S* – shear energy from a P-wave source

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ABSTRACT

The existence of an apparent source of S_V waves propagating in the underlying homogeneous isotropic medium resulting from an explosive P wave point source in the immediate vicinity of an interface, most often the earth's surface, has been shown in the literature to be a mathematical, numerical, and physical reality. This arrival, when observed on synthetic sections computed using the hybrid finite integral transform – finite-difference method, was designated as the S^* arrival. It had no true geometrical ray path, but rather an apparent path of energy transport and was termed, appropriately, a non-geometrical or inhomogeneous arrival. Its theoretical existence was subsequently confirmed by analytical methods using a zero order saddle point approximation to the Sommerfeld integral. These first analytical methods made a simplifying assumption that the saddle point was constrained to be on the real axis in the complex slowness (p) plane, eventhough it was known at the time that this was an idealized solution. However, the numerical results from this simplified solution showed reasonably good agreement with the purely numerical results, and as is often the case, the problem was not pursued further.

In the twenty years that have passed since this original investigation significant advances in data acquisition have been made and interest in this shear wave generation phenomenon has been shown. This has prompted a more mathematically intensive study of the problem with the idea that some of the original conjectures regarding the properties of this type of arrival be investigated in the light of a more comprehensive mathematical analysis. The investigation presented here of the S^* arrival indicates other instances where the zero-order asymptotic expansion, which is dependent on plane-wave reflection and transmission coefficients, to describe the particle displacement of body waves, is inadequate.

INTRODUCTION

When numerically considering Lamb's problem using a hybrid finite-difference – finite integral transform method of solution (Aleekseev and Mikhailenko, 1977, and Hron and Mikhailenko, 1981), an arrival of significant amplitude was observed when a point P source was located close to the free interface in an isotropic homogeneous halfspace. No geometrical ray path, based on a physical analysis, could be found to correspond to the transport of shear-wave energy from the source to a line of receivers buried in the isotropic halfspace. However, a path from an apparent source point was determined from traveltime measurements. This non-geometrical arrival was called the S^* arrival and from the numerical experiments using the computed synthetics, the following attributes of this disturbance were determined (Daley and Hron, 1983(a), (paper 1)):

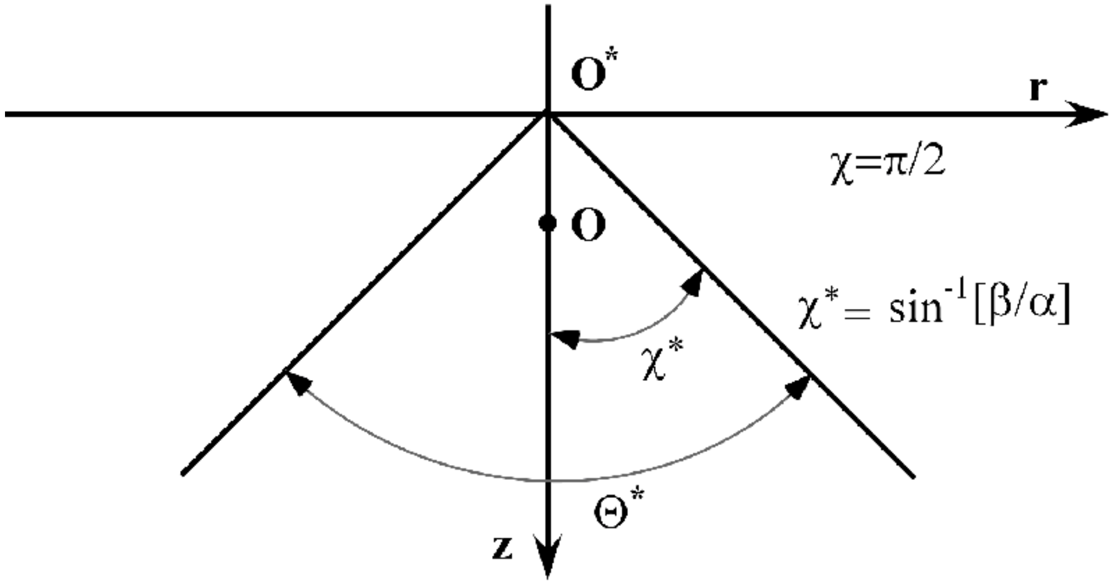

FIG. 1: Given a point source of *P* waves at O, the S^* arrival appears to emanate from the point O^*. The region of existence of the S^* arrival in the approximate saddle point case is confined to the region exterior to the conical surface with its apex at O^* and at an angle $\Theta^* = 2\sin^{-1}(\beta/\alpha)$ with the vertical axis. The angle $\chi = \sin^{-1}(\beta/\alpha)$ defines the distinct ray while $\chi = \pi/2$ corresponds to the surface ray.

