Finite difference modelling, Fourier analysis, and stability

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ABSTRACT

This paper uses Fourier analysis to present conclusions about stability and dispersion in finite difference modelling. The most elementary finite difference model is presented, one dimension in space with second order accuracy in space and time. For this one spatial dimension case formulae are derived to correct for the dispersion caused by finite grid sampling. The conclusions drawn are compatible with other discussions of stability in one dimension.

INTRODUCTION

There has been much work done on the stability of finite difference algorithms. In the literature on seismic modelling, examples can be found in Aki and Richards (1980), and Lines et al (1998). These studies usually follow the Von Neumann approach with direct use of the wave equation.

The approach outlined here is to make a direct Fourier analysis of the finite difference method. A single frequency wave is operated upon by the sequence of steps required to obtain a single finite difference time step, and this is compared to the continuous case. The continuous case is then used as a standard, and the adjustment required to make the finite difference step equal to the continuous case is regarded as a correction.

THEORY

The basic wave equation in one dimension can be written in the form:

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \phi}{\partial t^2} \quad . \tag{1}$$

If the solution is limited to a particular wave number k and frequency ω, and the sine function is chosen to represent the harmonic component, then:

$$\left(\frac{\partial^2 (\sin(kx - \omega t))}{\partial x^2}\right)_{cont} = -k^2 \sin(kx - \omega t) \tag{2}$$

$$\left(\frac{\partial^2 (\sin(kx - \omega t))}{\partial t^2}\right)_{cont} = -\omega^2 \sin(kx - \omega t) \quad . \tag{3}$$

This shows that the continuous solution has no attenuation or dispersion and requires only that $v = \omega/k$.

The finite difference second derivative in x (omitting the t dependence for brevity) can be specified as:

$$\left(\frac{\partial^2(\sin(kx))}{\partial x^2}\right)_{fd} = \frac{\dfrac{\sin(k(x+\Delta x))-\sin(kx)}{\Delta x} - \dfrac{\sin(kx)-\sin(k(x-\Delta x))}{\Delta x}}{\Delta x} \tag{4}$$

where the two adjacent first finite differences are subtracted and divided by Δx to obtain a second order finite difference. In the appendix it is shown that this leads directly to:

$$\left(\frac{\partial^2(\sin(kx))}{\partial x^2}\right)_{fd} = -\sin c^2\left(\frac{k\Delta x}{2}\right)k^2\sin(kx) \quad . \tag{5}$$

Comparison of equations 2) and 5) shows that the continuous and finite difference second derivatives in x are related by:

$$\left(\frac{\partial^2(\sin(kx))}{\partial x^2}\right)_{fd} = \sin c^2\left(\frac{k\Delta x}{2}\right)\left(\frac{\partial^2(\sin(kx))}{\partial x^2}\right)_{cont} . \tag{6}$$

Similarly, the second derivatives in time are related by:

$$\left(\frac{\partial^2(\sin(\omega t))}{\partial t^2}\right)_{fd} = \sin c^2\left(\frac{\omega\Delta t}{2}\right)\left(\frac{\partial^2(\sin(\omega t))}{\partial t^2}\right)_{cont} . \tag{7}$$

To step a wave field in time by the finite difference method we can begin with a time version of equation A2):

$$\left(\frac{\partial^2(\sin(\omega t))}{\partial t^2}\right)_{fd} = \frac{1}{(\Delta t)^2}\left[\sin(\omega(t+\Delta t)) - 2\sin(\omega t) + \sin(\omega(t-\Delta t))\right] \quad . \tag{8}$$

This expression may be turned around to get:

$$\sin(\omega(t+\Delta t)) = 2\sin(\omega t) - \sin(\omega(t-\Delta t)) + (\Delta t)^2\left(\frac{\partial^2(\sin(\omega t))}{\partial t^2}\right)_{fd} . \tag{9}$$

Substituting formula 7) gives:

$$\sin(\omega(t+\Delta t)) = 2\sin(\omega t) - \sin(\omega(t-\Delta t)) + (\Delta t)^2\sin c^2\left(\frac{\omega\Delta t}{2}\right)\left(\frac{\partial^2(\sin(\omega t))}{\partial t^2}\right)_{cont} . \tag{10}$$

Substituting formula 1) gives:

$$\sin(\omega(t+\Delta t)) = 2\sin(\omega t) - \sin(\omega(t-\Delta t)) + (\Delta t)^2\sin c^2\left(\frac{\omega\Delta t}{2}\right)v^2\left(\frac{\partial^2(\sin(kx))}{\partial x^2}\right)_{cont} \tag{11}$$

and using 6) gives:

$$\sin(\omega(t+\Delta t)) = 2\sin(\omega t) - \sin(\omega(t-\Delta t)) + (\Delta t)^2 \frac{\sin c^2\left(\frac{\omega\Delta t}{2}\right)}{\sin c^2\left(\frac{k\Delta x}{2}\right)} v^2 \left(\frac{\partial^2(\sin(kx))}{\partial x^2}\right)_{fd} . \quad (12)$$

