Elastic finite difference modelling with stability and dispersion corrections

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ABSTRACT

This paper presents a very general method to limit the dispersion and instability inherent within finite-difference elastic modelling in two dimensions. The method is based on an extension of the Von Neumann stability analysis. For a fixed frequency an analytic relationship is derived between the continuous derivatives in the elastic wave equation and their second order finite-difference approximations. Typically, the continuous derivative is equal to a finite-difference result divided by a correction factor that is a squared sinc function dependant on frequency and grid size. When the continuous derivatives are replaced by these expressions, an exact formulation of the elastic wave equation results that involves finite differences and correction factors. These correction factors are all frequency dependent. The frequency dependence can be converted to wavenumber dependence using P and S wave velocities. This allows the correction factors to be applied as spatial filters. Numerical tests show that these correction factors compensate for a wide range of dispersion and instability.

INTRODUCTION

The fact that a simple substitution of finite differences for continuous differentials is unstable or dispersive has long been understood, for example as Von Neumann stability analysis in Aki and Richards (1980). The early studies showed the nature and magnitude of the problem in order to allow an appropriate selection of sample rates. Later papers addressed the economics of solving real problems, for example by using higher order spatial derivatives to allow a more coarse spatial sample rate as in Levander (1988). This report presents another approach to improving the estimate of continuous differentials by the type of analysis presented in Press et al (1992) for stability estimation. It is very similar to, and an extension of, the one spatial dimension study in Manning and Margrave (1999).

The approach here will be to show that two dimensional finite difference formulae can be derived that will propagate pressure and shear plane waves in close approximation to the way that continuous equations will. The theoretical steps involved are presented in the following subsections:

- **The continuous wave equation**

- **Pressure wave continuous solution**

- **Equivalent finite difference pressure equation** (has the same solution as the continuous equation)

- **Shear wave continuous solution**

- **Equivalent finite difference shear equation** (has the same solution as the continuous equation and turns out to be the same as the pressure equation)

- **Formulation of frequency correction terms**

- **Equivalent equation test**

Wavefield propagation examples will be shown in the section following the theory.

THEORY

The continuous wave equation

The two dimensional, continuous, elastic wave equation is (along with a similar equation with **x** and **z** switched)

$$(\lambda + 2\mu)\frac{\partial^2 U_z}{\partial z^2} + (\lambda + \mu)\frac{\partial^2 U_x}{\partial x \partial z} + \mu \frac{\partial^2 U_z}{\partial x^2} = \rho \frac{\partial^2 U_z}{\partial t^2}. \tag{1}$$

Here λ and μ are the Lame elastic parameters for an isotropic medium and U_x and U_z are the components of the particle displacement.
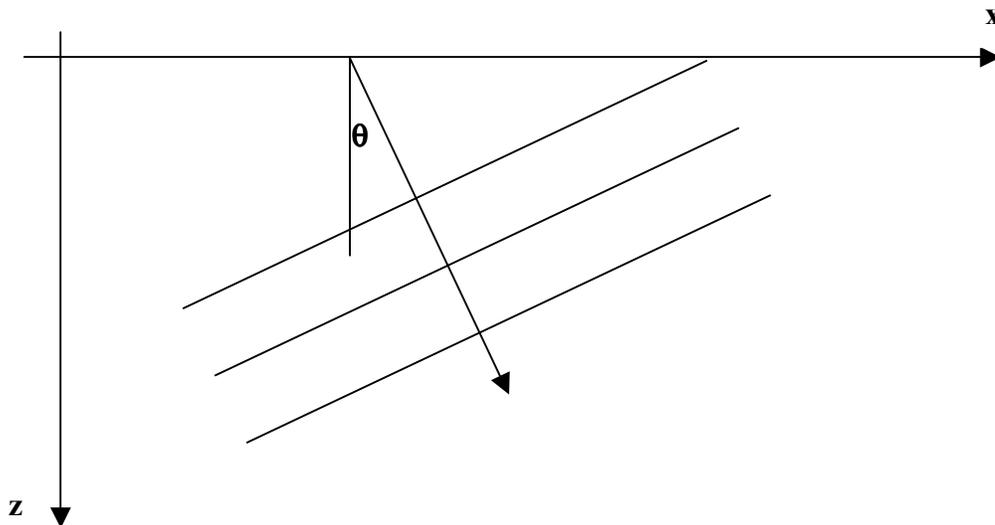

Figure 1. A monochromatic plane wave is shown propagating at an angle **θ**.

Pressure wave continuous solution

A monochromatic compressional plane wave propagating at an angle **θ** to the **z** axis is shown in Figure 1, and can be represented by the formula

$$\vec{U} = e^{i((z\cos\theta + x\sin\theta)k - \omega t)}\,\hat{k}, \qquad \hat{k} = \sin\theta\,\hat{x} + \cos\theta\,\hat{z}, \tag{2}$$

where the hat denotes a unit vector. In component notation

$$U_z = \cos\theta \; e^{i((z\cos\theta + x\sin\theta)k - \omega t)} \tag{3}$$

$$U_x = \sin\theta \; e^{i((z\cos\theta + x\sin\theta)k - \omega t)}. \tag{4}$$

For the continuous case, the following derivatives of the compressional wave can be calculated

$$\frac{\partial^2 U_z}{\partial z^2} = -\cos\theta \cos^2\theta \; k^2 e^{i((z\cos\theta + x\sin\theta)k - \omega t)} \tag{5}$$

$$\frac{\partial^2 U_z}{\partial x^2} = -\cos\theta \sin^2\theta \; k^2 e^{i((z\cos\theta + x\sin\theta)k - \omega t)} \tag{6}$$

$$\frac{\partial^2 U_x}{\partial x \partial z} = -\sin\theta \sin\theta \cos\theta \; k^2 e^{i((z\cos\theta + x\sin\theta)k - \omega t)} \tag{7}$$

$$\frac{\partial^2 U_z}{\partial t^2} = -\cos\theta \; \omega^2 \; e^{i((z\cos\theta + x\sin\theta)k - \omega t)}. \tag{8}$$

To check that the pair of equations (3) and (4) are a solution of (1), equations (5), (6), (7) and (8) can be substituted into (1) giving

$$-\cos\theta \; k^2 e^{i((z\cos\theta + x\sin\theta)k - \omega t)} \left\{ (\lambda + 2\mu)\cos^2\theta + (\lambda + \mu)\sin^2\theta + \mu\sin^2\theta \right\}$$
$$= -\cos\theta \; \omega^2 \rho \; e^{i((z\cos\theta + x\sin\theta)k - \omega t)} \tag{9}$$

This is true if

$$k^2 \left\{ (\lambda + 2\mu)\cos^2\theta + (\lambda + \mu)\sin^2\theta + \mu\sin^2\theta \right\} = \omega^2 \rho, \tag{10}$$

and true if

$$k^2 \left\{ \lambda\left(\cos^2\theta + \sin^2\theta\right) + 2\mu\left(\cos^2\theta + \sin^2\theta\right) \right\} = \omega^2 \rho, \tag{11}$$

and true if

$$\frac{\lambda + 2\mu}{\rho} = \frac{\omega^2}{k^2} = v_\alpha^{\;2}, \tag{12}$$

where v_α is the compressional wave velocity.

