Snell's law in transversely isotropic media

P.F. Daley

ABSTRACT

The problem of reflection and transmission of waves at a plane boundary separating two transversely isotropic media is considered subject to the general condition that the axes of anisotropy in both media need not align with the intervening interface. Slowness vectors and surfaces are employed in the treatment presented. The characteristics or rays, defining the direction of energy propagation, are normal to the slowness surface, which makes slowness space ideally suited for treatment of anisotropic problems, as both Snell's Law and ray properties are formulated in terms of slowness. Ray-vector magnitudes are not dealt with here as this topic warrants a separate treatment, and the inclusion of which would create needless complexity.

The exact eikonals (Hamiltonians) of the coupled quasi-compressional (qP) and quasi-shear (qS_V) wave propagation are used in the derivations and are homogeneous of order 2 in powers of slowness, which is a requisite for the use of the theory of characteristics. Once this condition is violated through simplifications of an eikonal, the theory of characteristics is not applicable. As it is an extremely useful mathematical tool for dealing with problems related to wave propagation, negating it seems counterproductive, unless it is done for a specific rather than general purpose.

The possibility of components of the slowness vector becoming complex is briefly considered. This is motivated by the fact that, for post-critical regions of quantities such as the PP reflection coefficient at an interface between two transversely isotropic media, this becomes a factor in the proper computation of certain quantities.

INTRODUCTION

Reflection and transmission of waves at an interface between two transversely isotropic media has been considered in numerous publications, one of the more recent being the work of Slawinski (1996). The treatment presented here differs from that work in that the media type considered do not in general have their anisotropy axes aligned with the plane boundary separating the two and that the exact rather than linearized eikonals (Thomsen, 1986) are used in the derivations and final formulae.

This report will be based almost exclusively on some basic properties applicable to anisotropic elastic medium:

- When considering reflection or refraction of wavefronts due to incidence at a plane interface, the horizontal component of slowness, p, is constant for all propagation modes involved. The vertical component of the slowness vector, q, may be obtained from the eikonal equation so that the slowness vector is defined as $\mathbf{p} = (p, q)$. The slowness surface is the inverse of the wavefront normal (phase) velocity surface, both being obtained from the eikonal equations discussed in the Appendix, Musgrave (1970) and Schoenberg and Helbig (1996).

- The degenerate ellipsoidal case is not an unreasonable point to start for most problems related to waves propagating in a transversely isotropic medium, as it may be dealt with in a general manner and insight obtained is directly transferable to the more general case.

- The ray (characteristic) corresponding to a given slowness vector, $\mathbf{p} = (p, q)$, has a direction normal to this point on the slowness surface. (Gassmann, 1965; Musgrave, 1970).

- The magnitude of the ray velocity, given the appropriate initial conditions, may be obtained from the solution of the characteristic equations (Courant and Hilbert, 1962) corresponding to a given eikonal equation which, apart from being homogeneous in powers of slowness, is a function of the anisotropic coefficients, A_{ij}, that have the dimensions of *velocity*^{2} and may be spatially dependent. (Vlaar, 1968; Cerveny, 1972; Cerveny and Psencik; 1972). As mentioned earlier, ray theory in general will not be dealt with in this report. Certain properties of rays will be used when convenient without rigorous mathematical derivations.

In what follows, two coordinate systems will be used: model and rotated or primed coordinates. Model coordinates are related to geological structures such as plane boundaries separating two media. Rotated (primed) coordinates are aligned with the axes of anisotropy, which need not be the same as the model coordinates. Vectors in one system may be expressed in the other using an orthonormal rotation by some angle ϕ, which preserves length. It is practical to require the constraint that $|\phi| < \pi/2$.

All of the figures shown here use surfaces which are applicable to the ellipsoidal case, and a reader may infer that the results are only useful in that context. This is not so, as this format was adopted to maintain a consistency in the figures, and when printed – without most of the annotation – are useful for obtaining solutions of reasonable accuracy using simple manual geometrical methods.

Another assumption incorporated here is that all of the slowness surfaces are convex, which, if not invoked, leads to possibility of up to three rays of one type corresponding to a single horizontal slowness (Musgrave, 1970). The imposition of this was done in an effort to keep the theoretical concepts to a minimum. The formulae obtained here may be used in cases where the slowness surface is not convex without modification. Some minimal additional theory would be required to facilitate pursuit of this problem area.

To introduce the notation used, the reader is referred to Figure 1. The angle ϕ is the angle that the primed system is rotated, with respect to the model coordinates; ϕ being measured positive counter-clockwise. The anisotropic parameters are defined with respect to the primed coordinate system, so that the half-length of the q' slowness axis is $\left(A_{33}\right)^{-1/2}$ and the corresponding quantity along the p' axis is $\left(A_{11}\right)^{-1/2}$ (Gassmann, 1965). The slowness vector components in the model and rotated coordinate systems may be expressed in terms of one another through the following orthonormal transformation

$$\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} p' \\ q' \end{bmatrix} \tag{1}$$

and its inverse

$$\begin{bmatrix} p' \\ q' \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \tag{2}$$

The characteristics (rays) are the solution of a coupled set of four, first-order, ordinary differential equations with appropriate initial conditions (Cerveny and Psencik, 1972). Their solution must be obtained in the primed system. Both the model, (p,q), and primed, (p',q'), coordinate axes will be indicated on the figures. Any boundary between two media will be assumed to be plane. Curved boundaries may be incorporated but in this discussion are an unnecessary complication.