(1). Its amplitude increased exponentially with decreasing source depth.

(2). In the vicinity of the free surface, the amplitude of the S^* increases in an almost linear manner with increasing receiver depth.

(3). The existence of the S^* arrival is restricted to that part of the elastic halfspace, which includes the outer domain of a conical surface, symmetric about the vertical axis passing through the source. The angle Θ^* satisfies the relation $\Theta^* = 2\sin^{-1}(\beta/\alpha)$ where α and β are, respectively, the phase velocities of the compressional (P) and shear (S_V) waves in the halfspace. [Figure 1 of this report.]

(4). The S^* arrival appears to originate at the vertex O^* of the above-mentioned conical surface and propagates with the velocity of the shear wave.

(5). The particle motion of the S^* disturbance is linearly polarized and at right angles to the direction of propagation.

Chapman (1985) addressed a canonical problem and obtained similar results. References to earlier reports of certain aspects of this phenomenon may also be found in paper 1, notably Lapwood (1949) and Burridge et al. (1964). Subsequent independent numerical experiments using purely finite-difference methods (Gutowski et al., 1984) confirmed the findings of Hron and Mikhailenko, (1981). The existence of the S^* wave was shown experimentally with physical modelling experiments performed by Kim and Behrens (1985,1986). Other confirmations of the results were given by Fertig (1984), Fertig and Psencik (1985), and Edelmann (1985).

As the disturbance was thought to be a contribution of that part of the Sommerfeld integral describing the reflected PS_V wavefield from a free interface not usually considered in high-frequency approximations to the integral the contour integral in question was re-examined in the context of the physical situation in which the arrival existed. Another saddle point defining an evanescent arrival type was found to be the contributing factor. As previously mentioned these arrival types are not usually included in the high frequency analysis of the Sommerfeld integral because in most cases they make no or insignificant contributions to the total computed wavefield.

In this report, the saddle point approximation described in paper 1 will be re-examined and a more mathematically correct determination of the saddle point and its contribution to the total Sommerfeld integral investigated. Some of the points, in particular, 3 and 4, given above regarding the properties of the S^* arrival will then be re-evaluated in light of the more mathematically correct solution. For completeness the solution for the PS_V reflected arrival will be discussed so as to provide a context, specifically regarding the integration contour, within which to approach the solution method for the S^* arrival. Only the vertical components of the disturbance will be developed in detail. The horizontal component may then be inferred from the vertical component solution for both the reflected PS_V and non-geometrical S^* arrivals. The zero order saddle point method will be employed in this discussion. A more rigorous, higher order treatment may be pursued at a later time as it may prove useful in dealing with further questions raised here.

THEORETICAL BACKGROUND

Consider a homogeneous isotropic halfspace with compressional and shear wave velocities, α and β respectively, and density ρ. A point source of *P* waves is assumed to be located a distance *h* below the vacuum/solid boundary and a receiver situated at (r,z) in the cylindrical coordinate system in which the problem is being considered (Figure 2). The elastodynamic equations in this medium type are

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2}-(\lambda+2\mu)\nabla(\nabla\cdot\mathbf{u})+\mu\left[\nabla\times(\nabla\times\mathbf{u})\right]=\mathbf{f} \tag{1}$$

The particle displacement, $\mathbf{u}(r,z,t)=\left[u_r(r,z,t),u_z(r,z,t)\right]$ may written in terms of P and S_V potentials, ϕ and ψ as

$$\mathbf{u}=\nabla\phi+\nabla\times\nabla\times(0,0,\psi) \tag{2}$$

with the related potentials, $\mathbf{\Phi}$ and $\mathbf{\Psi}$, of the source term $\mathbf{f}(x,z,t)$ being given by

$$\mathbf{f}(x,z,t)=\nabla\Phi+\nabla\times\nabla\times(0,0,\Psi),\quad \Phi=A(t)\delta(x)\delta(y)\delta(z-h),\quad \Psi=0. \quad (3)$$

In the above a P type source has been assumed so that any possible S_V source contribution due to Ψ is zero.