Expanding the finite difference derivative using formula A2) gives the final expression:

$$\begin{aligned}\sin(\omega(t+\Delta t)) &= 2\sin(\omega t) - \sin(\omega(t-\Delta t)) \\ &+ \frac{\sin c^2\left(\frac{\omega\Delta t}{2}\right)}{\sin c^2\left(\frac{k\Delta x}{2}\right)} v^2 \frac{(\Delta t)^2}{(\Delta x)^2}\left[\sin(k(x+\Delta x)) - 2\sin(kx) + \sin(k(x+\Delta x))\right].\end{aligned} \quad (13)$$

It is required that wave numbers and frequencies be related by kv=ω, so in order to bring the third term on the right hand side of 13) entirely to wave numbers, the substitution may be made to get:

$$\begin{aligned}\sin(\omega(t+\Delta t)) &= 2\sin(\omega t) - \sin(\omega(t-\Delta t)) \\ &+ \frac{\sin c^2\left(\frac{kv\Delta t}{2}\right)}{\sin c^2\left(\frac{k\Delta x}{2}\right)} v^2 \frac{(\Delta t)^2}{(\Delta x)^2}\left[\sin(k(x+\Delta x)) - 2\sin(kx) + \sin(k(x+\Delta x))\right].\end{aligned} \quad (14)$$

This is an improvement over the usual second order finite difference time stepping equation because the ratio of squared sinc functions makes it exact for a particular frequency/wave number. The usual expression corresponds to setting this ratio to unity. Therefore this ratio can be regarded as a correction factor to be applied after spatial differencing to make the time stepping exact.

APPLICATION – STABLE CONDITIONS

The normal one spatial dimension finite difference equation is stable where $\Delta t<\Delta x/v$, or $\omega\Delta t<k\Delta x$. The effect of the correction factor can be qualitatively investigated under these conditions.

An analysis of the correction factor requires an understanding of the sinc function, shown in Figure 1. It is symmetric about zero and takes the shape of a tapered sine wave at positive values except at zero where the function is 1. The largest wave number normally used is the Nyquist wave number given by $\pi/\Delta x$, so that $\mathrm{sinc}(k\Delta x/2)$ becomes $\mathrm{sinc}(\pi/2)$, or half way to the first zero of the function. For normal sampling ranges then, the function is always greater than zero and drops off from 1 monotonically in a positive or negative direction. A larger argument means the value of the function is smaller.

Under stable conditions the argument of the upper sinc function is less than the argument of the lower one, so the value of the upper sinc function is greater than the value of the lower one and the correction factor is greater than one. Omission of the correction factor means that the function at an advanced time is less than it should be. Continual stepping with this factor leads to exponential decay, with the higher frequencies decaying faster. All frequencies remain bounded under these conditions, consistent with the stability assumptions made above.

An alternative way to view the correction factor is to lump it in with the velocity to get a new 'pseudo' velocity (frequency dependent). Note that in the stable case the correction factor is greater than one, the 'pseudo' velocity is greater than the real velocity, and use of the real velocity will result in propagation at less than realistic rates. This lower rate has been labeled the 'uncorrected' velocity. Higher frequencies will propagate at velocities lower than low frequencies and therefore show numerical dispersion.

It is not obvious whether the correction factor should be absorbed into new 'pseudo' velocities, or result in reduced amplitudes. Model tests seem to indicate that either or both may result.

Modelling tests

Some tests have been carried out to assess the prediction value of the theory given above. Figure 2 shows a spatial wavelet with a very limited bandwidth (equivalent to a frequency of 60 Hz) in a model with a spatial Nyquist equivalent to 100 Hz. (Frequencies close to Nyquist are required to see significant effects). The wavelet was propagated through a model where $\Delta t=\Delta x/v$, and therefore the correction factor is exactly one. The initial wavelet is on the left and the wavelet was propagated to the right. Note the maximum amplitudes at 1500.

The result above can be compared to the example in Figure 3 where the same wavelet was propagated through twice as many time steps at half the sample rate without using the correction factor. The maximum amplitudes of the propagated wavelet appear here at about 1400. The lesser distance covered (the slower velocity) is obvious in a qualitative sense and is consistent with the explanation above.

Figure 4 plots the distance covered by the same wavelet under the conditions of Figure 3. The results at each time step are plotted as determined by a correlation algorithm. Also plotted are the material velocity and the 'uncorrected' velocity above. It is apparent that the 'uncorrected' velocity is more representative of the actual propagation. The 'chatter' on the curve seems to be an artifact of the correlation process.

Figure 5 is equivalent to Figure 4 except that the correction factor for a frequency of 60 Hz has been applied at each finite difference step. The correlation results follow the material velocity curve exactly except for the correlation artifacts.

