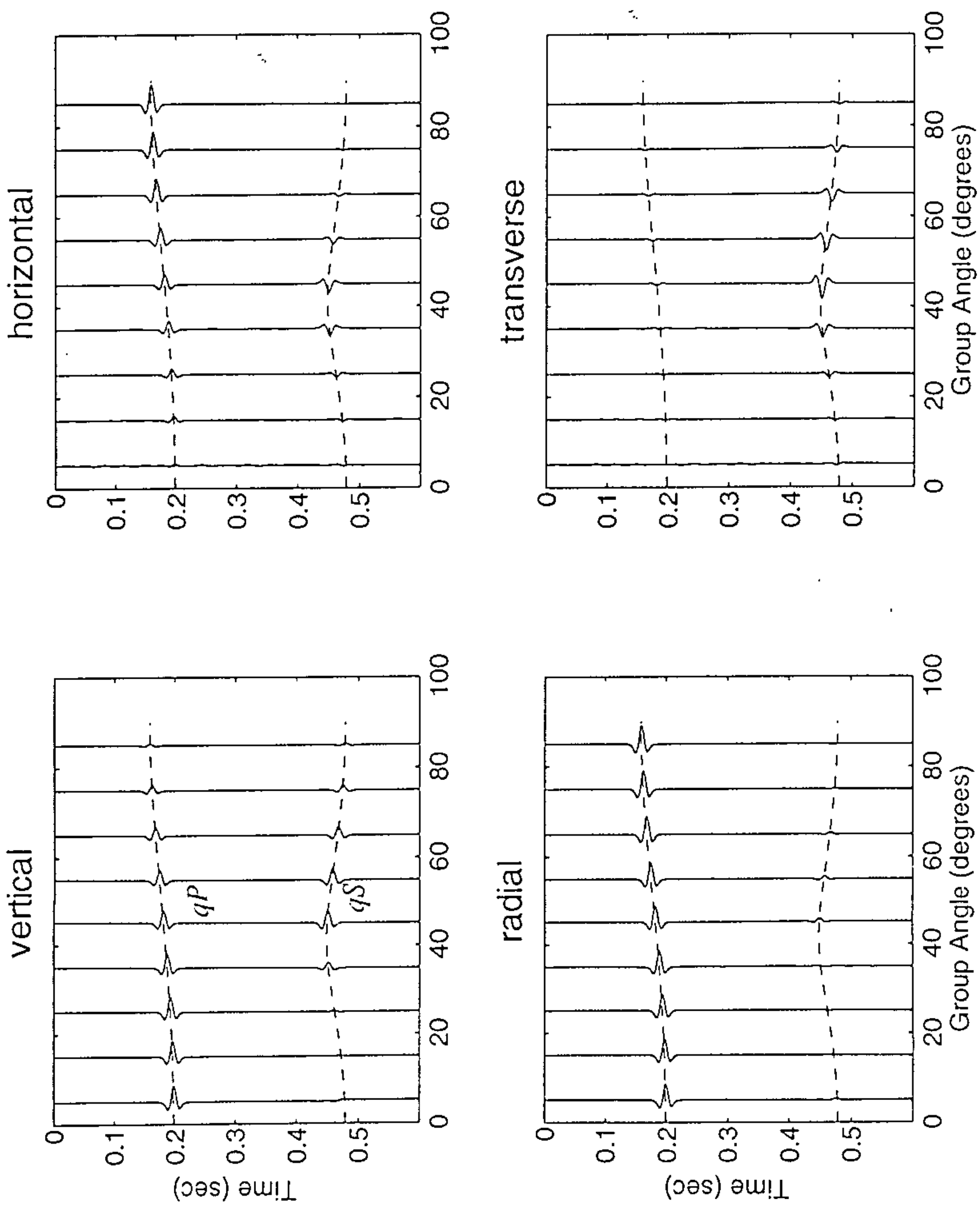

Figure 3: A profile of synthetic seismograms for a homogenous space, consisting of Austin chalk material. The receivers are located at a distance of 500m from the source, at different group angles.