Before proceeding with a more structured discussion of the problem, an example will be presented in Figure 2 which deals with horizontal slownesses, $\pm p$, corresponding to the grazing incidence of the qP ray or equivalently the horizontal slownesses of the qS_V in a T.I. medium which result in critical reflection of the qP ray. The qP rays are normal to the slowness surface and in general not parallel to the slowness vector. For the qS_V case, in the geometry shown, the rays are always normal to the slowness surface and parallel to the slowness vector. In a general transversely isotropic medium the qS_V rays behave in the same manner as the qP rays; i.e., not aligned with the slowness vector. As the magnitude of the ray velocities are not being discussed here, the ray vectors on the figures define only their direction. This will be the standard throughout this report.

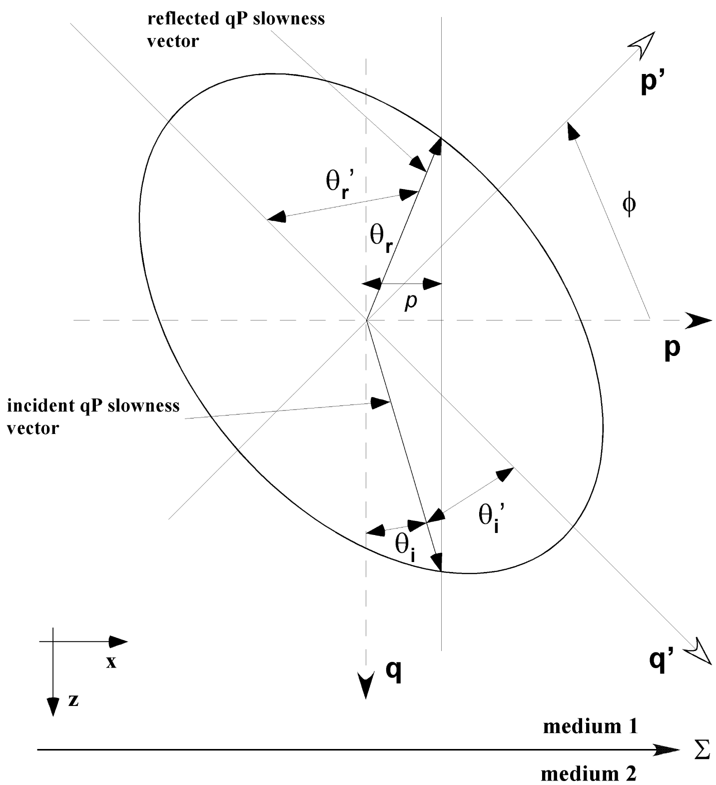

FIG. 1. Definition of annotation used. The primed system is aligned with the axes of anisotropy while the model system is defined with respect to the interface Σ. The primed system is rotated by an angle ϕ, positive counter-clockwise, with respect to the model system.

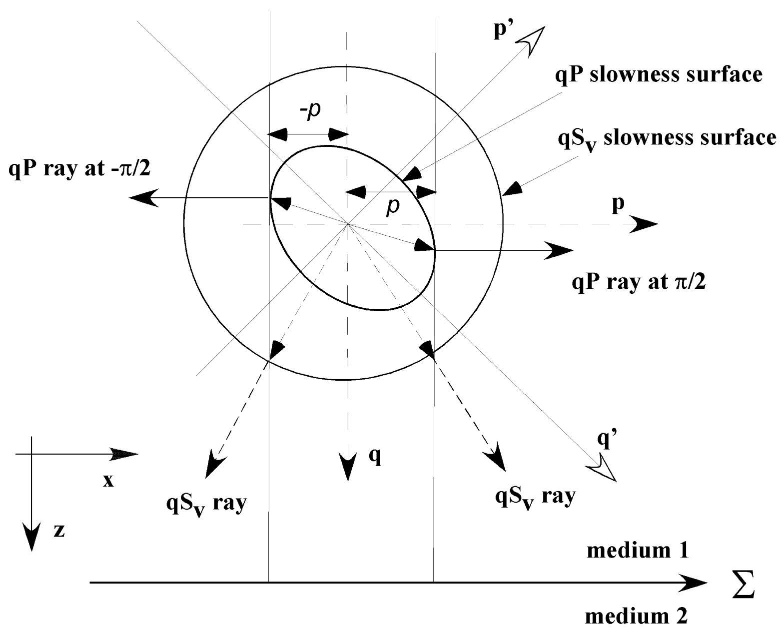

FIG. 2. The slowness vectors which result in the *qP* rays being at grazing incidence. Alternatively, this may be viewed as the incident qS_V slownesses which upon use of Snell's Law result in critically reflected *qP* rays.As mentioned in the text the ray vectors indicate direction but not magnitude.

SNELL'S LAW IN ANISOTROPIC MEDIA

A convention used in the text below, which could possibly lead to some confusion, is that even though the vertical slowness coordinate is defined to be positive *downwards*, the reflected slowness angles are measured from the negative vertical slowness axis. That is, the reflected angles are the acute angles which the reflected slowness vector makes with the vertical slowness axis. This is consistent with the manner in which this problem is treated in the isotropic homogeneous case but the use of two coordinate systems could tend to mask this.