APPLICATION – UNSTABLE CONDITIONS

The normal finite difference equation is unstable where $\Delta t > \Delta x/v$, or $\omega\Delta t > k\Delta x$. The effect of the correction factor under these conditions can also be investigated in qualitative and experimental ways.

With the above condition the correction factor is less than one, and its omission could lead to exponential growth instead of decay. Higher frequencies would grow more than lower frequencies, and the Nyquist frequency would grow most of all. This fits with the observations of models under unstable conditions, where the growth in amplitude of the high frequencies is very apparent.

Another factor which seems to play a role in the unstable region is the effect of insufficient sampling. A digitized wave must be adequately sampled in both space and time to propagate properly. The maximum frequency which can be represented at a sample rate of Δt is given by:

$$\omega_{\max} = \omega_{Nyquist} = \frac{1}{2\Delta t}. \tag{15}$$

The wavenumber which corresponds to this frequency is given by

$$k_{\max} = \frac{\omega_{\max}}{v} = \frac{1}{2v\Delta t}. \tag{16}$$

Since $v\Delta t > \Delta x$ (in the unstable region):

$$k_{\max} = \frac{1}{2v\Delta t} < \frac{1}{2\Delta x} = k_{Nyquist} \tag{17}$$

means that there are a range of possible spatial wave numbers between k_{max} and $k_{Nyquist}$ which can not propagate properly because they correspond to temporal frequencies greater than the temporal Nyquist frequency. The fact that these wave numbers tend to remain stationary combined with the tendency to exponential growth explains some of the features found in unstable models. Avoiding these conditions then provides stability conditions equivalent to those in other studies.

Modelling tests

A set of model parameters was chosen to illustrate normally unstable conditions. In this case the time sample rate was set at .005 seconds, twice the value for stability. Figure 6 shows the correction factor as a function of spatial wave number, and it is less than one everywhere as discussed above. This correction factor could be applied to the Fourier transform of the finite difference spatial calculations.

The Fourier transform of the above function is plotted in Figure 7 (it assumes that the imaginary components are zero, or the output will be zero phase). The actual coefficients of this transform are all zero except for the values (.25, .5, .25) centred at zero spatial shift.

A broad band wavelet was put into the model with the above parameters and propagated only four steps in the uncorrected mode. The result is shown in Figure 8. It shows the original and propagated wavelets at low amplitude at about 400 offset, and three typical unstable artifacts.

The same wavelet and model were used for the corrected modelling procedure. The correction was made by convolving the operator of Figure 7 with the result of the finite difference spatial calculations. The input wavelet and its propagated version (propagated through 100 steps) are shown in Figure 9. It is clear that the propagation is visually flawless. Although it is really not legitimate to use the correction factor at wave numbers that can not propagate properly, the attenuation it causes is sufficient in this case to eliminate the undesirable side effects.

INSTABILITY IN TWO DIMENSIONS

It seems reasonable to extend the analysis of equations 15), 16) and 17) to two dimensions to compare with other theories.

$$k_{max} = \sqrt{k_{x-max}^2 + k_{z-max}^2} \tag{18}$$

gives the relationship between the maximum model wave number and the maximum wave numbers in the x and z directions.

If we start with the assumption that time frequencies must be great enough to propagate all wave numbers in the model, then:

$$\omega_{max} \geq v k_{max} \tag{19}$$

implies

$$\omega_{max} \geq v\sqrt{k_{x-max}^2 + k_{z-max}^2} \tag{20}$$

implies

$$\frac{1}{2\Delta t} \geq v\sqrt{\frac{1}{4h^2} + \frac{1}{4h^2}} = \frac{v\sqrt{2}}{2h} \tag{21}$$

implies

$$\Delta t \leq \frac{h}{v\sqrt{2}} \tag{22}$$

where h is the sample rate in the x and z directions. This is equivalent to the condition in Lines et al (1998) for two dimensions when the spatial and temporal finite differences are of the same order.

CONCLUSIONS

Analysis of the finite difference method in the Fourier domain provides useful insights into why and how instability and dispersion occur.

In a time stepping algorithm, instability can be seen as caused, at least in part, by insufficient temporal sampling. This leads to the circumstance where some wave numbers propagate at aliased temporal frequencies.

Correction factors can be applied which work nearly perfectly in the one dimension case.

FURTHER WORK

It may be possible to understand better how the correction factor works as an amplitude modifier or as a velocity modifier.

Application of this method in two spatial dimensions should provide some insights.

ACKNOWLEDGEMENTS

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REFERENCES

Aki, K. and Richards, P. G., 1980, Quantitative seismology theory and methods: W. H. Freeman and company, New York.

Lines, L. R., Slawinski, R. and Bording, R. P., 1998, A recipe for stability analysis of finite-difference wave equation computations: Crewes 1998 research report.