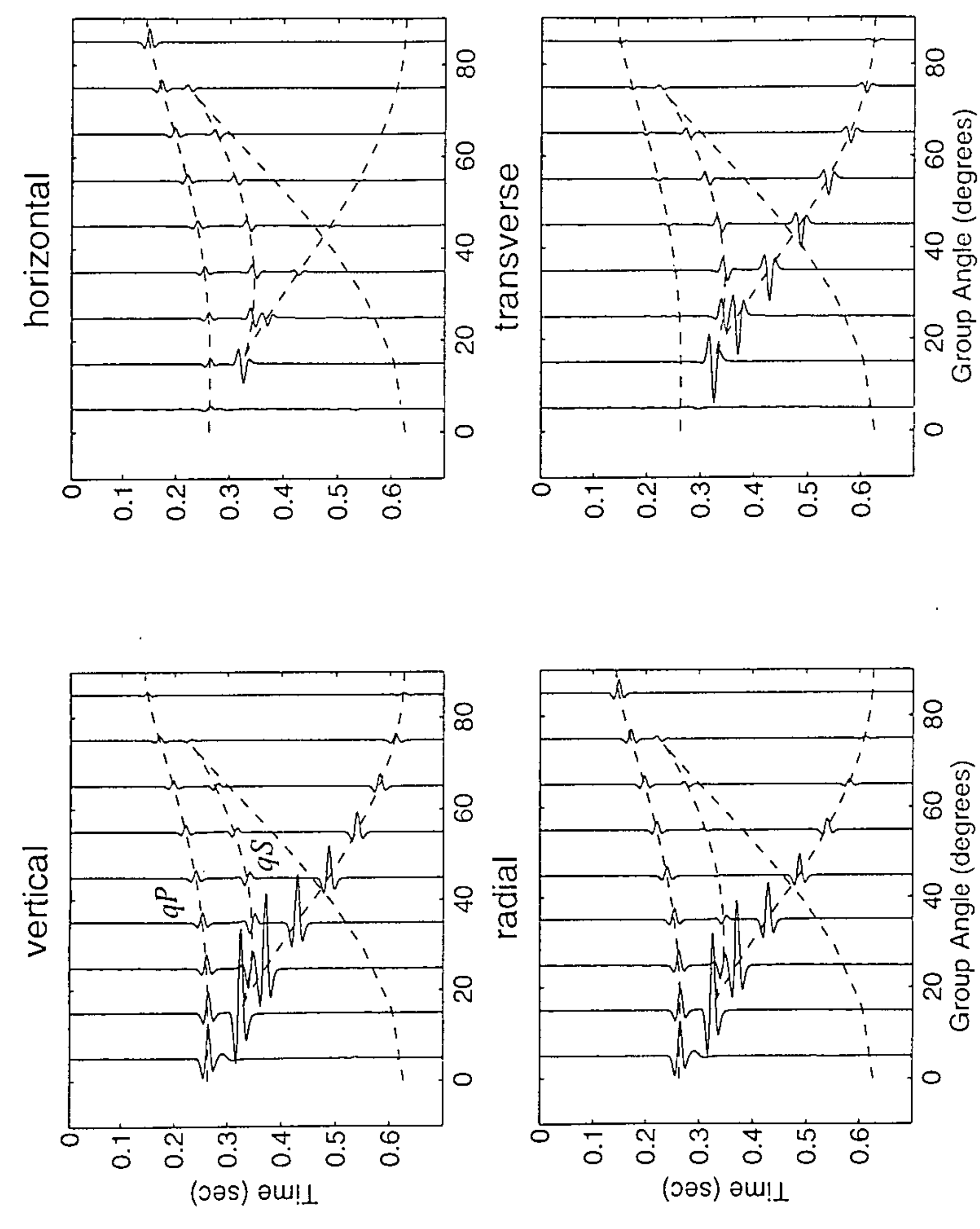

Figure 4: A profile of synthetic seismograms for a homogenous space, consisting of gypsum-soil material. The receivers are located at a distance of 500m from the source, at different group angles.