Equivalent finite difference pressure equation

To investigate the results of a two-dimensional finite difference operator, the finite difference formulae can be applied to the waveform of equations (3) and (4). The second derivative in the direction of the axes is the same as the one dimensional form, so that

$$D_{\Delta z}{}^2 U_z = \frac{\cos\theta}{(\Delta z)^2}\left\{ \begin{array}{l} e^{i\left(((z+\Delta z)\cos\theta+x\sin\theta)k-\omega t\right)} - e^{i\left((z\cos\theta+x\sin\theta)k-\omega t\right)} \\ -e^{i\left((z\cos\theta+x\sin\theta)k-\omega t\right)} + e^{i\left(((z-\Delta z)\cos\theta+x\sin\theta)k-\omega t\right)} \end{array} \right\} \tag{13}$$

$$D_{\Delta z}{}^2 U_z = \frac{\cos\theta}{(\Delta z)^2} e^{i\left((z\cos\theta+x\sin\theta)k-\omega t\right)}\left\{ \begin{array}{l} e^{i\frac{k\Delta z}{2}\cos\theta} \\ -e^{i\frac{k\Delta z}{2}\cos\theta} \end{array} \right\}\left\{ \begin{array}{l} e^{i\frac{k\Delta z}{2}\cos\theta} \\ -e^{i\frac{k\Delta z}{2}\cos\theta} \end{array} \right\} \tag{14}$$

$$D_{\Delta z}{}^2 U_z = \cos\theta\, \frac{\left\{2i\sin\left(\frac{k\Delta z}{2}\cos\theta\right)\right\}^2}{(\Delta z)^2} e^{i\left((z\cos\theta+x\sin\theta)k-\omega t\right)} \tag{15}$$

$$D_{\Delta z}{}^2 U_z = -\cos\theta\, k^2 \cos^2\theta\, \frac{\sin^2\left(\frac{k\Delta z}{2}\cos\theta\right)}{\left(\frac{k\Delta z}{2}\cos\theta\right)^2} e^{i\left((z\cos\theta+x\sin\theta)k-\omega t\right)} \tag{16}$$

$$D_{\Delta z}{}^2 U_z = -\cos\theta\, k^2 \cos^2\theta\, \operatorname{sinc}^2\left(\frac{k_z\Delta z}{2}\right) e^{i\left((z\cos\theta+x\sin\theta)k-\omega t\right)}, \tag{17}$$

which takes the same form as the continuous differential but with the extra sinc squared factor, specifically

$$D_{\Delta z}{}^2 U_z = \operatorname{sinc}^2\left(\frac{k_z\Delta z}{2}\right)\frac{\partial^2 U_z}{\partial z^2}. \tag{18}$$

Similarly it can be shown that

$$D_{\Delta x}{}^2 U_z = -\cos\theta\, k^2 \sin^2\theta\, \frac{\sin^2\left(\frac{k\Delta x}{2}\sin\theta\right)}{\left(\frac{k\Delta x}{2}\sin\theta\right)^2} e^{i\left((z\cos\theta+x\sin\theta)k-\omega t\right)} \tag{19}$$

$$D_{\Delta x}{}^2 U_z = -\cos\theta\, k^2 \sin^2\theta\, \operatorname{sinc}^2\left(\frac{k_x\Delta x}{2}\right) e^{i\left((z\cos\theta+x\sin\theta)k-\omega t\right)} \tag{20}$$

$$D_{\Delta x}{}^2 U_z = \operatorname{sinc}^2\left(\frac{k_x\Delta x}{2}\right)\frac{\partial^2 U_z}{\partial x^2}. \tag{21}$$

It should be emphasized that equations (18) and (21) are true only for a monochromatic plane wave characterized by the wave number **k**.

Equations (18) and (21) give the contribution of displacement in the **z** direction to acceleration in the **z** direction. The contribution from U_x has single differentials in both **x** and **z**. The staggered-grid representation displaces this **x** component one half grid spacing in **x** and **z**, and the output point then can be made to coincide with the output points for the terms in (18) and (21) as shown in Figure 2.
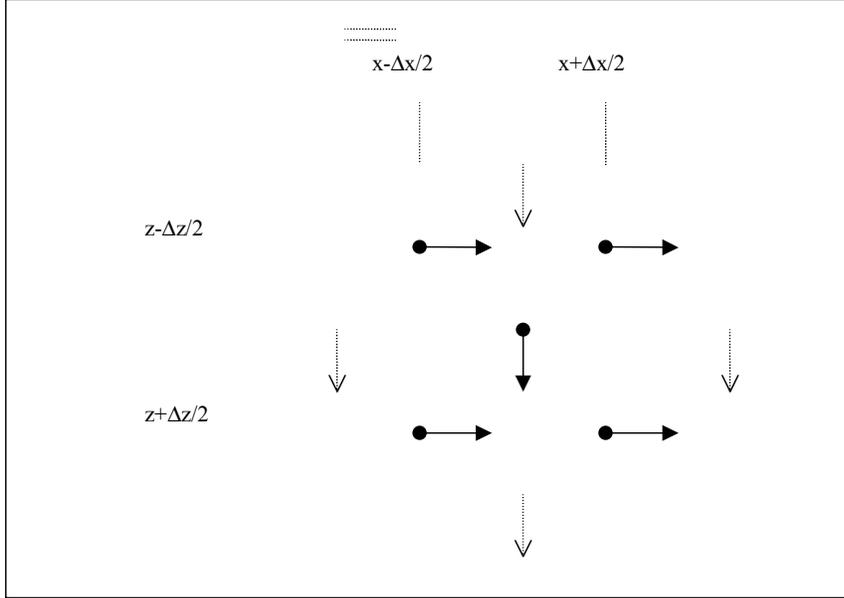

Figure 2. The arrows show the relative positions of the input and output displacements for the second term of equation (1) using the staggered grid. The output Z acceleration is in the centre. The four contributing input X displacements surround it evenly.