Horizontal Slowness, p_i or p_i', Known

There are many instances in practice when only the horizontal component, p, of the incident slowness vector $\mathbf{p_i}$ is known. In this case it is possible to determine the slowness vectors of the three other wave types at a solid/solid interface. This involves solving a non-linear equation numerically. If only p' is known and $\phi \neq 0$, q' may be obtained using equation (A.18). The slowness in model coordinates, $\mathbf{p_i} = (p_i, q_i)$, may then be calculated using (1).

Referring to Figure 3, where the problem of an incident qP wavefront and reflected qS_V is depicted, it is clear that Snell's Law requires $p = p_i = p_r$. The unknown is θ_r from which θ_r' results as; $\theta_r' = \theta_r + \phi$. Utilizing the right-angle triangle diagram insert in the figure the following relation is obtained

$$p\left[V_N^{(qS_V)}\left(\theta_r + \phi\right)\right] - \sin\theta_r = 0 \tag{3}$$

which in general requires a numerical solution for the unknown θ_r. The sub- and superscripts on the velocity quantity, V, indicate that the qS_V wavefront normal (phase) velocity is being employed. (See (A.11)). Once θ_r has been determined, p' and q' may be calculated with (A.9) and (A.10) or p' calculated and q' obtained from (A.18). The return to model coordinates is achieved using (1).

Another example of the use of this numerical technique is when considering a transmitted disturbance into medium 2 due to incidence from medium 1. In this case, when $\phi_1 \neq \phi_2$, the values, which require a solution, are obtained from an equation almost identical to that in (3) and is also numerical in nature. This case is shown schematically in Figure 4 for a transmitted qP wave in medium 2. The basic equation, which must be considered, and from which all other relevant quantities are obtained is (Figure 4.c)

$$p\left[V_N^{(qP)}\left(\theta_t - \phi_2\right)\right] - \sin\theta_t = 0 \tag{4}$$

As is the previous case, p' and q', may be obtained in the manner described above and the return to model coordinates, (p, q), is also done using (1).

The numerical algorithm used in the solution of these nonlinear equations is similar to that attributed to Dekker-Brent (Press et al., 1997). A routine based on the *regula falsi* method (Press et al., 1997), from the F66 version of the IBM Scientific Subroutine Package, archived as (D)RTMI, is employed in the solution. This algorithm requires a function of a single variable as one of the calling parameters

Complex Vertical Slowness

If the *PP* reflection from the boundary is required for all angles of incidence in the upper medium, $-\pi/2 \le \theta \le \pi/2$, θ being the phase angle, the slowness vector components for both the *qP-wave* and qS_V*-wave* in the lower medium as well as for the qS_V*-wave* in the upper medium are required in the range of incident *qP* phase angles in medium 1. Schematics for two different orientations of the *qP-wave* slowness surface in medium 2 are shown in Figures 5.b and 5.c.

The motivation for considering this case is to introduce complex slowness components. In medium 1, ϕ_1 has been chosen equal to 0, and the incident-wavefront type as, qP; both without loss of generality. The transmitted wavefront in medium 2 is also of the qP type. The meridional axes will be chosen at angles of ϕ_1 and ϕ_2 with respect to the model coordinates, which define the plane interface Σ at which the refraction takes place.

Further, the *qP-wave* phase velocity is chosen to always be larger in medium 2 than in medium 1. It should be noted that this may not always be the case. It is possible to have two media as described above where the faster layer changes depending on the angle. The first case to be considered is that of the transmitted qP wavefront being elliptical with its axes of anisotropy aligned with the inter-medium boundary (Figure 5.a). The schematic resulting in critical refraction in medium 2 of the qP wavefront is shown.

The introduction of the *imaginary slowness surface* in the figure may produce some confusion, as this should be really be done by adding another dimension to the schematic. However, it is hoped that a reader will understand that this has been included in an effort to illustrate what occurs in medium 2 for horizontal slownesses greater than that corresponding to critical refraction.

As θ_t is measured from the vertical (q) axis, it is apparent in Figure 5b that the vertical slowness for the *qP-wave* is real for $\theta_t < \theta_C$ at which point the vertical component of slowness in medium 2 for the *qP-wave* becomes imaginary. The reason for this is that it is required that the horizontal components of slowness of all wave types, whether incident, reflected, or refracted at an interface, must be the same. This is a generalization of Snell's Law. To accommodate this, together with the fact that the eikonal equation for all wave types must be satisfied, the only possibility is that the vertical component of slowness must become imaginary. In general, for an ellipsoidal qP slowness surface in medium 2, the vertical slowness is defined as

$$q = \left[\frac{1 - A_{11}^{(2)} p^2}{A_{33}^{(2)}}\right]^{1/2} \tag{5}$$

with the superscripts on A_{ii} indicating the medium. At critical incidence $p = \left[A_{11}^{(2)}\right]^{-1/2}$ so that $q = 0$. For $p > \left[A_{11}^{(2)}\right]^{-1/2}$ q is purely imaginary* and given by

$$q = \pm i \left[\frac{\left|1 - A_{11}^{(2)} p^2\right|}{A_{33}^{(2)}}\right]^{1/2} \qquad i^2 = 1 \tag{6}$$

The sign preceding the term on the right hand side of (6) is chosen such that as the associated wave moves away from the interface it decays in amplitude, thus satisfying physical radiation conditions. This in turn, dependents on how the exponential spatial and time dependence of the wave has been defined; [$\exp\left\{\pm i\omega\left(\mathbf{p}\cdot\mathbf{x} - t\right)\right\}$].