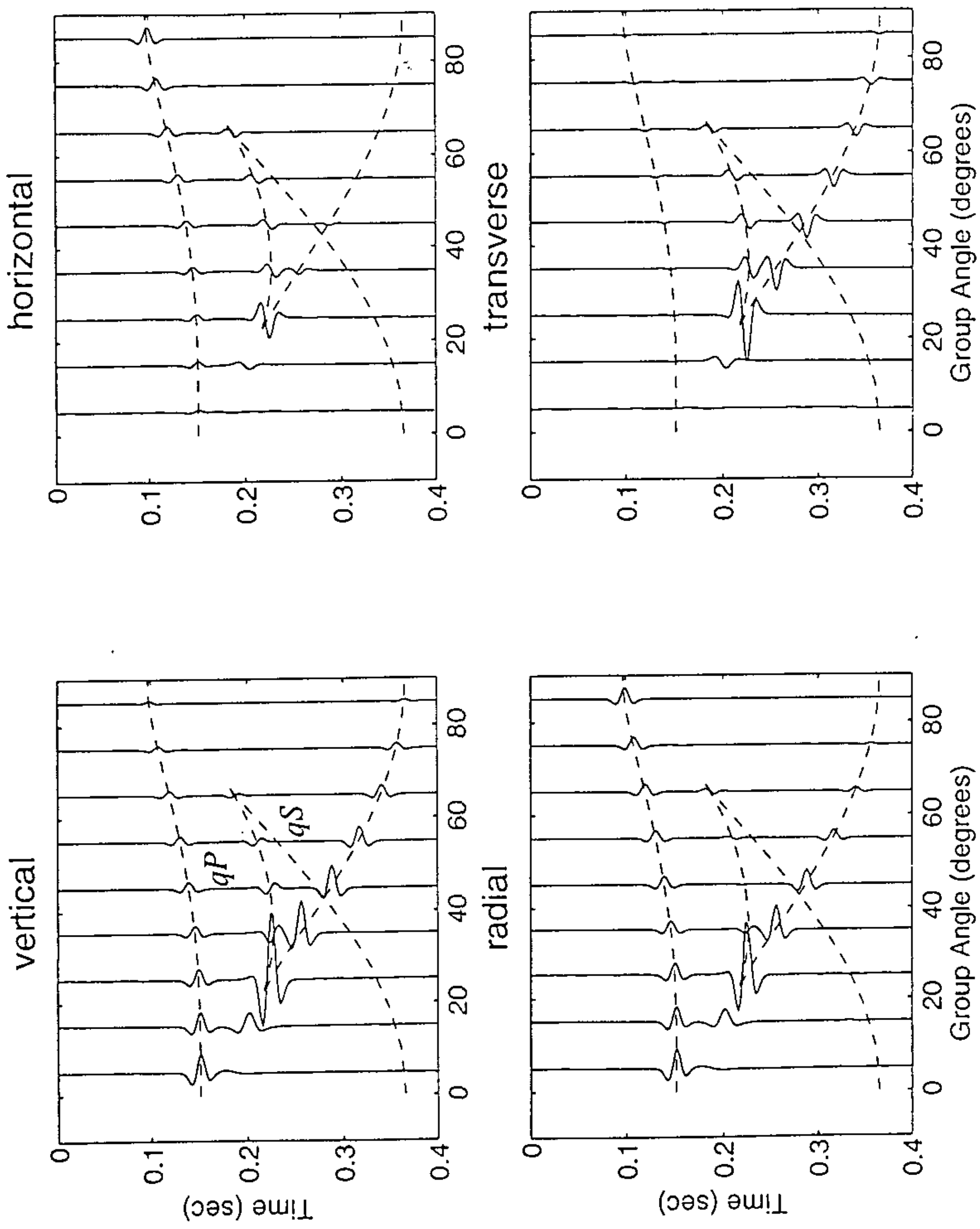

Figure 5: A profile of synthetic seismograms for a homogenous space, consisting of the composite material plexiglas-aluminum. The receivers are located at a distance of 500m from the source, at different group angles.