APPENDIX

The derivation of formula 5) from formula 4) is a straightforward use of algebra and trigonometric identities. The starting point is:

$$\left(\frac{\partial^2(\sin(kx))}{\partial x^2}\right)_{fd} = \frac{\frac{\sin(k(x+\Delta x))-\sin(kx)}{\Delta x} - \frac{\sin(kx)-\sin(k(x-\Delta x)}{\Delta x}}{\Delta x}. \quad (4)$$

This is easily modified to:

$$\left(\frac{\partial^2(\sin(kx))}{\partial x^2}\right)_{fd} = \frac{1}{(\Delta x)^2}\left[\left(\sin(k(x+\Delta x)-\sin(kx)\right)-\left(\sin(kx)-\sin(k(x-\Delta x))\right)\right]. \quad (A1)$$

A third form of this equation (which is required later in the text) is:

$$\left(\frac{\partial^2(\sin(kx))}{\partial x^2}\right)_{fd} = \frac{1}{(\Delta x)^2}\left[\sin(k(x+\Delta x))-2\sin(kx)+\sin(k(x-\Delta x))\right]. \quad (A2)$$

The following steps follow from A1) using trigonometric formulae and algebra:

$$\left(\frac{\partial^2(\sin(kx))}{\partial x^2}\right)_{fd} = \frac{1}{(\Delta x)^2}\left[2\cos\frac{k}{2}(2x+\Delta x)\sin\frac{k}{2}(\Delta x) - 2\cos\frac{k}{2}(2x-\Delta x)\sin\frac{k}{2}\Delta x\right] \quad (A3)$$

$$\left(\frac{\partial^2(\sin(kx))}{\partial x^2}\right)_{fd} = \frac{1}{(\Delta x)^2}2\sin\left(\frac{k\Delta x}{2}\right)\left[\cos\frac{k}{2}(2x+\Delta x)-\cos\frac{k}{2}(2x-\Delta x)\right] \quad (A4)$$

$$\left(\frac{\partial^2(\sin(kx))}{\partial x^2}\right)_{fd} = \frac{1}{(\Delta x)^2}2\sin\left(\frac{k\Delta x}{2}\right)\left[2\sin(kx)\sin\left(-\frac{k\Delta x}{2}\right)\right] \quad (A5)$$

$$\left(\frac{\partial^2(\sin(kx))}{\partial x^2}\right)_{fd} = -\left(\frac{2\sin\left(\frac{k\Delta x}{2}\right)}{\Delta x}\right)^2 \sin(kx) \quad (A6)$$

$$\left(\frac{\partial^2(\sin(kx))}{\partial x^2}\right)_{fd} = -\left(\frac{\sin\left(\frac{k\Delta x}{2}\right)}{\frac{k\Delta x}{2}}\right)^2 k^2\sin(kx) \quad (A7)$$

$$\left(\frac{\partial^2(\sin(kx))}{\partial x^2}\right)_{fd} = -\sin c^2\left(\frac{k\Delta x}{2}\right)k^2\sin(kx). \quad (A8)$$

This equation (A8) is the same as equation (5) in the main text.

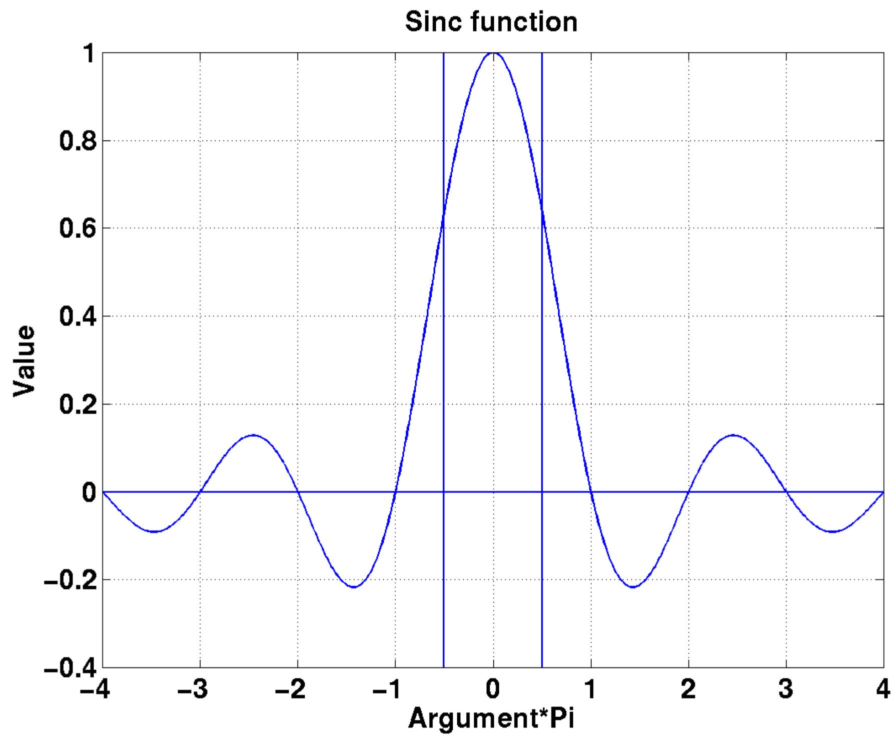

Figure 1. The central part of the sinc function. The range of interest for a correction factor is mainly from $-\pi/2$ to $\pi/2$.