The U_x contribution then takes the following form

$$D_{\Delta x \Delta z} U_x = \frac{\sin \theta}{(\Delta x \Delta z)} \left\{ \begin{array}{l} e^{i\left(\left[\left(z+\frac{\Delta z}{2}\right)\cos\theta + \left(x+\frac{\Delta x}{2}\right)\sin\theta\right]k - \omega t\right)} \\ -e^{i\left(\left[\left(z+\frac{\Delta z}{2}\right)\cos\theta + \left(x-\frac{\Delta x}{2}\right)\sin\theta\right]k - \omega t\right)} \\ -e^{i\left(\left[\left(z-\frac{\Delta z}{2}\right)\cos\theta + \left(x+\frac{\Delta x}{2}\right)\sin\theta\right]k - \omega t\right)} \\ +e^{i\left(\left[\left(z-\frac{\Delta z}{2}\right)\cos\theta + \left(x-\frac{\Delta x}{2}\right)\sin\theta\right]k - \omega t\right)} \end{array} \right\}$$

(22)

$$D_{\Delta x \Delta z} U_x = \frac{\sin\theta}{(\Delta x \Delta z)} \left[\begin{array}{c} e^{i\left(\left[\left(z+\frac{\Delta z}{2}\right)\cos\theta + x\sin\theta\right]k - \omega t\right)} \left\{ e^{i\frac{k\Delta x}{2}\sin\theta} - e^{-i\frac{k\Delta x}{2}\sin\theta} \right\} \\ -e^{i\left(\left[\left(z-\frac{\Delta z}{2}\right)\cos\theta + x\sin\theta\right]k - \omega t\right)} \left\{ e^{i\frac{k\Delta x}{2}\sin\theta} - e^{-i\frac{k\Delta x}{2}\sin\theta} \right\} \end{array} \right] \tag{23}$$

$$D_{\Delta x \Delta z} U_x = \sin\theta \frac{2i\sin\left(\frac{k\Delta x}{2}\sin\theta\right)}{(\Delta x \Delta z)} e^{i((z\cos\theta + x\sin\theta)k - \omega t)} \left\{ \begin{array}{c} e^{i\frac{k\Delta z}{2}\cos\theta} \\ -e^{-i\frac{k\Delta z}{2}\cos\theta} \end{array} \right\} \tag{24}$$

$$D_{\Delta x \Delta z} U_x =$$

$$-\sin\theta\, k^2 \sin\theta \cos\theta \frac{\sin\left(\frac{k\Delta x}{2}\sin\theta\right)}{\frac{k\Delta x}{2}\sin\theta} \frac{\sin\left(\frac{k\Delta z}{2}\cos\theta\right)}{\frac{k\Delta z}{2}\cos\theta} e^{i((z\cos\theta + x\sin\theta)k - \omega t)} \tag{25}$$

$$D_{\Delta x \Delta z} U_x = -\sin\theta\, k^2 \sin\theta \cos\theta\, \text{sinc}\left(\frac{k_x \Delta x}{2}\right) \text{sinc}\left(\frac{k_z \Delta z}{2}\right) e^{i((z\cos\theta + x\sin\theta)k - \omega t)}. \tag{26}$$

Again, this is similar to the continuous differential with the two sinc functions as extra terms, so

$$D_{\Delta x \Delta z} U_x = \text{sinc}\left(\frac{k_x \Delta x}{2}\right) \text{sinc}\left(\frac{k_z \Delta z}{2}\right) \frac{\partial^2 U_x}{\partial x \partial z}. \tag{27}$$

The time differential is identical to the one dimensional case, where

$$D_{\Delta t}{}^2 U_z = -\cos\theta \frac{\sin^2\left(\frac{\omega\Delta t}{2}\right)}{\left(\frac{\omega\Delta t}{2}\right)^2} \omega^2 e^{i((z\cos\theta + x\sin\theta)k - \omega t)} \tag{28}$$

$$D_{\Delta t}{}^2 U_z = -\cos\theta\, \text{sinc}^2\left(\frac{\omega\Delta t}{2}\right) \omega^2 e^{i((z\cos\theta + x\sin\theta)k - \omega t)} \tag{29}$$

$$D_{\Delta t}{}^2 U_z = \text{sinc}^2\left(\frac{\omega\Delta t}{2}\right) \frac{\partial^2 U_z}{\partial t^2}. \tag{30}$$

Shear wave continuous solution

A two dimensional grid can also be expected to propagate a shear wave, and the **z** and **x** components of a possible shear wave (similar to expressions (3) and (4)) are

$$U_z = \sin\theta \; e^{i((z\cos\theta + x\sin\theta)k - \omega t)} \tag{31}$$

$$U_x = -\cos\theta \; e^{i((z\cos\theta + x\sin\theta)k - \omega t)}. \tag{32}$$

The negative sign in (32) is required to give a displacement perpendicular to the propagation direction. Again, for the continuous case, similar to equations (5), (6) and (7), we have

$$\frac{\partial^2 U_z}{\partial z^2} = -\sin\theta\cos^2\theta \; k^2 e^{i((z\cos\theta + x\sin\theta)k - \omega t)} \tag{33}$$

$$\frac{\partial^2 U_z}{\partial x^2} = -\sin\theta\sin^2\theta \; k^2 e^{i((z\cos\theta + x\sin\theta)k - \omega t)} \tag{34}$$

$$\frac{\partial^2 U_x}{\partial x \partial z} = \cos\theta\sin\theta\cos\theta \; k^2 e^{i((z\cos\theta + x\sin\theta)k - \omega t)} \tag{35}$$

$$\frac{\partial^2 U_z}{\partial t^2} = -\sin\theta\,\omega^2 e^{i((z\cos\theta + x\sin\theta)k - \omega t)}. \tag{36}$$

These four equations are substituted into (1) giving

$$-\sin\theta \; k^2 e^{i((z\cos\theta + x\sin\theta)k - \omega t)} \left\{ (\lambda + 2\mu)\cos^2\theta - (\lambda + \mu)\cos^2\theta + \mu\sin^2\theta \right\}$$
$$= -\sin\theta \; \omega^2 \rho \; e^{i((z\cos\theta + x\sin\theta)k - \omega t)} \tag{37}$$

This is true if

$$k^2 \left\{ (\lambda + 2\mu)\cos^2\theta - (\lambda + \mu)\cos^2\theta + \mu\sin^2\theta \right\} = \omega^2 \rho, \tag{38}$$

and true if

$$k^2 \left\{ (\lambda - \lambda)\cos^2\theta + (2\mu - \mu)\cos^2\theta + \mu\sin^2\theta \right\} = \omega^2 \rho, \tag{39}$$

and true if

$$\frac{\mu}{\rho} = \frac{\omega^2}{k^2} = v_\beta^2, \tag{40}$$

where v_β here is now the shear velocity.