The situation discussed above is depicted in Figure 5b, where the axes of anisotropy in medium 2 are aligned with the model coordinates. In medium 2 the critical slowness vector and critically refracted ray are in the same direction, parallel to the plane boundary Σ. As the slowness vector in this case is parallel to the boundary, $\phi_2 = 0$ and as a consequence when q becomes imaginary its value may be simply determined by equation (A.18), using the current value of p. This is a result of the corresponding slowness vector being aligned with an axis of the slowness surface.

In the situation shown in Figure 5c the computation of imaginary q is marginally more complicated, as the slowness vector associated with critical refraction is not situated along a meridional axis of the slowness surface in medium 2 and $\phi_2 \neq 0$. It is required that θ_C be computed by methods in the previous section, and the value of q (q') determined in the primed coordinate system and relative to the slowness vector associated with θ_C.

The above discussion is also valid for the general transversely isotropic case and the required expressions for q may be found in the Appendix.

* It should be noted that the term *imaginary* rather than *complex* has been used, as in both the ellipsoidal and general transversely isotropic problems, the vertical slowness, q, is either real or purely imaginary.

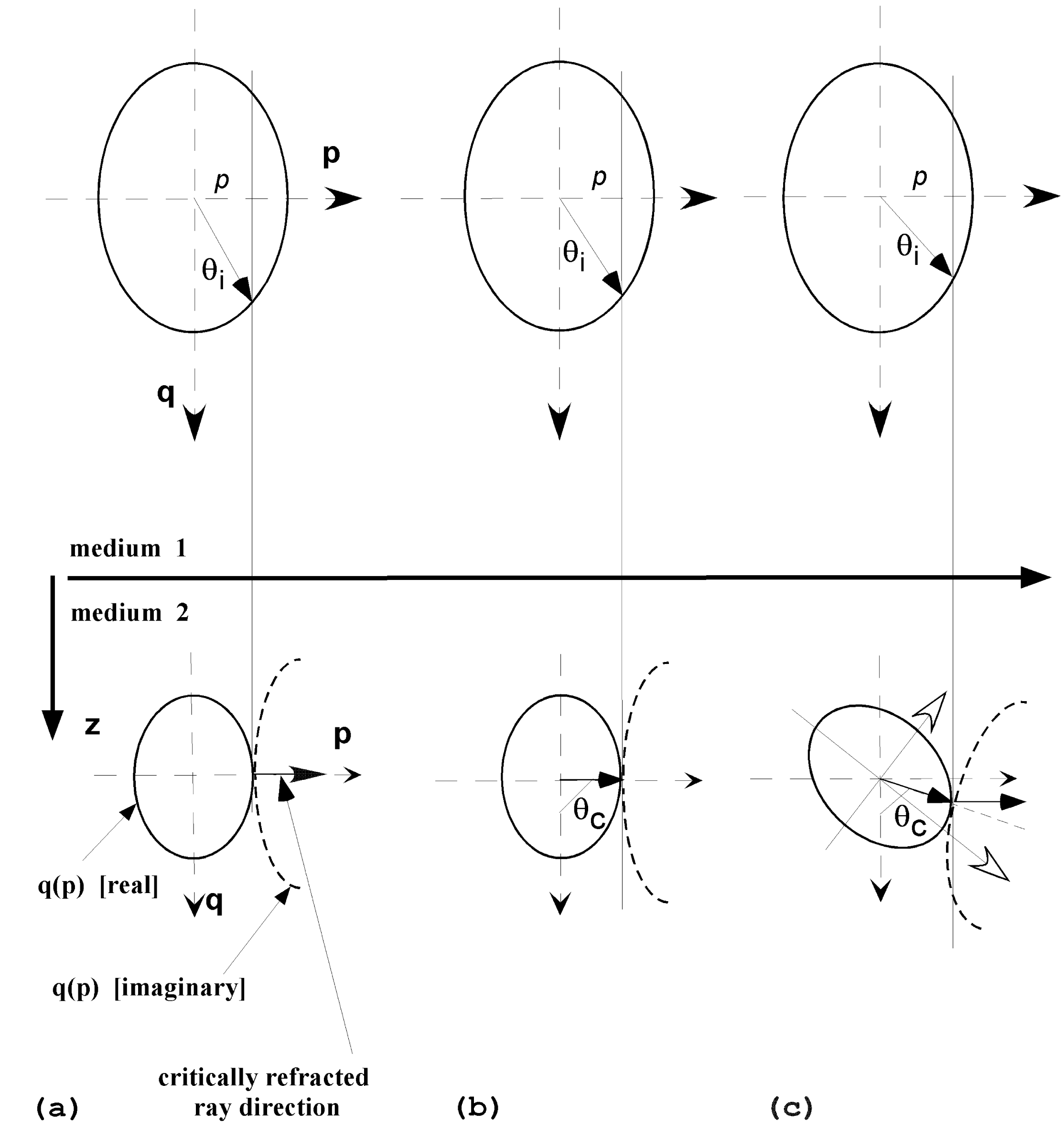

FIG. 5. Schematics of the case when the vertical component of the slowness vector becomes imaginary. Basic definitions are given in (a) and the instance of alignment of the plane interface and the axis of anisotropy is depicted in (b). Non alignment of the two is shown in (c), where q becoming imaginary is related to the critically refracted slowness vector.