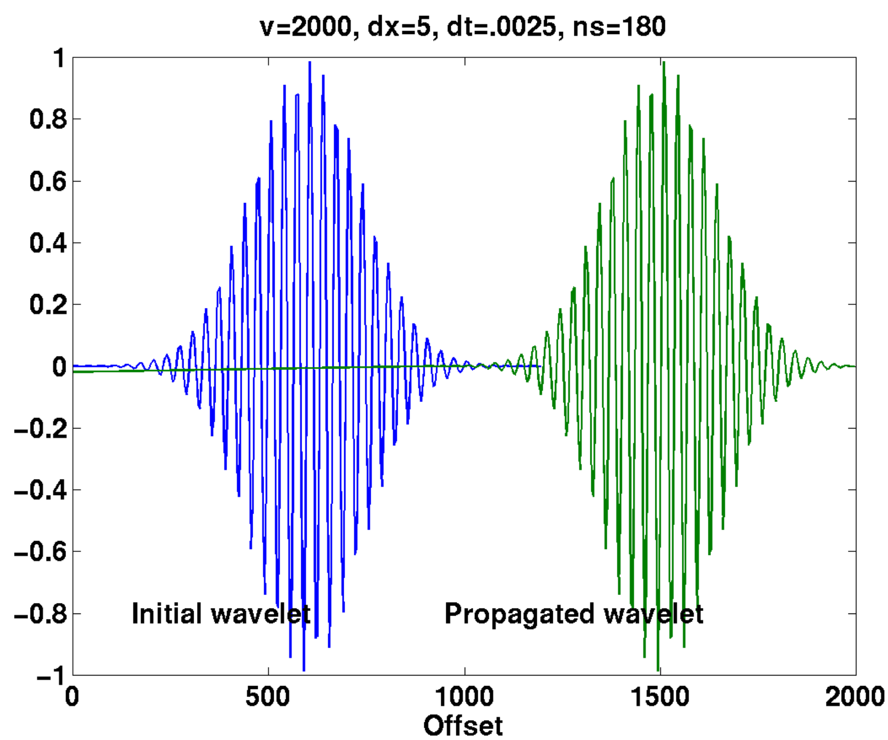

Figure 2. The single frequency wavelet propagated with no dispersion. ($\Delta t = \Delta x / v$)

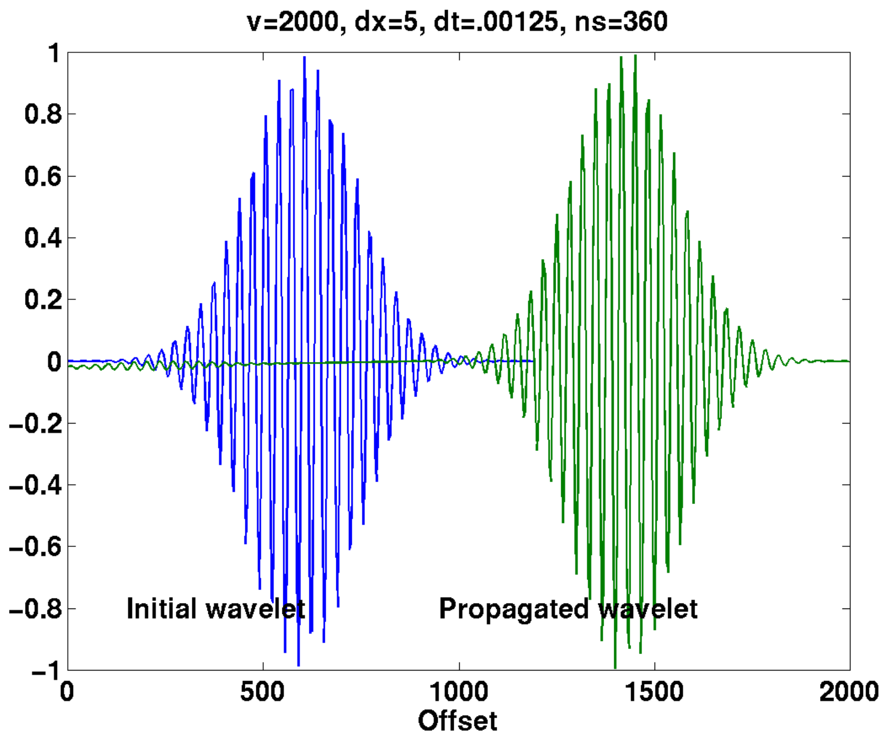

Figure 3: The single frequency wavelet propagated through the same conditions as Figure 2 except with twice as many time steps of half the size. Note lower distance covered.