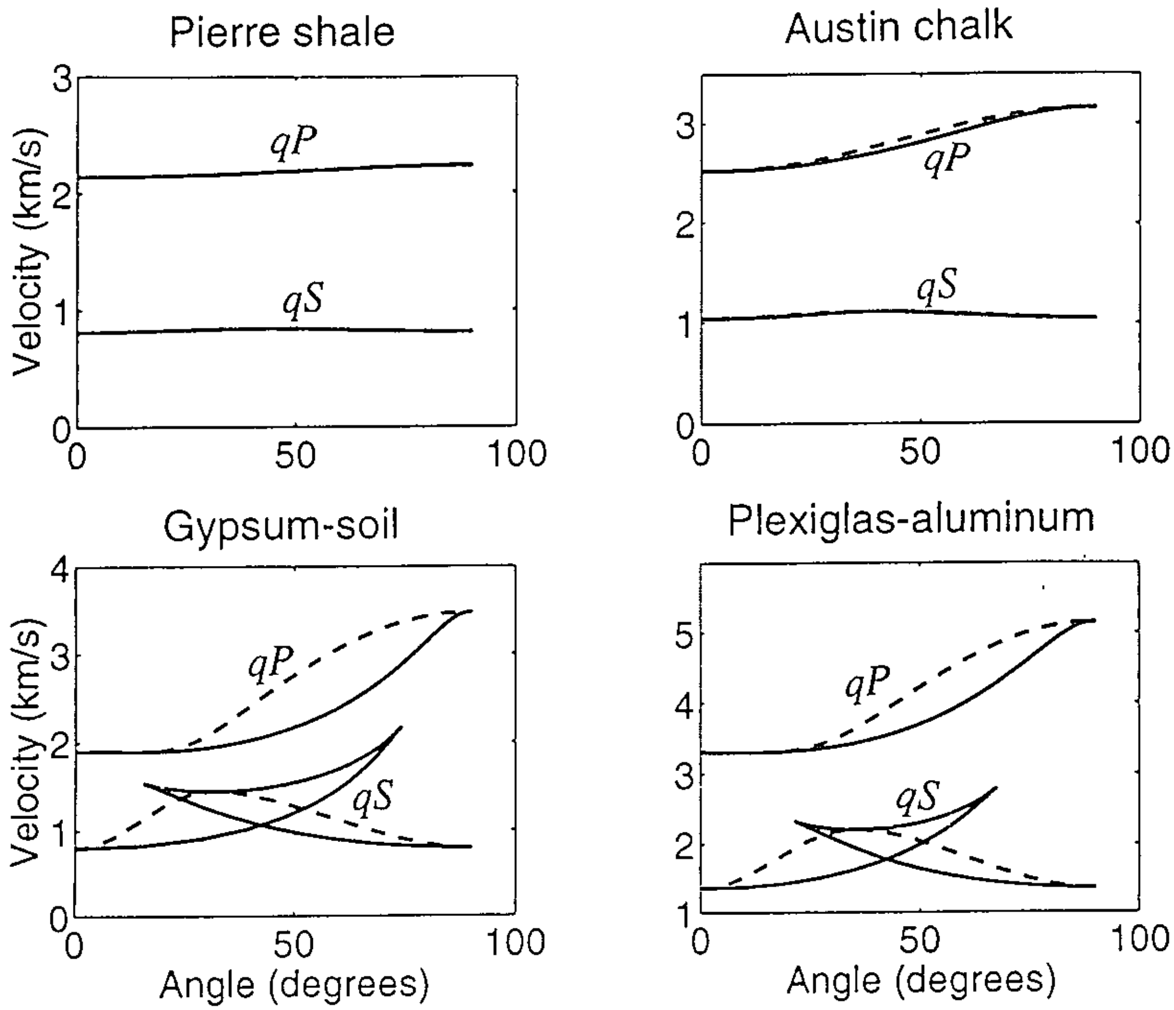

Figure 6: Group (thick curves) and phase (dashed curves) velocities of the quasi-P and quasi-S waves for the four transversely isotropic materials listed in Table 1.