EXAMPLES

Two simple examples of reflection and mode conversion, near vertical incidence at an interface with the axes of anisotropy in the incident medium not aligned with the interface, will be briefly examined. In addition, a schematic showing the previously mentioned instance when the faster medium varies with angle of incidence is shown. The *qP* slowness surfaces in media 1 and 2 have been designed specifically to illustrate this.

The two related Figures 6 and 7, depicting a reflected mode conversion at the interface Σ, are fairly self-explanatory. The reflected qS_V ray associated with a normally incident qP ray is shown in Figure 6, and in Figure 7, a normally reflected qS_V ray is the result of non-normal incidence of a qP ray. It is the rays that are of interest in seismological applications as it is along the rays that energy is transported from one point in a medium to another and it is an energy related quantity, particle displacement, which is measured by recording devices.

An overlay of slowness surfaces in media 1 and 2 at the interface Σ is presented in Figure 8 with $\phi_1 = \phi_2 = 0$. The slowness vectors in both media make the same angle with the vertical, $\theta_i = \theta_t$ and further $\mathbf{p_i} = \mathbf{p_t}$, that is, the slowness vectors are not only equal but their components are equal: $p = p_i = p_t$, $q = q_i = q_t$. The incident and transmitted rays, however, make different angles with the model coordinates as they are normal to their respective slowness surfaces. For angles of incidence in medium 1, $\theta < \theta_i$, the *faster* medium is medium 2 while after the intersection of the two slowness surfaces, for angles greater than θ_i, medium 1 is the *faster*. Also, if $\phi_1 = 0$, the reflected slowness vector, $\mathbf{p_r}$, is, apart from the sign of q_r, equal to the incident slowness, $\mathbf{p_i}$, and $\theta_t = \theta_i$. This condition holds even if $\phi_2 \neq 0$.

CONCLUSIONS

Using the exact eikonals for both qP and qS_V in two transversely isotropic medium, separated by a plane interface, and not requiring the axes of anisotropy in either medium to be aligned with the model coordinate system in which the interface Σ is defined, methods have been presented for determining the reflected and transmitted slowness vectors for all resulting related modes of propagation. The formulae derived may be incorporated into computer programs with little difficulty and are of some possible use in exploration seismology as there are numerous examples from seismic data where the alignment of anisotropy axes differs from that of neighbouring horizons of interest. Once the components of the slowness vector are obtained for an incident, reflected, or transmitted wave type they may be used in determining related quantities of the associated rays using the theory of characteristics. Previous work in this area is being refocused and will be the topic of a future report.

No approximations have been made in the derivations of the formulae presented with exceptions of the assumptions of a convex slowness surface and a plane interface separating the two media. These two extensions of the theory may be implemented with relative ease and were omitted to keep this report from becoming unnecessarily complicated. The problem of the vertical component of slowness becoming imaginary was given cursory treatment.

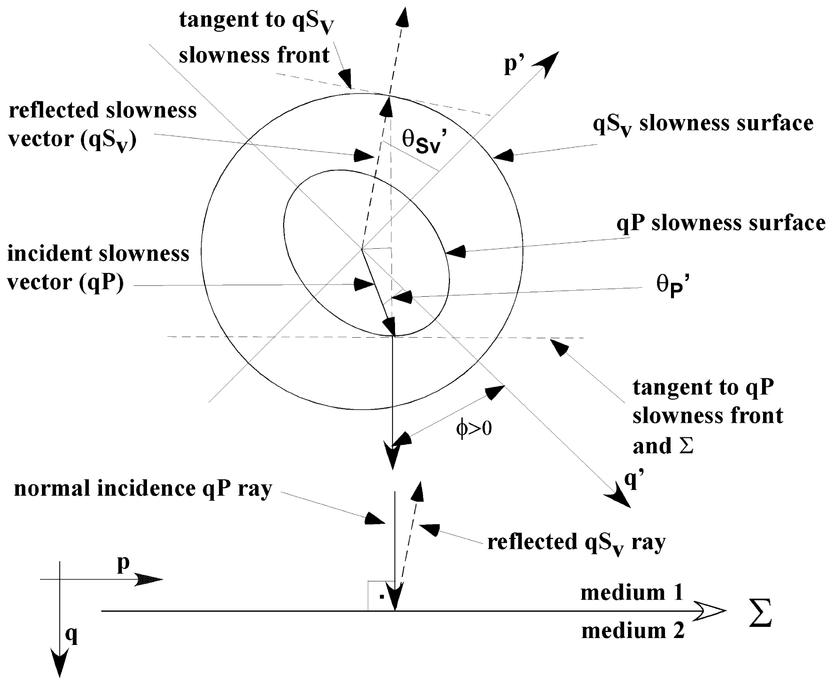

FIG. 6. In this example a normal incident *qP* ray at the interface Σ results in the reflected qS_V ray being reflected back into the medium at a non-zero angle. Both *qP* and qS_V ray segments are normal to their respective slowness surfaces.