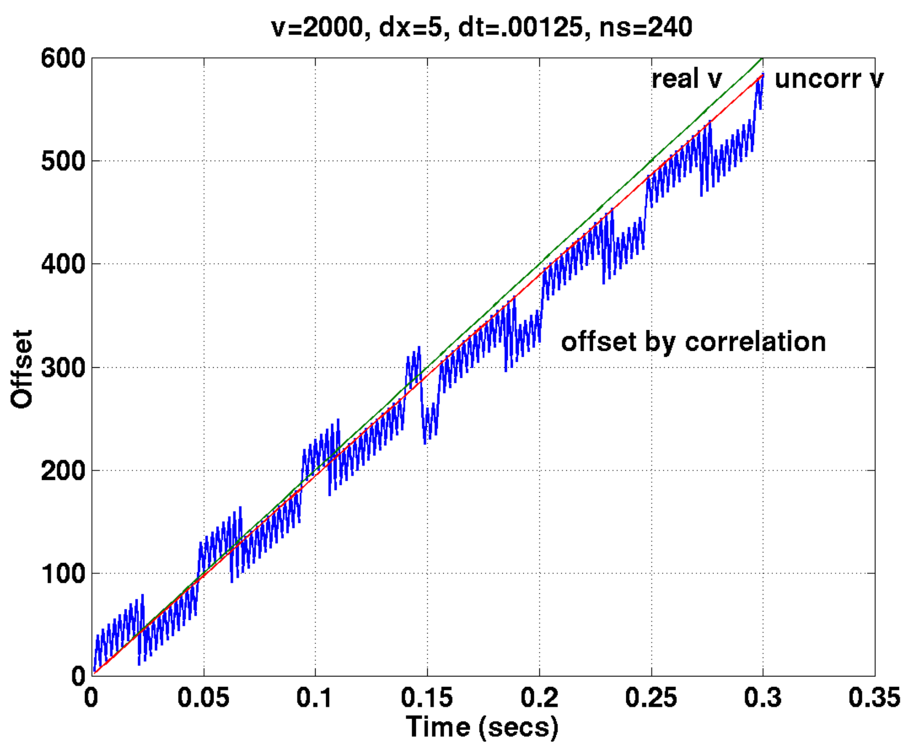

Figure 4: Offset of the propagated wavelet as determined by correlation. The ‘uncorrected’ velocity represents the offset better than the real velocity.

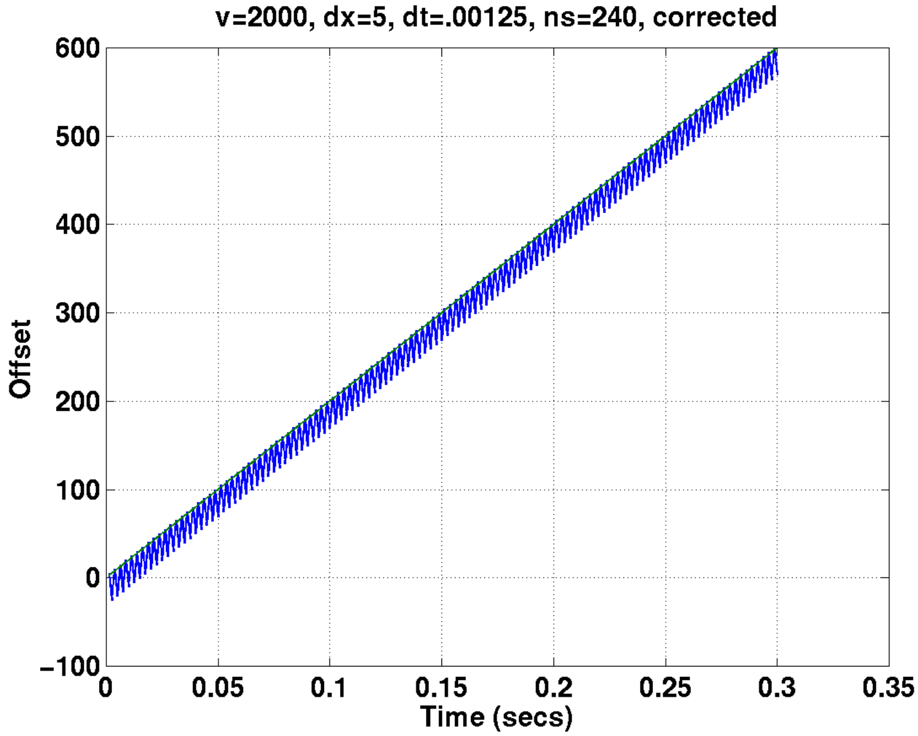

Figure 5: Offset of the propagated wavelet with the correction applied. The straight line on the plot is the real velocity.