Equivalent finite difference shear equation

Application of the finite difference formulae for the first term of the differential equation to the shear plane wave results in an expression that is similar to (17)

$$D_{\Delta z}{}^2 U_z = -\sin\theta \ k^2 \cos^2\theta \ \mathrm{sinc}^2\left(\frac{k_z\Delta z}{2}\right) e^{i((z\cos\theta + x\sin\theta)k - \omega t)}. \tag{41}$$

Note that comparison of the finite difference and continuous differentials gives

$$D_{\Delta z}{}^2 U_z = \ \mathrm{sinc}^2\left(\frac{k_z\Delta z}{2}\right)\frac{\partial^2 U_z}{\partial z^2}, \tag{42}$$

which is exactly the same as equation (18). Similarly

$$D_{\Delta x}{}^2 U_z = -\sin\theta \ k^2 \sin^2\theta \ \mathrm{sinc}^2\left(\frac{k_x\Delta x}{2}\right) e^{i((z\cos\theta + x\sin\theta)k - \omega t)} \tag{43}$$

$$D_{\Delta x\Delta z} U_x = \cos\theta \ k^2 \sin\theta\cos\theta \ \mathrm{sinc}\left(\frac{k_x\Delta x}{2}\right)\mathrm{sinc}\left(\frac{k_z\Delta z}{2}\right) e^{i((z\cos\theta + x\sin\theta)k - \omega t)} \tag{44}$$

$$D_{\Delta t}{}^2 U_z = -\sin\theta \ \mathrm{sinc}^2\left(\frac{\omega\Delta t}{2}\right)\omega^2 e^{i((z\cos\theta + x\sin\theta)k - \omega t)}. \tag{45}$$

Note that equations (21), (27) and (30) relating the finite difference and continuous differentials still apply. It appears that in general the continuous equation (1) can be represented by the following finite difference equation

$$\left[\begin{matrix}(\lambda+2\mu)\dfrac{D_{\Delta z}{}^2 U_z}{\mathrm{sinc}^2\left(\dfrac{k_z\Delta z}{2}\right)} \\[2em] +(\lambda+\mu)\dfrac{D_{\Delta x\Delta z} U_x}{\mathrm{sinc}\left(\dfrac{k_x\Delta x}{2}\right)\mathrm{sinc}\left(\dfrac{k_z\Delta z}{2}\right)} \quad +\mu\dfrac{D_{\Delta x}{}^2 U_z}{\mathrm{sinc}^2\left(\dfrac{k_x\Delta x}{2}\right)}\end{matrix}\right] = \rho\frac{D_{\Delta t}{}^2 U_z}{\mathrm{sinc}^2\left(\dfrac{\omega\Delta t}{2}\right)}. \tag{46}$$

The spatial derivatives here will result in the same values as those calculated by the pseudo-spectral approach from Fornberg (1987).

Formulation of frequency correction terms

Note that the left side of equation (46) is fully determined because the complete wavefield in space is available to calculate the spatial wavenumbers in **x** and **z**. Therefore the acceleration can be calculated exactly. The sinc correction on the right side of the equation can not be made in a straightforward way because the temporal frequencies are not yet available.

In the one-dimensional case the **ω** can be determined from the formula **ω=vk,** as in Manning and Margrave (1999). This works because there is only one velocity and it can be used to relate the spatial and time frequencies. In the two-dimensional case it can also be expected to work in the case where only one type of wave can propagate.

In the case of a compressional wave travelling in the **z** direction, the displacement in the **x** direction is zero, and the variation of the **z** displacement in the **x** direction is also zero. The only non-zero terms from equation (1) are then the first and the last so that

$$(\lambda + 2\mu)\frac{\partial^2 U_z}{\partial z^2} = \rho \frac{\partial^2 U_z}{\partial t^2}. \tag{47}$$

Substitution of the finite difference second derivatives yields the equation

$$(\lambda + 2\mu)\frac{D_{\Delta z}{}^2 U_z}{\mathrm{sinc}^2\left(\dfrac{k_z \Delta z}{2}\right)} = \rho \frac{D_{\Delta t}{}^2 U_z}{\mathrm{sinc}^2\left(\dfrac{\omega \Delta t}{2}\right)} = \rho \frac{D_{\Delta t}{}^2 U_z}{\mathrm{sinc}^2\left(\dfrac{v_\alpha k \Delta t}{2}\right)}. \tag{48}$$

So the contribution of the first term of (46) must be corrected by the sinc function on the right above, yielding

$$(\lambda + 2\mu)\frac{\mathrm{sinc}^2\left(\dfrac{v_\alpha k \Delta t}{2}\right)}{\mathrm{sinc}^2\left(\dfrac{k_z \Delta z}{2}\right)} D_{\Delta z}{}^2 U_z = \rho D_{\Delta t}{}^2 U_z. \tag{49}$$

Similarly, only the third term on the left of equation 1) is relevant to a shear wave propagating in the **x** direction, with the result

$$\mu \frac{\mathrm{sinc}^2\left(\dfrac{v_\beta k \Delta t}{2}\right)}{\mathrm{sinc}^2\left(\dfrac{k_x \Delta x}{2}\right)} D_{\Delta x}{}^2 U_z = \rho D_{\Delta t}{}^2 U_z. \tag{50}$$

The factor for the centre spatial term of the equation was found by inspection, and the fully corrected finite-difference elastic wave equation is

$$\left\{ \begin{array}{l} (\lambda + 2\mu)\dfrac{\text{sinc}^2\left(\dfrac{v_\alpha k\Delta t}{2}\right)}{\text{sinc}^2\left(\dfrac{k_z\Delta z}{2}\right)}D_{\Delta z}{}^2 U_z \\[4ex] + \dfrac{(\lambda + 2\mu)\text{sinc}^2\left(\dfrac{v_\alpha k\Delta t}{2}\right) - \mu\text{sinc}^2\left(\dfrac{v_\beta k\Delta t}{2}\right)}{\text{sinc}\left(\dfrac{k_x\Delta x}{2}\right)\text{sinc}\left(\dfrac{k_z\Delta z}{2}\right)}D_{\Delta x\Delta z}U_x \\[4ex] +\mu\dfrac{\text{sinc}^2\left(\dfrac{v_\beta k\Delta t}{2}\right)}{\text{sinc}^2\left(\dfrac{k_x\Delta x}{2}\right)}D_{\Delta x}{}^2 U_z \end{array} \right\} = \rho D_{\Delta t}{}^2 U_z. \qquad (51)$$

In principle, this equation is an exact reformulation of the elastic wave equation. No approximations have been made to accommodate finite differencing. Each additive term consists of a product of elastic constants, a finite difference operator, and a ratio of sinc functions. We refer to these various sinc function ratios as 'correction factors'. If all of the sinc functions are set to unity, then the conventional, second order, finite difference elastic wave equation results. If the sinc functions are directly evaluated, they 'correct' the finite difference for its dispersion and instability. However, the correction factors are frequency dependent and so must be applied as spatial filters. Furthermore, the correction factors all involve sinc functions that have zeros. We expect that the occurance of these zeros will limit the ability of these correction factors to fix dispersion and instability effects.