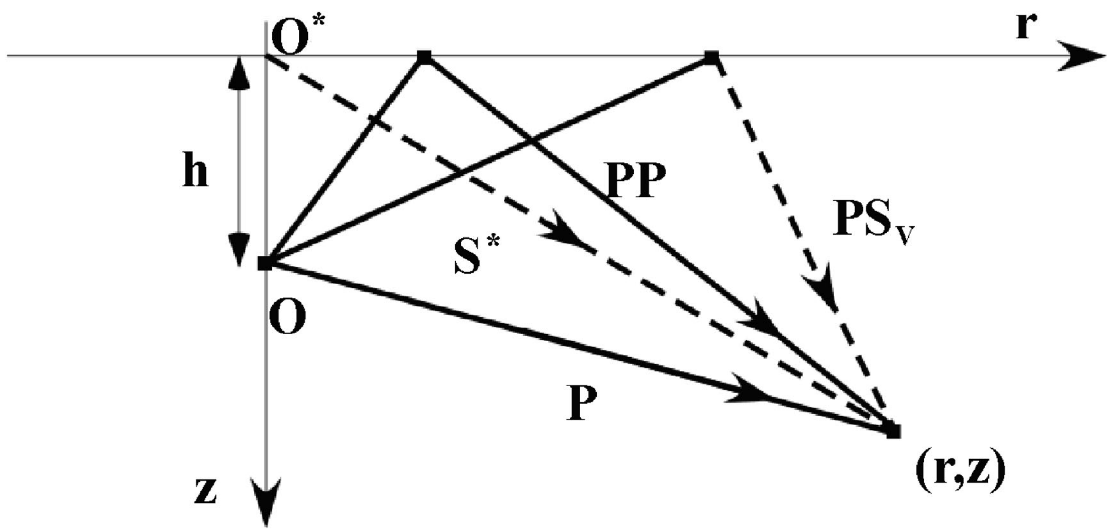

FIG. 2. A schematic of the geometry of the problem being considered. A point source of *P* waves is located at O in an isotropic homogeneous halfspace a distance h below the free surface. The source depth is assumed to be of the order of less than half a wavelength measured with respect to the predominant frequency of the source pulse. A line of receivers is located at some depth z below the vacuum/solid interface. The apparent location of a source of shear wave energy is at O^* on the surface.

The reflected PS_V displacement potential, defined above, may be written exactly as

$$\psi(r,z,\omega)=A(\omega)\int_0^\infty\left(\frac{1}{i\omega p}\frac{\beta}{\alpha}R_{PS}(p)\right)J_0(\omega pr)\exp\left[i\omega(\eta h+\xi z)\right]\frac{p\,dp}{\eta} \quad (4)$$

(Cerveny and Ravindra, 1970 and Aki and Richards, 1980) where the quantities requiring definition are the radicals η and ξ, which are related to the P and S_V velocities and propagation angles, θ_P and θ_{Sv} in the following manner

$$\eta=\left(\alpha^{-2}-p^2\right)^{1/2}=\frac{\cos\theta_P}{\alpha},\quad \mathrm{Im}(\eta)\geq 0 \quad (5)$$

and

$$\xi=\left(\beta^{-2}-p^2\right)^{1/2}=\frac{\cos\theta_{Sv}}{\beta},\quad \mathrm{Im}(\xi)\geq 0. \quad (6)$$

The free surface PS_V displacement reflection coefficient as derived in Aki and Richards, (1980) with η and ξ introduced is

$$R_{PS}(p) = \frac{4\alpha\beta p\eta\left(1-2\beta^2p^2\right)}{\left(1-2\beta^2p^2\right)^2 + 4\beta^4p^2\eta\xi} \tag{7}$$

where all quantities have been previously defined. It should be noted that the denominator of equation (7) is the Rayleigh function (Aki and Richards, 1980), $J_0(\kappa)$ is the zero order Bessel function and $A(\omega)$ is a frequency dependent quantity, which may contain the Fourier time transform of the source wavelet.