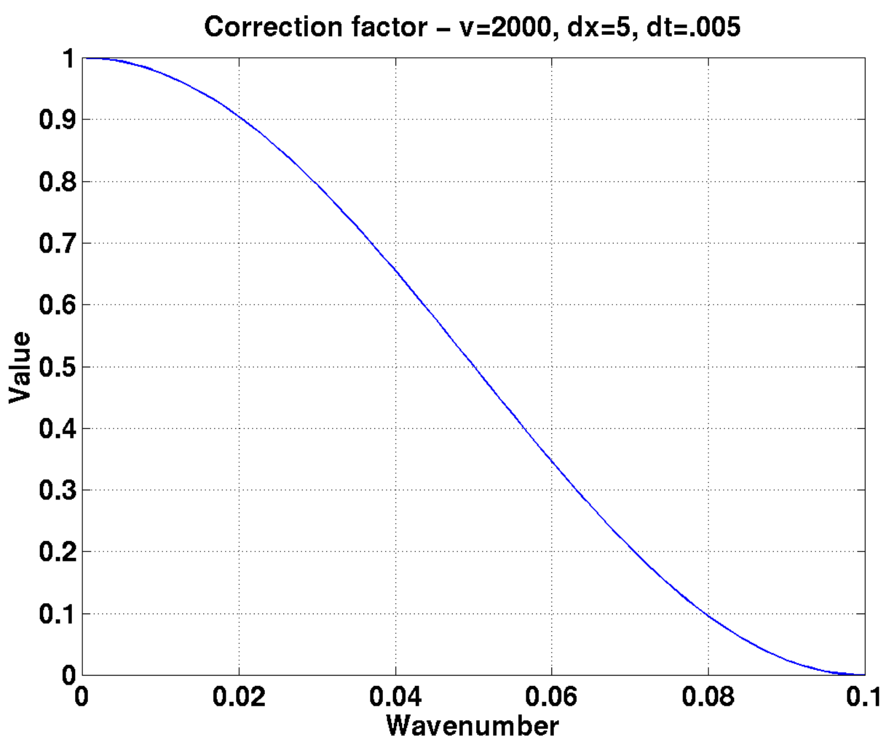

Figure 6: Correction factor in wave number space for an unstable model (with the time sample rate twice as large as it should be).

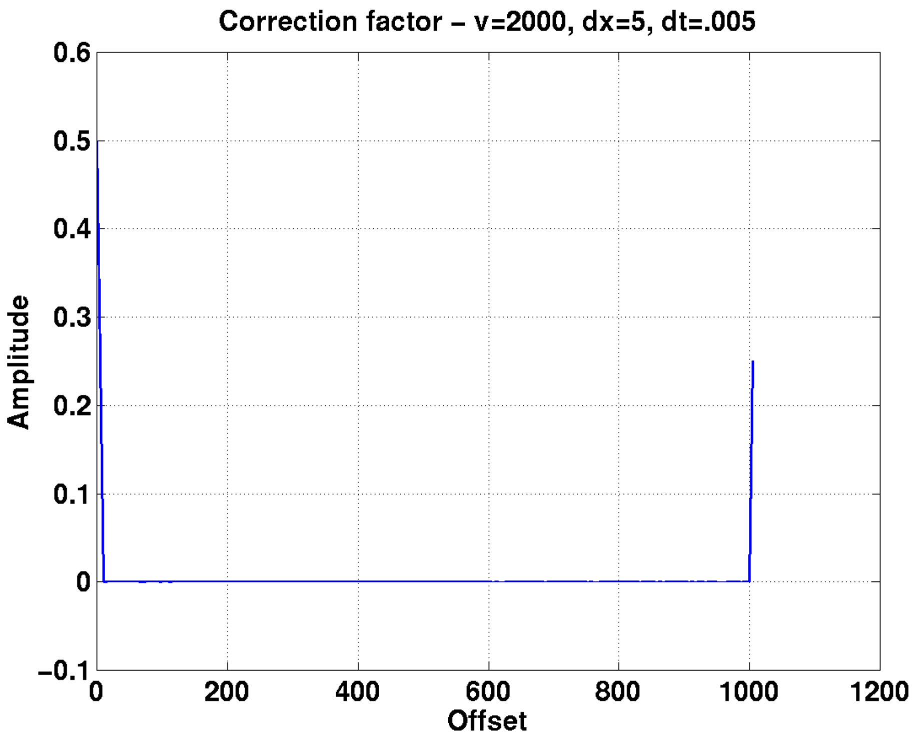

Figure 7: Correction factor in space for the unstable model parameters above. These turn out to be (.25, .5, .25) centred at zero lag.