Equivalent equation test

To show that equation (51) propagates a pressure wave properly, a wave described by equations (3) and (4) can be used. Equations (17), (26), (20), and (29) describe how the finite difference operators apply to this wave. The right hand side of these equations may be substituted into equation (51). The spatial sinc functions obviously cancel out, and when the factor $\cos\theta\, e^{i((z\cos\theta + x\sin\theta)k - \omega t)}$ is divided from both sides, the remaining terms are

$$\begin{aligned} &-(\lambda + 2\mu)\text{sinc}^2\left(\frac{v_\alpha k\Delta t}{2}\right)\cos^2\theta\, k^2 - (\lambda + 2\mu)\text{sinc}^2\left(\frac{v_\alpha k\Delta t}{2}\right)\sin^2\theta\, k^2 \\ &+\mu\text{sinc}^2\left(\frac{v_\beta k\Delta t}{2}\right)\sin^2\theta\, k^2 - \mu\text{sinc}^2\left(\frac{v_\beta k\Delta t}{2}\right)\sin^2\theta\, k^2 = -\rho\omega^2\text{sinc}^2\left(\frac{\omega\Delta t}{2}\right) \end{aligned} \qquad (52)$$

This is true if

$$(\lambda + 2\mu)\ k^2 \text{sinc}^2\left(\frac{v_\alpha k \Delta t}{2}\right) = \rho\omega^2 \text{sinc}^2\left(\frac{\omega \Delta t}{2}\right), \qquad (53)$$

and true if $v_\alpha = \omega / k$ and $(\lambda + 2\mu)/\rho = v_\alpha^2 = \omega^2 / k^2$, consistent with the definition of pressure wave velocity. Note that this result is independent of θ, and so it is valid for a pressure wave propagating in any direction.

In a similar fashion, equations (31) and (32) describe an arbitrary shear wave. Equations (41), (44), (43), and (45) describe the results of applying the finite difference operators to this wave. This set of consistent equations may also be substituted into equation (51). The spatial sinc functions cancel out as before, and when the factor $\sin\theta\ e^{i((z\cos\theta + x\sin\theta)k - \omega t)}$ is divided from both sides, the remaining terms leave

$$\begin{aligned}-(\lambda + 2\mu)\text{sinc}^2\left(\frac{v_\alpha k \Delta t}{2}\right)\cos^2\theta\ k^2 + (\lambda + 2\mu)\text{sinc}^2\left(\frac{v_\alpha k \Delta t}{2}\right)\cos^2\theta\ k^2 \\ -\mu\text{sinc}^2\left(\frac{v_\beta k \Delta t}{2}\right)\cos^2\theta\ k^2 - \mu\text{sinc}^2\left(\frac{v_\beta k \Delta t}{2}\right)\sin^2\theta\ k^2 = -\rho\omega^2\text{sinc}^2\left(\frac{\omega \Delta t}{2}\right)\end{aligned}. \qquad (54)$$

This is true if

$$\mu k^2 \text{sinc}^2\left(\frac{v_\beta k \Delta t}{2}\right) = \rho\omega^2 \text{sinc}^2\left(\frac{\omega \Delta t}{2}\right), \qquad (55)$$

and true if $v_\beta = \omega / k$ and $\mu / \rho = v_\beta^2 = \omega^2 / k^2$, consistent with the definition of shear wave velocity. Note that this result is also independent of θ.

APPLICATIONS

The method described here has been tested by a straightforward Fourier domain approach. For each component, the result of each of the three finite difference terms were transformed into the Fourier domain and multiplied by the appropriate wavenumber surface. The components were then added together and inverse transformed for use in time stepping. Thus the correction factors are applied after each time step.

An example of a wavenumber surface for the first term in equation (51) is shown in Figure 3. The two sinc functions correct for a net dispersion effect in the **z** direction, and a net unstable effect in the **x** direction. Figure 4 shows one quarter of the correction in space (it is symmetric in **x** and **z**).
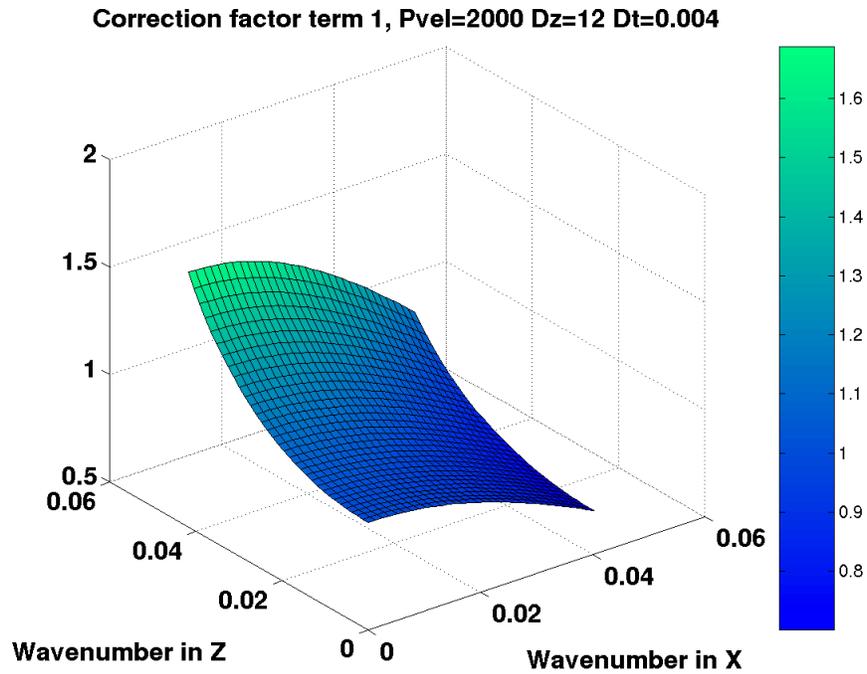
Correction factor term 1, Pvel=2000 Dz=12 Dt=0.004

Figure 3. The correction factor for term 1 in equation (51) is shown in the wavenumber domain