The reflected PS_V horizontal (radial) component of displacement may, utilizing the definition of the S_V potential in equation (1), be written as

$$u_r(r,z,\omega) = \frac{\partial^2\psi(r,z,\omega)}{\partial r\,\partial z} \tag{8}$$

$$u_r(r,z,\omega) = A(\omega)\int_0^\infty \left(\frac{1}{i\omega p}\frac{\beta}{\alpha}R_{PS}(p)\right)\left(-i\omega^2 p\xi\right)J_1(\omega pr)\exp\left[i\omega(\eta h+\xi z)\right]\frac{p\,dp}{\eta}. \tag{9}$$

and in the same manner, the reflected PS_V vertical component of displacement has the form

$$u_z(r,z,\omega) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi(r,z,\omega)}{\partial r}\right) \tag{10}$$

$$u_z(r,z,\omega) = A(\omega)\int_0^\infty \left(\frac{1}{i\omega p}\frac{\beta}{\alpha}R_{PS}(p)\right)\left(-\omega^2 p^2\right)J_0(\omega pr)\exp\left[i\omega(\eta h+\xi z)\right]\frac{p\,dp}{\eta}. \tag{11}$$

A zero order saddle point solution will be presented only for the vertical displacement component as the expression for the horizontal component may be obtained quite easily from the vertical problem solution. Replacing the Bessel function $J_0(\kappa)$ by Hankel function $H_0^{(1)}(\kappa)$ in equation (11) (Abramowitz and Stegun, 1980) yields

$$u_z(r,z,\omega) = A(\omega)\frac{i\omega\beta}{2\alpha}\int_{-\infty}^{\infty} R_{PS}(p)H_0^{(1)}(\omega pr)\exp\left[i\omega(\eta h+\xi z)\right]\frac{p^2\,dp}{\eta} \tag{12}$$

Introducing the asymptotic expansion for large argument for $H_0^{(1)}(\kappa)$ and retaining only the leading term results in

$$u_z(r,z,\omega) = A(\omega)\frac{i\beta}{\alpha}\sqrt{\frac{\omega}{2\pi r}}e^{-i\pi/4}\int_{-\infty}^{\infty} R_{PS}(p)\exp\left[i\omega(pr+\eta h+\xi z)\right]\frac{p^{3/2}\,dp}{\eta} \tag{13}$$

The integral along the real axis, $(-\infty < p < \infty)$, can be replaced by one more suitable for a saddle point approximation to the integral. Also, the branch cuts corresponding to the branch points at $p = \pm\alpha^{-1}$ and $p = \pm\beta^{-1}$ may be moved off the real axis into the first and third quadrants of the complex p-plane. Using standard closed contour methods