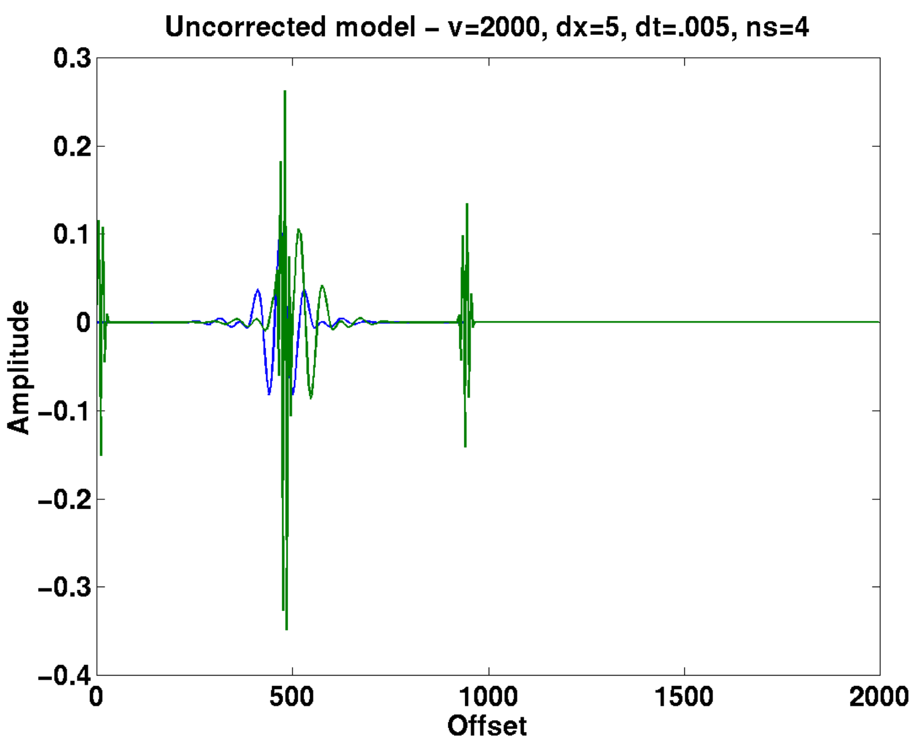

Figure 8: Uncorrected unstable model, propagated only four steps. The signature of unstable parameters appears as high frequencies at three places.

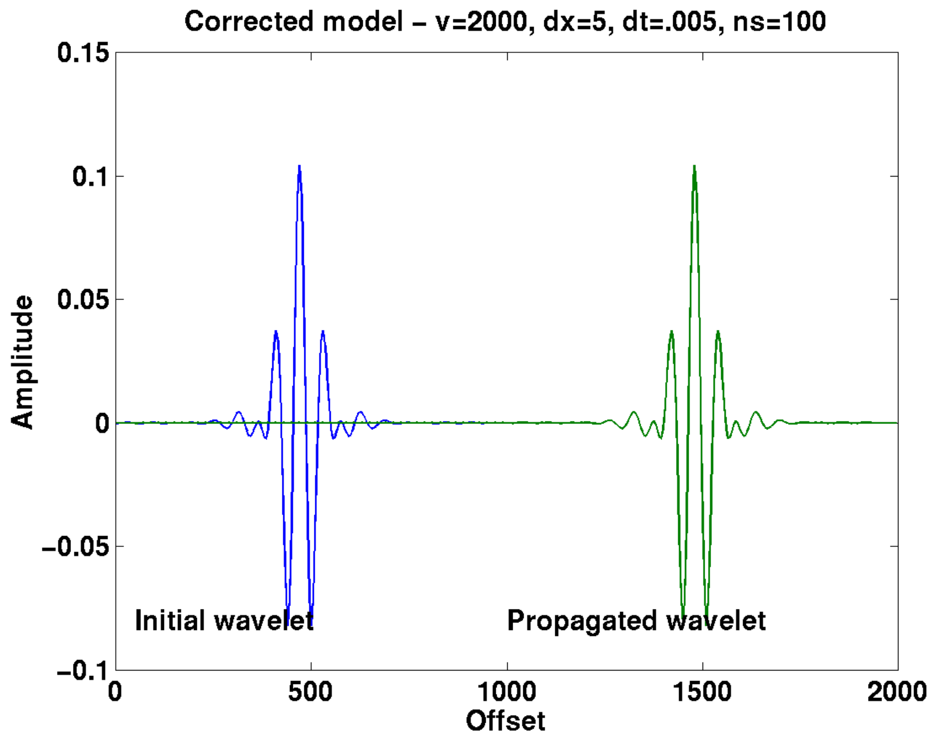

Figure 9: Corrected unstable model, propagated 100 steps. The correction factor (.25, .5, .25) was convolved at each step with the result of the spatial operation.