Correction factor term 1, Pvel=2000 Dz=12 Dt=0.004
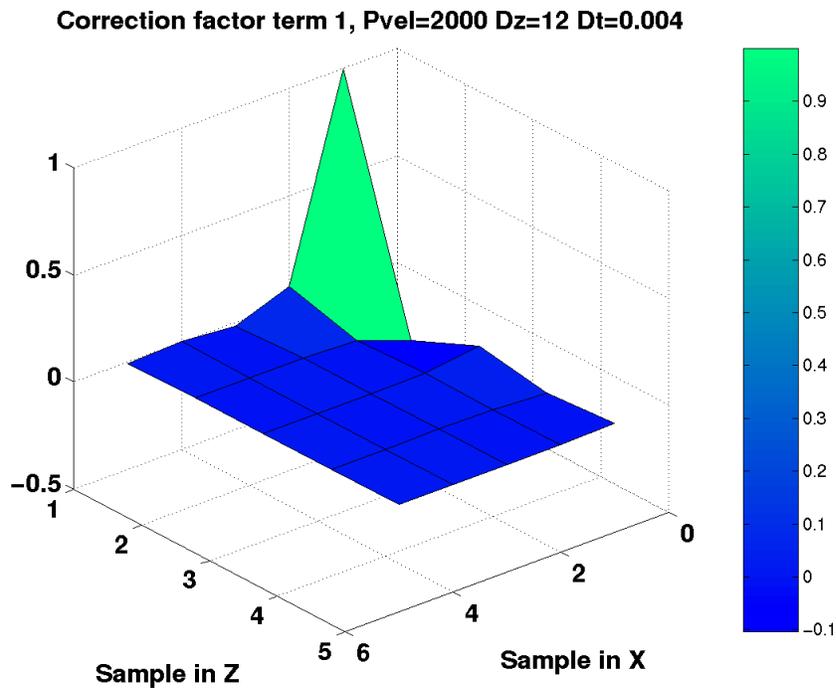

Figure 4. The correction factor for term 1 in equation (51) is shown in space.

Several comparisons were made of finite-difference wave propagation with and without the correction factors. The first example is a 30 Hz P wave initiated in the centre of the model and propagated, uncorrected in Figure 5 and corrected in Figure 6.

P wave, V=2000, Dt=.004, Dx=12
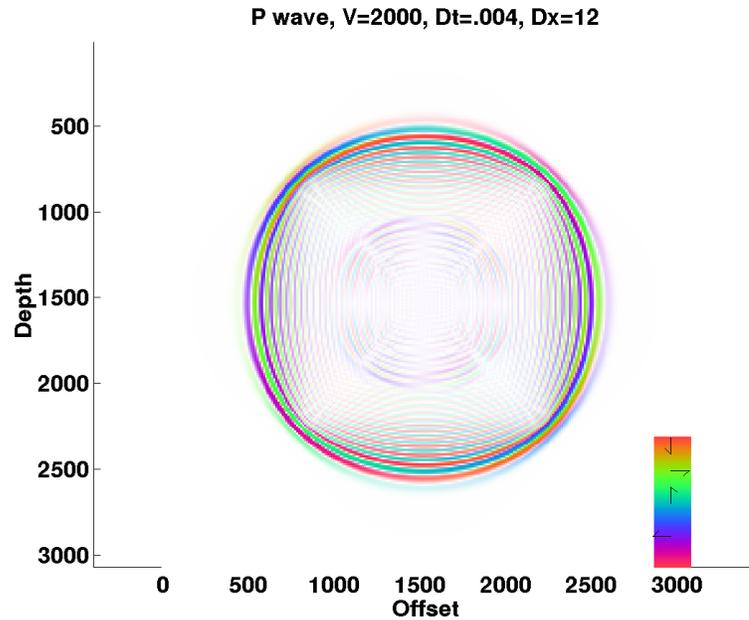
Figure 5. An uncorrected P wave is shown after propagation through 120 time steps. Note the 'square' shape to the dispersion pattern within the wavefront.
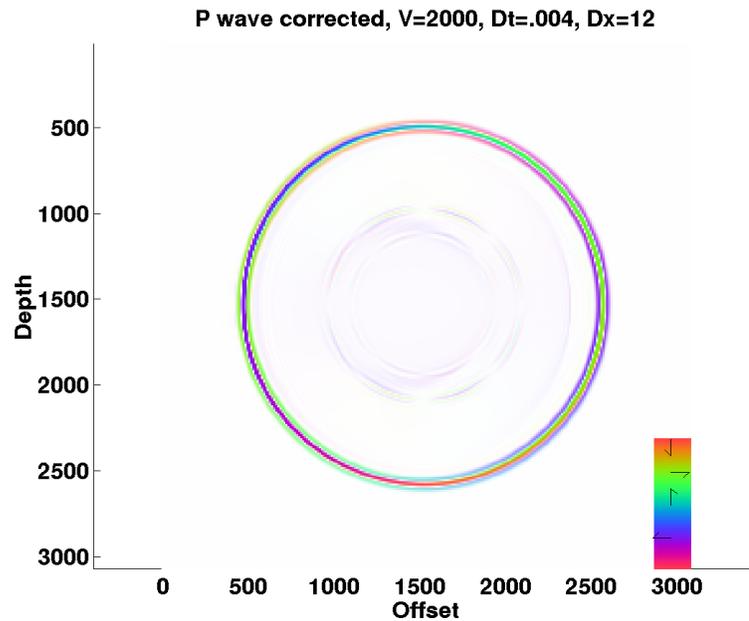
P wave corrected, V=2000, Dt=.004, Dx=12

Figure 6. This is similar to Figure 5 except that the correction factors have been applied after each time step.

The second example is a combination of two cylindrical source waves, one a pressure wave and the other a shear wave. The waves were initiated by defining them at two times separated by an interval equal to the sample rate. The definition at time zero is plotted in Figure 7. Both waves are 30 Hz Ricker wavelets in time.
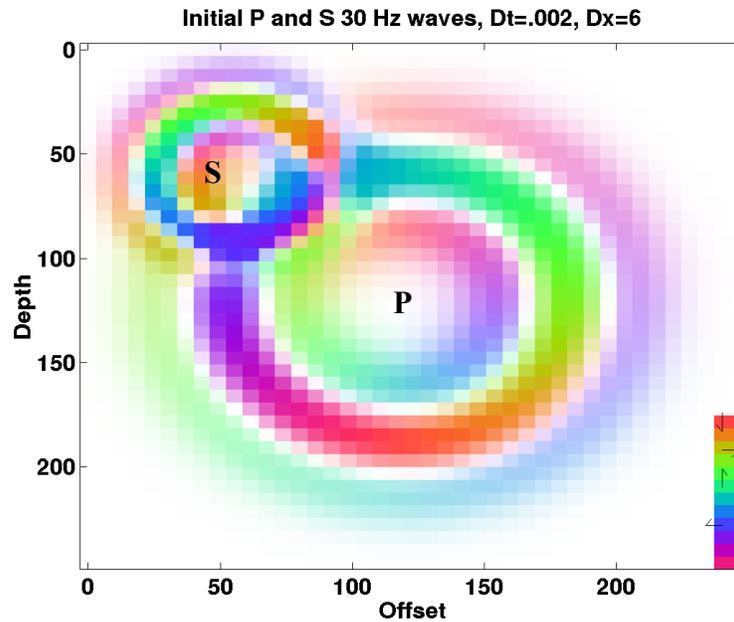