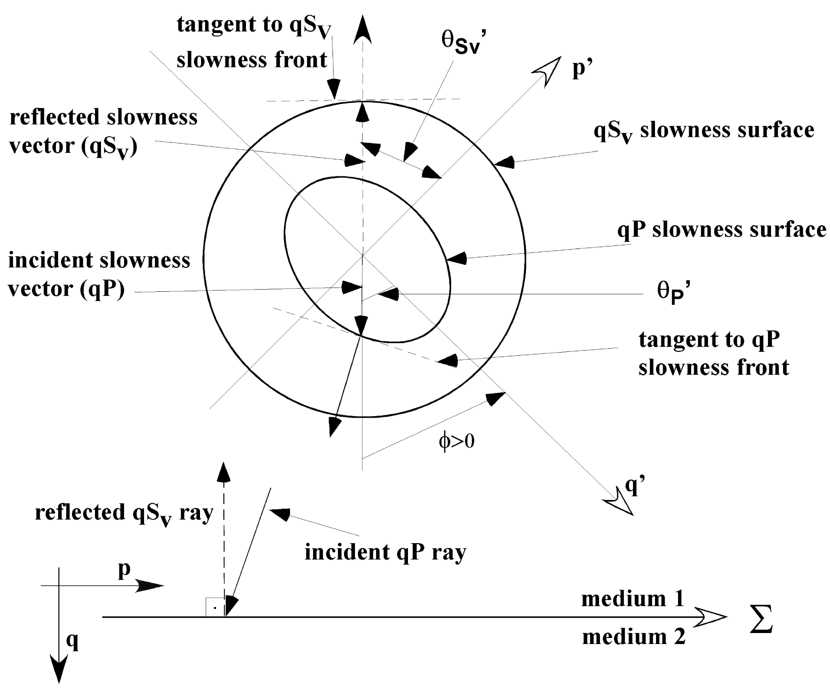

FIG. 7. A normally reflected qS_V ray at the interface Σ resulting from a *qP* ray that is not at normal incidence. The incident *qP* slowness vector is, however, at normal incidence to the interface, Σ.

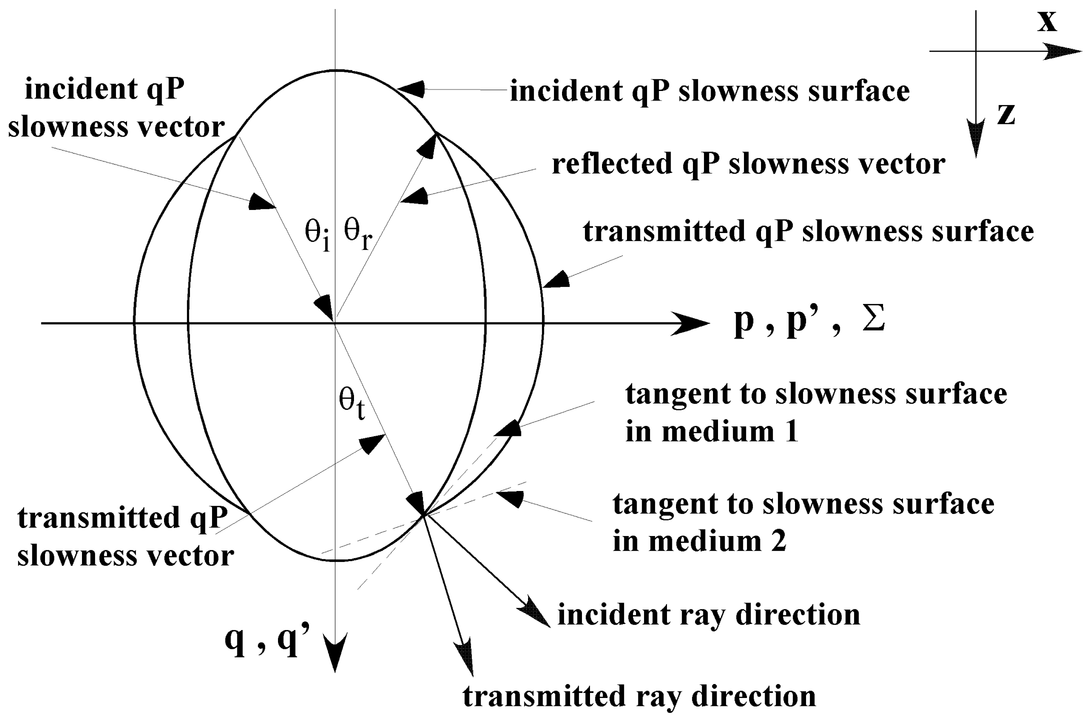

FIG. 8. Overlay of slowness surfaces in media 1 and 2 at the interface Σ. $\phi_1 = \phi_2 = 0$ in both media. The slowness vectors in both media make the same angle with the vertical, $\theta_i = \theta_t$ and further $\mathbf{p_i} = \mathbf{p_t}$, that is, the slowness vectors are not only equal but their components are equal: $p = p_i = p_t$, $q = q_i = q_t$. The incident and transmitted rays, however, make different angles with the model coordinates as they are normal to their respective slowness surfaces. Also, as $\phi_1 = 0$, $\theta_r = \theta_i$.