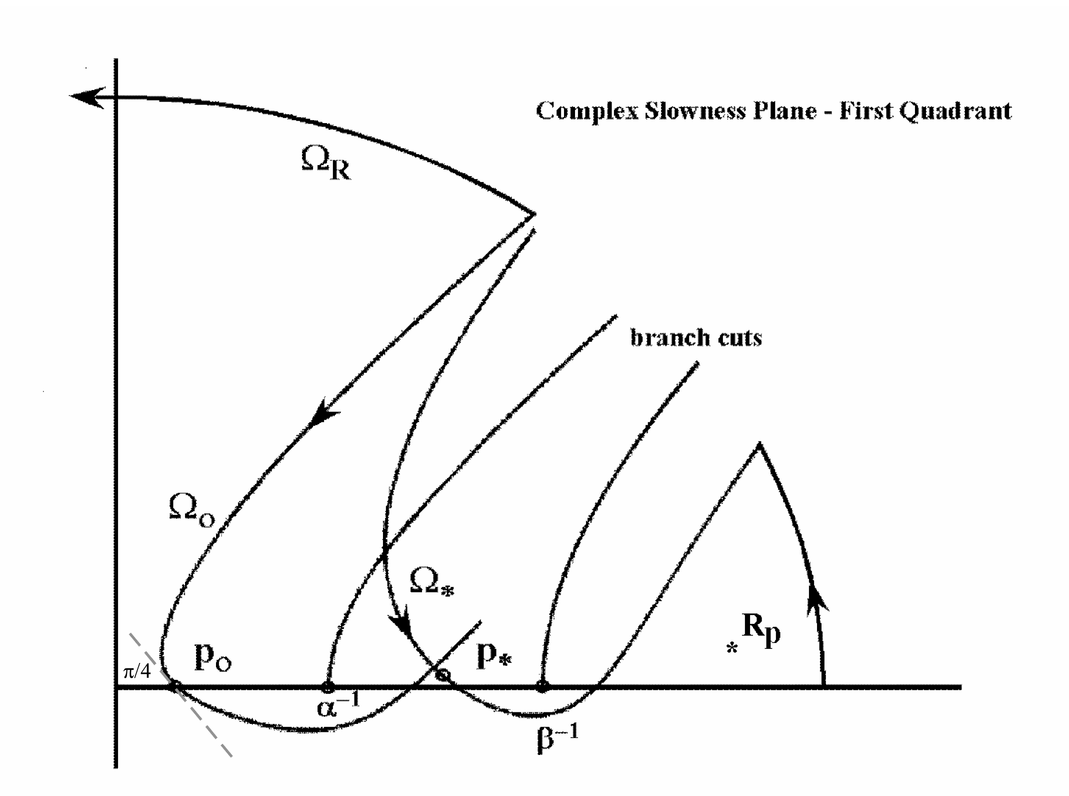

FIG. 3. Integration contour used in obtaining zero order saddle point solutions for the reflected PS_V and non-geometrical S^* arrivals. The saddle point contours have been parameterized in terms of a real variable defined in the text. The branch cuts have been moved off the real axis so that they lie in the first quadrant of the complex p-plane using a similar parameterization as for the branch points. The original integral has been along the real p axis $(-\infty < p < \infty)$ transformed into its constituent arrivals. The Rayleigh pole is indicated as R_P but its contribution will not be considered here (see Aki and Richards, 1980) nor will the branch cut integrals. The saddle point that produces the PS reflected arrival lies on the real p axis while that for the S^* arrival lies in the first quadrant of the complex p-plane.

$$\Omega = \int_{-\infty}^{\infty} + \Omega_0 + \Omega_* + \Omega_R = \sum Residues \tag{14}$$

where as indicated in Figure (3)

$\int_{-\infty}^{\infty}$ - original contour along the real p axis,

Ω_0 - geometrical PS_V reflected arrival saddle point contribution to the integral,

Ω_* - non-geometrical (inhomogeneous) S^* arrival saddle point contour,

Ω_R - integral around the semicircle of radius R in the upper half of the p-plane which as a consequence of the radiation conditions imposed tends to zero as $R \to \infty$, and

$\sum Residues$ - the residues contained within the closed contour Ω which is the Rayleigh pole contribution (Aki and Richards, 1980).

The only branch points which need to be considered for this contour choice are those at $p = \alpha^{-1}$ and $p = \beta^{-1}$ as those in the third quadrant in the complex p-plane are not included within the integration contour closed in the upper half of the p-plane as they are bypassed by the saddle point contours. If the problem had been formulated in a manner such that $H_0^{(2)}(\kappa)$ was needed to be introduced (complex conjugate case) they would be have to be considered rather than those in the first quadrant, as it would be required that the contour be closed in the lower half of the p-plane. The saddle point contour for the S^* arrival is isolated in Figure 4.

REFLECTED PS$_V$ ARRIVAL

A parameterization of the saddle point contour often used in these types of problems, y being a real quantity such that $y = 0$ corresponds to $p = p_0$, is

$$\left(\alpha^{-2} - p^2\right)^{1/2} = \left(\alpha^{-2} - p_0^2\right)^{1/2} - y\, e^{-i\pi/4} \quad , \; -\infty < y < \infty \tag{15}$$

or equivalently, with η_0 indicating the value of η at the saddle point, p_0

$$\eta = \eta_0 - y\, e^{-i\pi/4} \quad , \; -\infty < y < \infty . \tag{16}$$

The corresponding mapping of the branch cut from the real p axis into the first quadrant in the complex p-plane and originating at $p = \alpha^{-1}$ is accomplished through the variable change

$$\left(\alpha^{-2} - p^2\right)^{1/2} = y\, e^{i\pi/4} \quad , \; 0 < y < \infty , \tag{17}$$

y again is a real valued quantity and in an analogous manner to the saddle point parameterization, $y = 0$ corresponds to $p = \alpha^{-1}$.