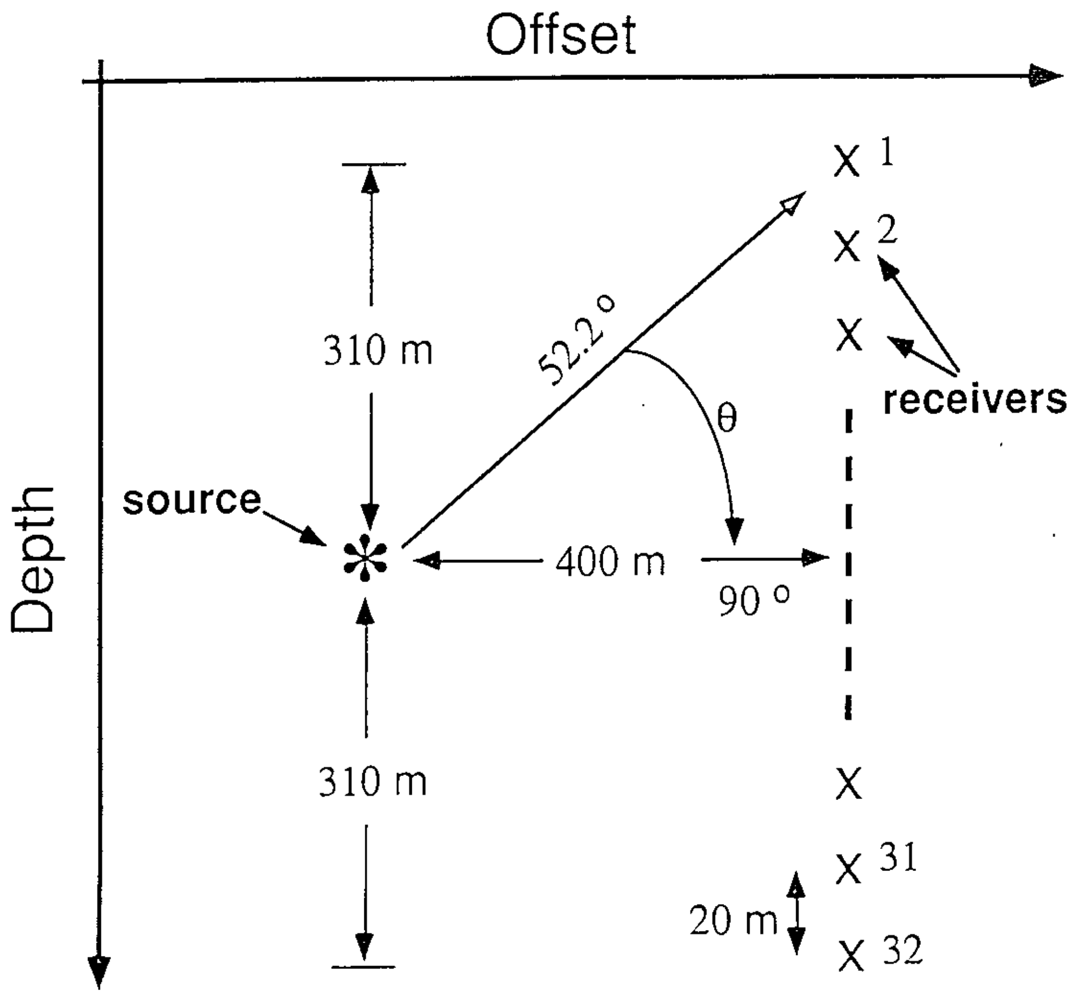

Figure 7: The location of source and receivers for a vertical seismic profile.