Figure 7. The initial P and S waves about to be propagated are shown in detail. The S wave is at upper left, the P wave is centred.
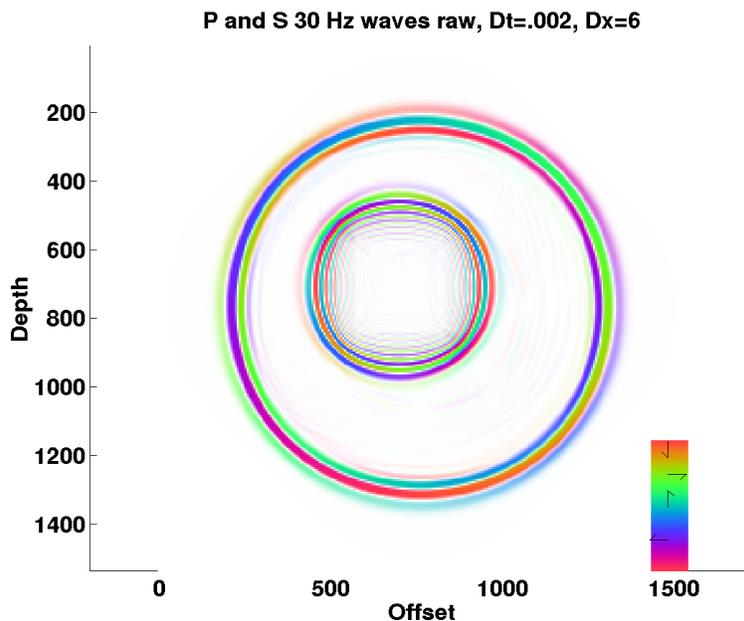

Figure 8. The uncorrected model propagated 120 steps from Figure 7 is shown. Note that the scale has changed. The finer sample rate in space and time compared with Figure 5 has allowed the P wave to propagate quite evenly, but the S wave remains distorted.
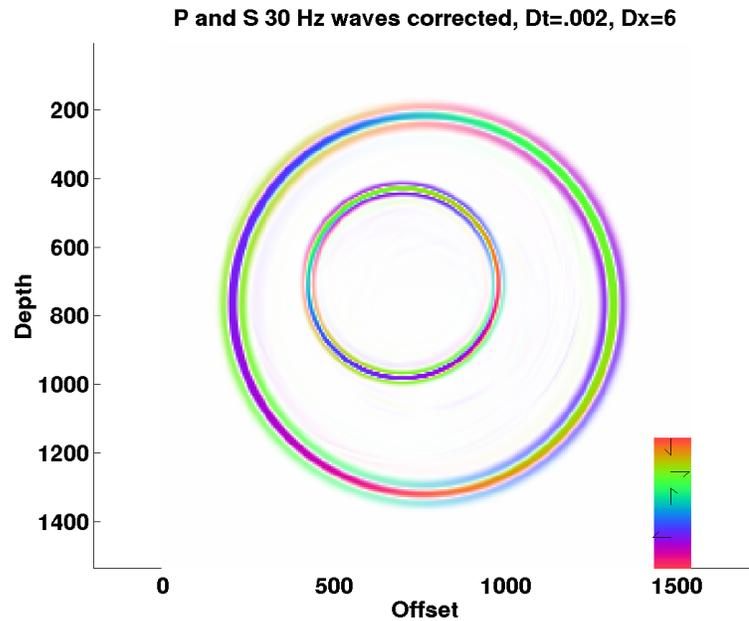
Figure 9. This shows the corrected model propagation from Figure 7. The S wave has retained its circular shape and the P wave has retained its zero phase character.

The propagated models are shown in the following figures. Figure 8 is uncorrected and Figure 9 is corrected.

It is obvious that the uncorrected modelling has not propagated all the frequencies of the shear wave accurately, but the corrected modelling has. The pressure wave has been propagated by the uncorrected modelling in Figure 8 much better than in Figure 6 because of the finer sample rate. However a close inspection of the uncorrected pressure wave in Figure 8 shows that it hasn't preserved the zero phase nature of the wavelet evident in the corrected version of Figure 9. Another method of comparing these results is shown in Figures 10 and 11 where individual traces through the model are displayed. The preserved zero phase nature of the pressure wave in Figure 11 is very obvious.
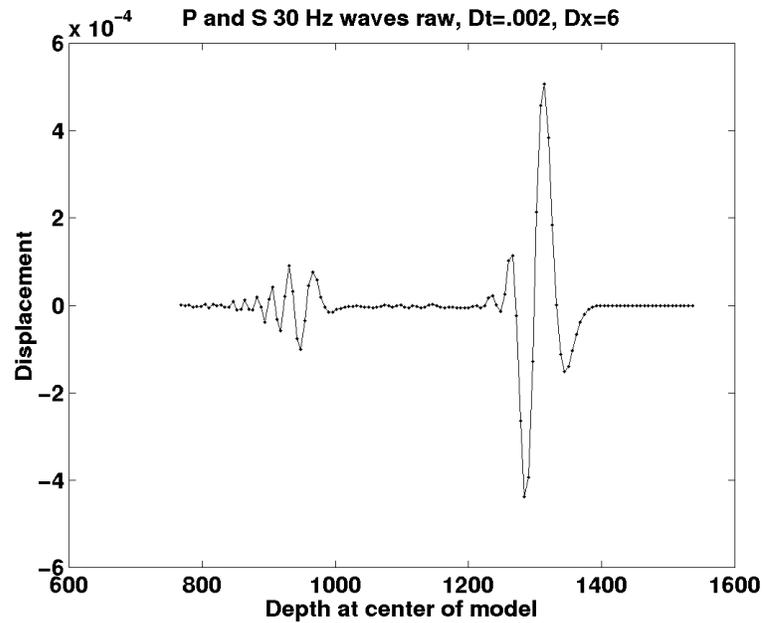
P and S 30 Hz waves raw, Dt=.002, Dx=6

Figure 10. The uncorrected S wave (left) and P wave (right) are shown on a trace from Figure 8. The S wave has been badly distorted and the P wave has been phase shifted.