From equation (15) the following relation used in the change of variable from p to y is obtained

$$\frac{p\,dp}{\eta} = dy\,e^{-i\pi/4} \qquad \left(\frac{dp}{dy} = \frac{\eta}{p}e^{-i\pi/4}\right). \tag{18}$$

The equation providing the saddle point definition for the reflected PS_V arrival is the phase function

$$\tau(r,z,p) = r\,p + h\eta + z\xi\ . \tag{19}$$

The saddle point location is obtained, with p_0 being real and corresponding to the geometrical reflected PS_V arrival $\left(0 \le p_0 < p_\alpha = \alpha^{-1}\right)$, from the derivative of the phase function τ with respect to p

$$\left.\frac{d\tau}{dp}\right|_{p=p_0} = \left[r - p\left(\frac{h}{\eta} + \frac{z}{\xi}\right)\right]_{p=p_0} = 0 \tag{20}$$

This apparently fairly simple equation cannot be solved analytically and numerical methods must be used. The solution of this equation for a real value of p yields the saddle point corresponding to the geometrical PS_V reflected arrival. For this case the value of p_0 is located on the real axis interval, $\left(0 \le p_0 < \alpha^{-1}\right)$. As in any saddle point approximation the validity of the solution within the region in which it is defined should be investigated before implementation. For example, as $p_0 \to \alpha^{-1}$, $r \to \infty$. This is a problematic area as it is a special case of the saddle point approximation where the saddle point is in the vicinity of a branch point. Higher order saddle point approximations to integrals in this region are usually in the form of parabolic cylinder functions or retaining more terms in the asymptotic expansion of the integrand (Felsen and Marcuvitz, 1973).

This is also the case at the other extreme of the interval of the existence of the saddle point, near $p_0 \approx 0\left(r \approx 0\right)$. In most instances the zero order saddle point is sufficiently accurate in this offset range even though the asymptotic expansion of the Hankel function, $H_0^{(1)}\left(\kappa\right)$, for large values of κ is used in this solution method. However, if h is small, of the order of less than a wavelength, the plane wave approximation is not valid as the curvature of the wavefront from a point source of P waves must be taken into consideration.(Daley and Hron, 1987 and confirmed by Psensik, 1988). Such is the case here, as this problem has been addressed in the above two papers it will not be repeated as it is fairly mathematically intensive and not particularly useful in the present discussion. However, the zero order saddle point approximation is included for completeness, but more specifically to introduce the saddle point integration contour parameterization employed.

Expanding the exponential term $\tau(r,z,p)$ in a Taylor series about $p = p_0$, $(y=0)$, results in the following

$$\tau(r,z,p) \approx \tau(r,z,p_0) + \frac{1}{2}\left[\frac{d^2\tau}{dp^2}\left(\frac{dp}{dy}\right)^2\right]_{p=p_0} y^2 + \cdots. \tag{21}$$

where the saddle point condition that $d\tau/dp = 0$ at $p = p_0$ has been used to simplify the above expansion. For the problem under consideration $\partial^2\tau/dp^2$ does not become zero within the range of applicability so that higher order terms need not be introduced in equation (21). In the case of the geometrical reflected PS_V arrival, $\tau(r,z,p_0)$ is the reflected travel time from the source to the free interface and then to the receiver. As p_0 is real it follows that $\tau(r,z,p_0)$ is also real. Using the expressions derived above the zero order saddle point approximation for the vertical component of particle displacement of the reflected PS_V arrival is obtained in the standard manner (Felsen and Marcuvitz, 1973).

$$u_z(r,z,\omega) = -A(\omega)\frac{\beta}{\alpha}\sqrt{\frac{\omega p_0}{2\pi r}}R_{PS}(p_0)e^{-i\pi/2}e^{i\omega\tau_{PS_V}}\int_{-\infty}^{\infty}\exp\left[-a_0^2y^2/2\right]dy \tag{22}$$

where

$$\tau_{PS_V} = \tau(r,z,p_0) \tag{23}$$

with a_0^2 defined as

$$a_0^2 = \omega\left(\frac{h}{\alpha^2\eta_0^3} + \frac{z}{\beta^2\xi_0^3}\right)\left(\frac{\eta_0^2}{p_0^2}\right) = \omega\,\tilde{a}_0^2\left(\frac{\eta_0^2}{p_0^2}\right) \tag{24}$$

The variables subscripted with "0" indicate that they are to be evaluated at p_0. This results in

$$u_z(r,z,\omega) = A(\omega)\,\frac{\beta}{\alpha}\sqrt{\frac{p_0}{r}}\,\frac{R_{PS}(p_0)}{\tilde{a}_0}\frac{p_0}{\eta_0}\;e^{i\omega\tau_{PS_V}}\,. \tag{25}$$

From this equation for the vertical component of the reflected PS_V particle displacement at a free surface, the horizontal component may be obtained in a similar manner from equation (9) and is given b

$$u_r(r,z,\omega) = A(\omega)\,\frac{\beta}{\alpha}\sqrt{\frac{p_0}{r}}\,\frac{R_{PS}(p_0)}{\tilde{a}_0}\frac{\xi_0}{\eta_0}\;e^{i\omega\tau_{PS_V}}\,. \tag{26}$$

The variation of the ray parameter for the reflected PS_V versus offset is shown in Figure 5.