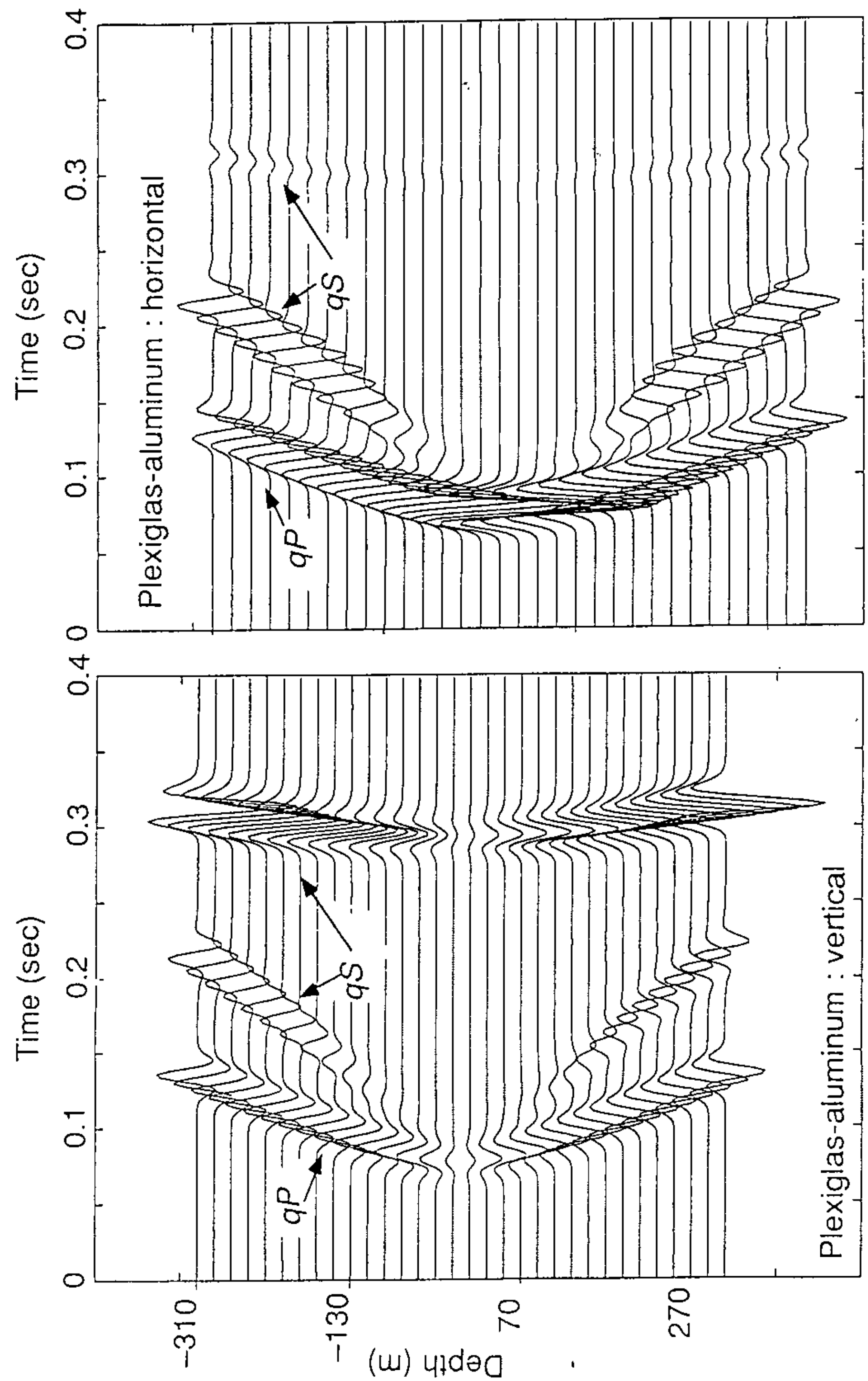

Figure 8: Vertical seismic profile in a homogeneous space consisting of the composite material plexiglas-aluminum.

