The characteristic biquadratic: An illustration of transmission in anisotropic media.

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ABSTRACT

Methods describing ray bending at interfaces between anisotropic media are often very complicated. Under the assumption of weak anisotropy, and considering quasi-compressional waves, a concept of graphical illustration of analytical expressions governing the ray bending is presented. The main purpose of the presented approach is to gain an intuitive understanding of anisotropic phenomena.

PHASE VELOCITY SLOWNESS AND RAY PARAMETER IN WEAKLY ANISOTROPIC MEDIA

The very appearance of equations involving phenomena of wave propagation through anisotropic media is often intimidating. It is of great benefit to gain an intuitive insight into some of these formulæ. This can be achieved, at times, with the help of graphical illustrations. Graphical illustrations can often allow one to observe the effects of a smooth transition between isotropic and anisotropic cases, i.e., from a well known scenario to a less intuitive one. There exist various approximations rendering some of these equations more manageable. Notably, Thomsen (1986), under the assumption of weak anisotropy, provided a set of formulæ which achieve the required simplicity of form, while retaining their validity in the context of most situations encountered in exploration geophysics. A weakly anisotropic medium, as far as compressional waves are concerned, can be characterized by a vertical speed and a pair of anisotropic parameters. The notation of Thomsen (1986) is strictly followed in this note.

Consequently, the phase velocity, v, of a compressional wave is given in terms of the vertical speed, α_0, anisotropic parameters, δ and ε, and the phase angle, θ, measured with respect to the normal to the interface, so that:

$$v(\theta) = \alpha_0\left(1 + \delta \sin^2\theta \cos^2\theta + \varepsilon \sin^4\theta\right). \qquad (1)$$

The reciprocal of the phase velocity, v, i.e., the phase slowness, plays an important rôle in various studies of anisotropic phenomena, notably in ray-tracing methods for layered media. The horizontal component of phase slowness (for horizontal interfaces) is equal across all boundaries. It is referred to as the ray parameter, p.

Using the expression for this horizontal slowness component in terms of polar coordinates, one can write the equation for the ray parameter, p, in weakly anisotropic media:

$$p = \frac{\sin\theta}{\alpha_0(1+\delta\sin^2\theta\cos^2\theta+\varepsilon\sin^4\theta)}. \tag{2}$$

QUARTIC EQUATION AND THE CHARACTERISTIC BIQUADRATIC

Expressing all trigonometric functions in terms of $\sin\theta$, and rearranging, one can write equation (2) as a fourth-order polynomial:

$$p\alpha_o(\varepsilon-\delta)\sin^4\theta + p\alpha_o\delta\sin^2\theta - \sin\theta + p\alpha_o = 0. \tag{3}$$

A general solution of a fourth-order polynomial is a very laborious task. But, a rather trivial manipulation gives a great insight into the character of the solution of equation (3). One can write:

$$\alpha_0(\varepsilon-\delta)\sin^4\theta + \alpha_0\delta\sin^2\theta + \alpha_0 = \frac{1}{p}\sin\theta. \tag{4}$$

The left-hand side of equation (4) is a biquadratic expression with coefficients dependent only upon the vertical speed, α_0, and anisotropic parameters of the medium, ε and δ. The coefficients are independent of the ray direction, and are characteristic of a given medium (as long as one is considering compressional waves only).

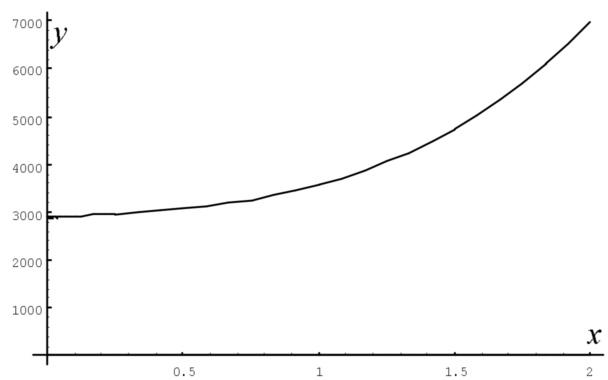

Fig. 1. A graph of a characteristic biquadratic for a medium with the following parameters: α_0 = 2925 m/s, ε = 0.224 and δ = 0.183, i.e., corresponding to the phenolic CE laminate used in many laboratory studies of anisotropy at The University of Calgary, e.g., Cheadle et al. (1991). The units of the vertical axis are metres per second, while the horizontal axis is dimensionless. The plot corresponds to the expression on the right-hand side of equation (4) $y = \alpha_0(\varepsilon-\delta)\sin^4\theta + \alpha_0\delta\sin^2\theta + \alpha_0$ plotted against $x = \sin\theta$.

The coefficient on the right-hand side of equation (4) depends only on the ray parameter, p, which is a function of the angle of incidence. Letting both sides of equation (4) equal y, and setting $\sin\theta \equiv x$, and plotting both sides of equation (4) separately versus x, one notices that the left-hand side is a curve whose y-intercept $y(0) = \alpha_0$, as illustrated in Figure 1. The right-hand side is a straight line passing through the origin, with a slope equal to the inverse of the ray parameter, p.