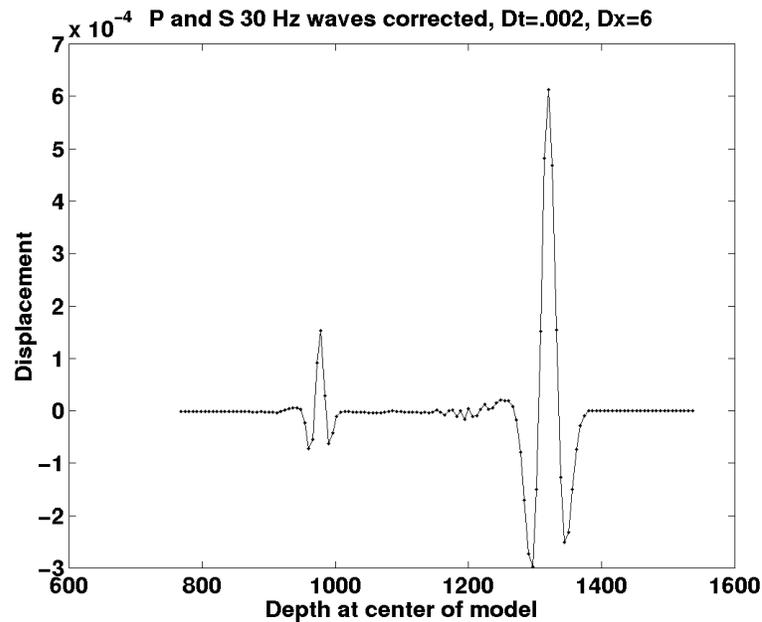**P and S 30 Hz waves corrected, Dt=.002, Dx=6**

Figure 11. The corrected S wave (left) and P wave (right) are shown on a trace from Figure 9. Both S and P waves are zero phase.

The corrected modelling can also allow for slightly unstable sampling. A pressure wave velocity of 2000 and a spatial sample rate of 6 requires a time sampling of less than .0021 (Lines et al (1999)). The uncorrected modelling in Figure 12 shows this instability after 6 steps because the sample rate is .0025. The corrected modelling in Figure 13 is stable (for 40 steps) until edge effects build up.

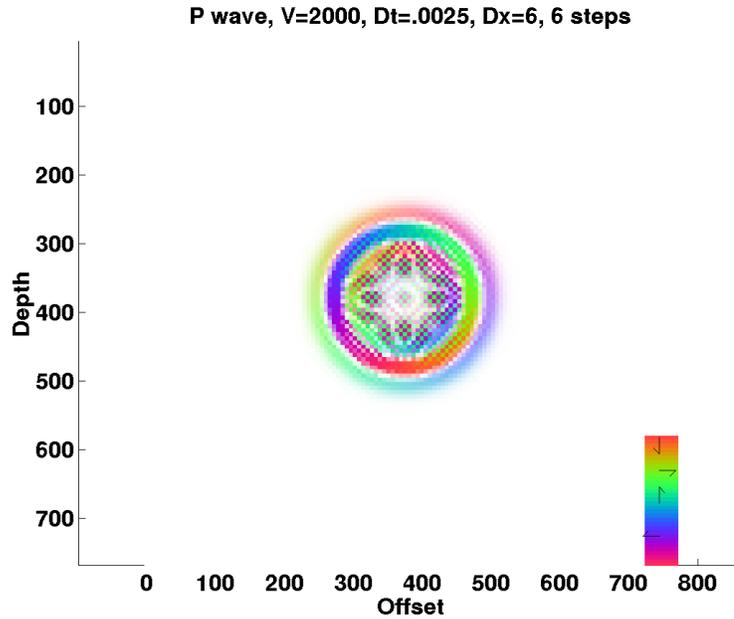**P wave, V=2000, Dt=.0025, Dx=6, 6 steps**

Figure 12. The start of unstable artifacts in an uncorrected model is shown here in the centre of the figure. These are caused by a time sample rate that is too large.

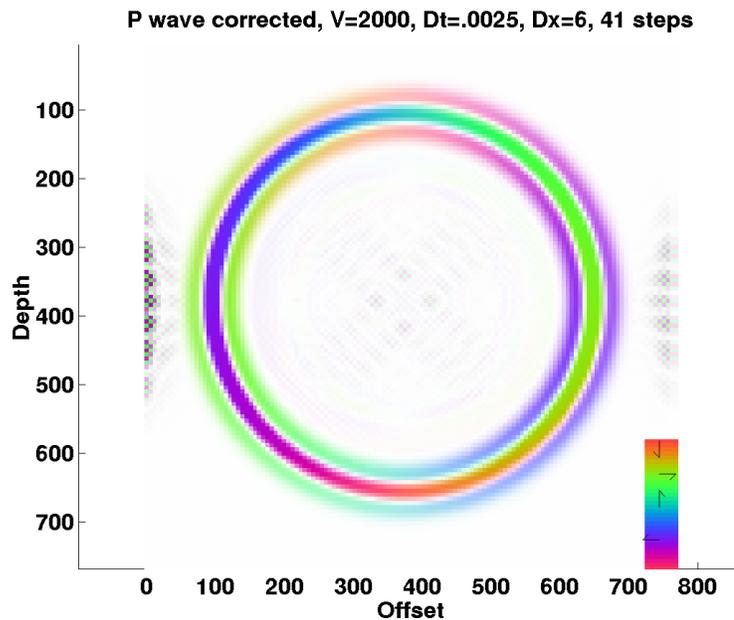**P wave corrected, V=2000, Dt=.0025, Dx=6, 41 steps**

Figure 13. The results of corrected modelling is shown here. The correction allows for an excessive time sample rate until edge effects grow too large.

The methods developed here should improve the economics of finite-difference modelling. In timing comparisons, the corrected modelling of the P wave in Figure 6 took 2.5 times the computation time of the uncorrected model of Figure 5. A halving of the sample rate and doubling of the number of samples in x, z, and t would give results equivalent to those for the uncorrected P wave in Figure 8 and cost 8 times as

much for lower quality. If short correction operators in the space domain can be developed, the economics could be improved much more by avoiding the long Fourier transforms used for the tests here.

CONCLUSIONS

The corrections developed in this paper for finite–difference modelling have been proved to propagate plane waves in arbitrary directions exactly like the continuous solution would. Since the continuous solutions have no dispersion or instability, the equivalent finite difference solutions should be stable and non-dispersive also. Many non-plane waves can be composed from plane waves, and they too should be propagated correctly.

Application of the theoretical corrections to practical models have confirmed the stable and non-dispersive properties. The corrections were applied in the frequency domain after Fourier transforms, and even though no special coding was used to suppress edge effects or wraparound, the results were satisfactory except in the basic unstable case.

The corrected modelling proved to be much more efficient at providing high quality results than smaller sampling rates with uncorrected modelling. Far more efficient methods of applying corrections can be expected in the future.

FURTHER WORK

The highest priority will be given to making corrections in the space domain instead of the Fourier domain. Figure 4 shows an operator in the space domain, and the small size of the coefficients is typical. This should make space domain convolutions quite practical.

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