ACKNOWLEDGEMENTS

The financial support of CREWES sponsors and NSERC is gratefully acknowledged.

REFERENCES

Abramowitz, M. and Stegen, I., 1970, Handbook of mathematical functions: Dover Publications Inc

Cerveny, V., 1972, Seismic rays and ray intensities in inhomogeneous anisotropic medium: Geophysical Journal of the Royal Astronomical Society, **29**, 1-19.

Cerveny, V. and I. Psencik, 1972, Rays and travel-time curves in inhomogeneous anisotropic media: Zeitschrift fur Geophysik, **38**, 565-577.

Courant, R. and Hilbert, D., 1962, Methods of mathematical physics, vol. II: partial differential equations: Interscience Publications.

Daley, P.F., Marfurt, K.J. and McCarron, E.B., 1999, Finite-element ray tracing through structurally deformed transversely isotropic formations: Geophysics, **64**, 954-962.

Gassmann, F., 1964, Introduction to seismic travel time methods in anisotropic media: Pure and Applied Geophysics, **58**, 63-112.

Musgrave, M.J.P., 1970, Crystal acoustics: Holden-Day.

Press, W.H., Teukolsky, S.A., Vetterling, W.T. and Flannery, B.P., 1997, Numerical recipes for Fortran 77: Press Syndicate.

Schoenberg, M. and Helbig, K., 1996,Orthorhombic media: modeling elastic behavior in a vertically fractured earth: Geophysics, **62**, 1954-1974.

Slawinski, M.A., 1996, On elastic-wave propagation in anisotropic media: reflection/refraction laws, raytracing, and traveltime inversion: Ph.D. Thesis, University of Calgary.
Thomsen, L., 1986, Weak elastic anisotropy: Geophysics, **51**, 1954-1966.
Vlaar, N.J., 1968, Ray theory for an anisotropic inhomogeneous medium: Bulletin of the Seismological Society of America, **58**, 2053-2072.

APPENDIX: EIKONAL EQUATIONS

The eikonal equation related to the qP slowness surface in a transversely isotropic medium may be written as

$$G_{qP}\left(p,q,x,z\right)=A_{11}p^2+A_{33}q^2+\frac{A_\alpha}{2}\left\{\left(1+4\varepsilon_D\right)^{1/2}-1\right\}=1 \tag{A.1}$$

or equivalently

$$G_{qP}\left(p,q,x,z\right)=G_{qP}^{(e)}+\frac{A_\alpha}{2}\left\{\left(1+4\varepsilon_D\right)^{1/2}-1\right\}=1 \tag{A.2}$$

$G_{qP}^{(e)}$ being the ellipsoidal part of the eikonal and the additional term specifying the deviation of the associated wavefront from the ellipsoidal. The quantities in the deviation term are defined as

$$A_\alpha=\left(A_{11}-A_{55}\right)p^2+\left(A_{33}-A_{55}\right)q^2 \tag{A.3}$$

$$\varepsilon_D=\frac{A_D p^2 q^2}{A_\alpha^2} \tag{A.4}$$

$$A_D=\left(A_{13}+A_{55}\right)^2-\left(A_{11}-A_{55}\right)\left(A_{33}-A_{55}\right) \tag{A.5}$$

The quantity A_D, which has the dimensions of $velocity^2$, may be either positive or negative. If it is equal to zero, the eikonal equation degenerates to the ellipsoidal case. An approximation of the exact eikonal equation in the quasi-compressional, qP, case for what has been referred to as *mild* anisotropy (Schoenberg and Helbig, 1996) may be written as

$$\begin{aligned} G_{qP}\left(p,q,x,z\right) &\approx A_{11}p^2+A_{33}q^2+\frac{A_D p^2 q^2}{A_\alpha}\approx 1 \\ &\approx G_{qP}^{(e)}+\frac{A_D p^2 q^2}{A_\alpha}\approx 1 \end{aligned} \tag{A.6}$$

Not much is gained in the simplification process, unless further approximations are made. The problem with making additional simplifications is that the eikonal is usually no longer homogeneous in powers of slowness, which was to be retained to facilitate the use of the theory of characteristics.

For the quasi-shear, qS_V, propagation mode, the exact eikonal in a transversely isotropic medium is

$$
\begin{aligned}
G_{qS_V}(p,q,x,z) &= A_{11}p^2 + A_{33}q^2 - \frac{A_\alpha}{2}\left\{\left(1+4\varepsilon_D\right)^{1/2} - 1\right\} = 1 \\
&= G_{qP}^{(e)} - \frac{A_\alpha}{2}\left\{\left(1+4\varepsilon_D\right)^{1/2} - 1\right\} = 1
\end{aligned}
\tag{A.7}
$$

with an approximate expression being

$$
\begin{aligned}
G_{qS_V}(p,q,x,z) &\approx A_{55}\left(p^2+q^2\right) - \frac{A_D p^2 q^2}{A_\alpha} \approx 1 \\
&\approx G_{qS_V}^{(e)} - \frac{A_D p^2 q^2}{A_\alpha} \approx 1
\end{aligned}
\tag{A.8}
$$

where all quantities in the above equations have been previously defined.