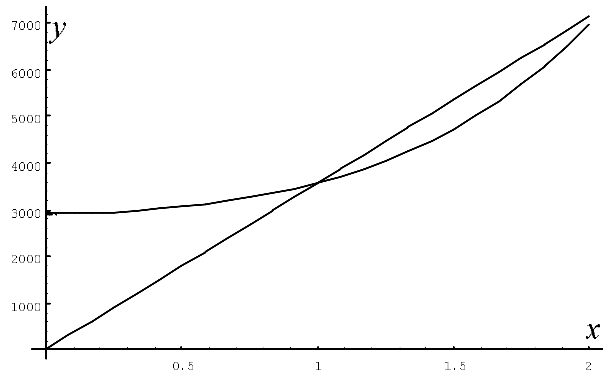

Fig.2. A graph of a characteristic biquadratic curve (for parameters see Figure 1) and the straight line plotted versus $x \equiv \sin\theta$. The inclination of the straight line corresponds to critical incidence, calculated using equation (6) as $\theta_c = 39^{\circ}$, i.e., it is such that the intercept occurs at $x \equiv \sin\theta = 1$. Counterclockwise rotation of the straight line would yield normal transmission, while clockwise rotation would yield postcritical refraction. For this illustration, the medium of incidence is assumed to be isotropic with a compressional-wave velocity, $v = 2250$, i.e., PVC used in numerous laboratory studies at The University of Calgary.

Such innocent transformation leads immediately to several interesting conclusions. The original quartic (equation (3)) has at most two real solutions. They correspond to the points of intersection of the straight line and the curve corresponding to the right-hand side, which often resembles a parabola. These two real roots can degenerate to just one real when the straight line is tangent to the graph. Finally, the original equation may have no real solutions if the straight line and the graph never touch.

For a given medium, the parameter that determines which case is applicable is the slope of the straight line. The biquadratic remains constant for all angles of incidence as long as one is considering compressional waves. Hence, it is referred to as a *characteristic biquadratic*.

THE CRITICAL ANGLE

For a physically meaningful solution, i.e., real θ, it is required that $\sin\theta$ not be greater than unity. This leads to the formulation of the critical-angle expression. The critical angle corresponds to the point where $\theta = \pi/2$ in equation (2). This gives a value of the ray parameter, p, that corresponds to the critical angle for compressional waves at the boundary between weakly anisotropic media:

$$p = \frac{1}{\alpha_0(\varepsilon + 1)}. \tag{5}$$

Recalling the definition of the ray parameter in isotropic media, i.e., $p \equiv \sin\theta_i/v$, one obtains the critical angle for compressional waves at the isotropic/anisotropic interface:

$$\theta_c = \arcsin\left\{\frac{v}{\alpha_0(\varepsilon+1)}\right\}. \tag{6}$$

Setting the anisotropic parameter, ε, to zero reduces equation (6) to the case of an isotropic/isotropic interface. One can easily rewrite equation (6) as:

$$\theta_c = \arcsin\left\{\frac{v_1}{v_2}\right\}, \tag{7}$$

where $v \equiv v_1$ is the velocity in the medium of incidence and $\alpha_0 \equiv v_2$ is the velocity in the medium of transmission. Equation (7) is the well known formula for critical angle, θ_c, at the boundary between two isotropic media.

REDUCTION TO THE PURELY ISOTROPIC CASE

To complete the concept, one notices that in the limiting case, $\varepsilon = \delta = 0$, i.e., isotropy, equation (4) reduces to the standard form of Snell's law. Thus, similar graphs can be obtained for isotropic media. As the values, δ and ε, approach zero, tending towards isotropy, the curve opens up. For a perfectly isotropic case, the curve of the left-hand side of equation (4) becomes a horizontal straight line, $y(x) = \alpha_o$. Also, in the context of isotropy, just as in the anisotropic case, transmission occurs for values of $\sin\theta < 1$, the critical angle at $\sin\theta = 1$, and postcritical incidence for $\sin\theta > 1$, as illustrated in Figure 3.

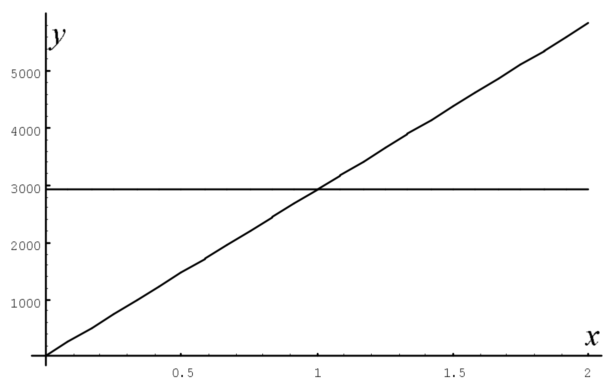

Fig.3. Degenerate biquadratic, the graph of a fourth-order polynomial becomes a horizontal straight line. A sloping line corresponds to critical incidence at the interface between two isotropic media with velocities of 2250 m/s and 2925 m/s, i.e., it is such that the intercept occurs at $x \equiv \sin\theta = 1$. Counterclockwise rotation of the straight line would yield a normal transmission, while clockwise rotation would yield postcritical refraction.

CONCLUSIONS

A graphical illustration corresponding to the fourth-order polynomial governing the transmission/refraction of compressional waves at the boundary between weakly anisotropic media has been presented. The concept of a characteristic biquadratic, a curve whose shape depends only on anisotropic parameters of a given medium, has been introduced. The presented graphical approach allows one to gain a more intuitive understanding of the phenomenon in question than offered by equations alone.

REFERENCES

Cheadle, S.P., Brown, R.J., and Lawton, D.C., 1991, Orthorhombic anisotropy: A physical seismic modeling study: Geophysics, **56**, 1603-1613.

Thomsen, L., 1986, Weak elastic anisotropy: Geophysics, **51**, 1954-1966.