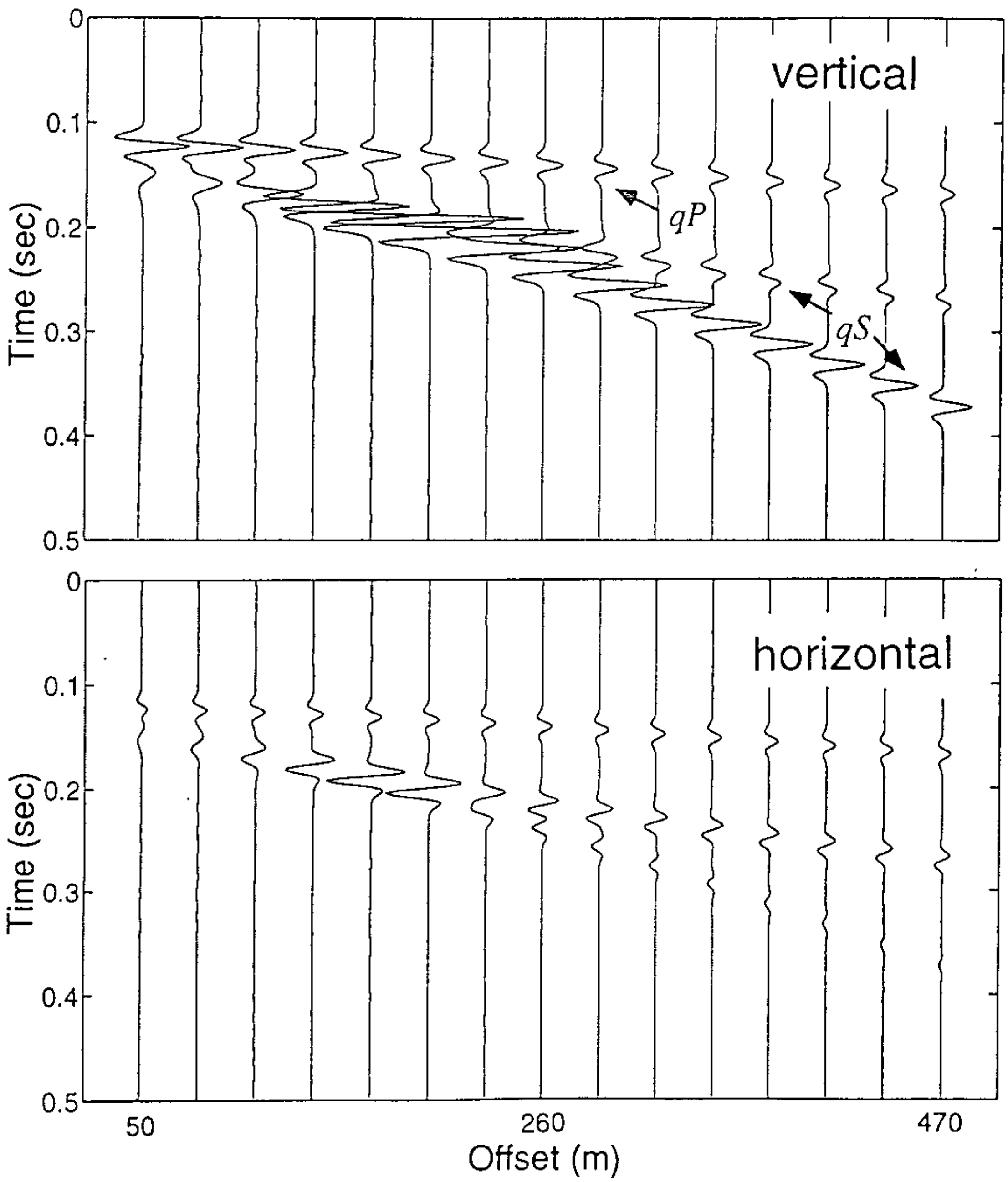

Figure 9: A profile of synthetic seismograms in a homogeneous space, with receivers forming a horizontal array. The medium is assumed to be made of the composite material plexiglas-aluminum.