Using the definitions of the slowness vector components, p and q, (Gassmann, 1965)

$$
p = \frac{\sin\theta_j}{V_N^j\left(\theta_j\right)} \qquad j = qP, qS_V
\tag{A.9}
$$

and

$$
q = \frac{\cos\theta_j}{V_N^j\left(\theta_j\right)} \qquad j = qP, qS_V
\tag{A.10}
$$

with θ_j being the wavefront normal (phase) or equivalently the slowness vector angle. The expressions for the phase velocities, V_N, may be written as

$$
V_N^j\left(\theta_j\right) = \left[A_{11}\sin^2\theta_j + A_{33}\cos^2\theta_j \pm \frac{A_\alpha^j}{2}\left\{\left(1+4\varepsilon_D^j\right)^{1/2} - 1\right\}\right]^{1/2}
\tag{A.11}
$$

with $j = qP$ or qS_V and the "+" and "−" signs corresponding to the qP and qS_V velocities, respectively. The modified definitions of A_α^j and ε_D^j are

$$
A_\alpha^j = \left(A_{11} - A_{55}\right)\sin^2\theta_j + \left(A_{33} - A_{55}\right)\cos^2\theta_j
\tag{A.12}
$$

$$
\varepsilon_D^j = \frac{A_D \sin^2\theta_j \cos^2\theta_j}{\left(A_\alpha^j\right)^2}
\tag{A.13}
$$

with A_D being as before.

For the degenerate (ellipsoidal) case of the coupled $qP - qS_V$ problem, $A_D \equiv 0$, the qP slowness surface is an ellipsoid of revolution, while the corresponding qS_V surface is a sphere. The eikonal equation for qP is

$$G_{qP}^{(e)}\left(p,q,x,z\right) = A_{11}p^2 + A_{33}q^2 = 1 , \tag{A.14}$$

while the qS_V eikonal is

$$G_{qSv}^{(e)}\left(p,q,x,z\right) = A_{55}\left(p^2 + q^2\right) = 1 \tag{A.15}$$

which is the same as the isotropic case, as $A_{55} = \beta^2$, β being the shear wave velocity in the medium. However, it is to be remembered that even in this case, the qP and qS_V wave motions are coupled.

In the ellipsoidal case, as in the general transversely isotropic case, it is required to have an expression for q in terms of p and the anisotropic parameters, $A_{ii}\ (i = 1,3,5)$. This results in

$$q = \left(1 - A_{11}p^2\right)^{1/2} \Big/ \sqrt{A_{33}} \tag{A.16}$$

for the qP case and

$$q = 1 \Big/ \sqrt{A_{55}} \tag{A.17}$$

The general transversely isotropic expression for q is, as would be expected, more complicated than for the elliptical case. It is a quartic equation in even powers of q. In addition, the equations for both the qP and qS_V eikonals in terms of powers of q are the same. This should not be of any surprise considering that the two waves are coupled and the associated eikonals are the two positive roots of a quartic equation. The equations defining the vertical slowness, q, for both the qP and qS_V wave types are

$$q_j = \left\{ \frac{-b \mp \left(b^2 - 4ac\right)}{2a} \right\}^{1/2} \tag{A.18}$$

with $j = qP$ or qS_V and the "$-$" and "$+$" signs corresponding to qP and qS_V.

The quantities in equations (A.18) are defined as

$$a = A_{33}A_{55} \tag{A.19}$$

$$b = -\left(A_{33} + A_{55}\right) + A_{55}\left(A_{11} + A_{33}\right)p^2 - A_D p^2 \tag{A.20}$$

$$c = 1 - \left(A_{11} + A_{55}\right)p^2 + A_{11}A_{55}p^4 \tag{A.21}$$

The above equations were obtained using the exact eikonal, and the solution for the vertical component of slowness is quite straightforward, as the quartic equation has no terms in odd powers of q. Employing an approximation, the eikonal results of the form derived by Thomsen (1986) for an analogous purpose requires that a similar quartic with the exception that it also contains an odd power of the independent variable (Slawinski, 1996). The result of this is a problem, which may be solved analytically but is more amenable to numerical solution (Abramowitz and Stegen, 1970).

For completeness, as recent publications involving anisotropic topics tend to use notation introduced by Thomsen (1986) rather than historical notation, the following definitions are included (Daley et al., 1999) to indicate the relationship between the two:

$$\alpha_0 = \sqrt{A_{33}} \tag{A.22}$$

$$\beta_0 = \sqrt{A_{55}} \tag{A.23}$$

$$\varepsilon = \frac{A_{11} - A_{33}}{2A_{33}} \tag{A.24}$$

$$\delta = \frac{\left(A_{13} + A_{55}\right)^2 - \left(A_{33} - A_{55}\right)^2}{2A_{33}\left(A_{33} - A_{55}\right)} \tag{A.25}$$

$$A_{11} = \left(1 + 2\varepsilon\right)A_{33} \tag{A.26}$$

$$A_D^2 = 2A_{33}\left(A_{33} - A_{55}\right)\left(\delta - \varepsilon\right) \tag{A.27}$$