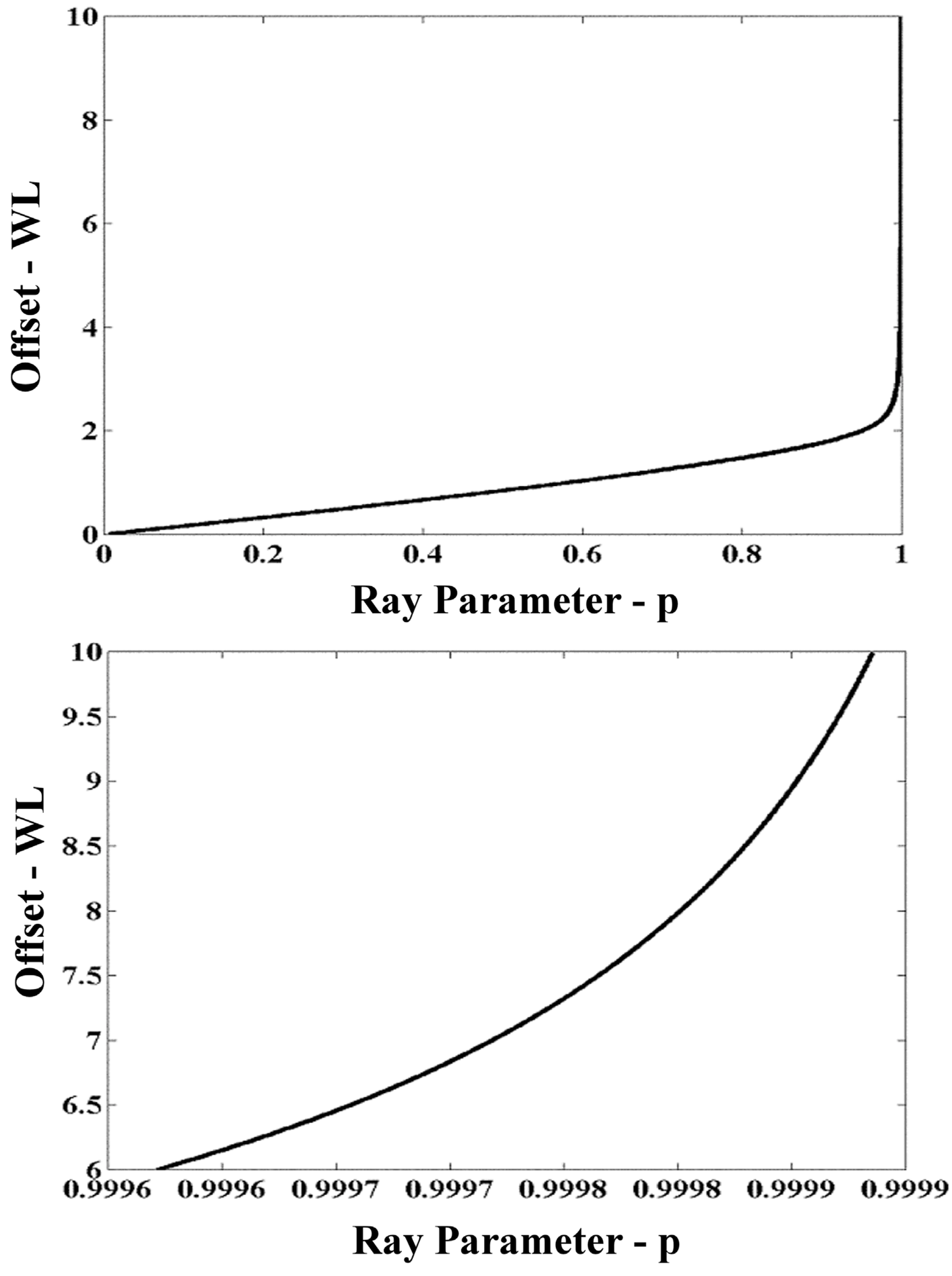

FIG. 5. The location of the real saddle point, p_0, corresponding to the PS_V reflected arrival from the free surface. The source is located at a depth of 0.125 WL in an isotropic homogeneous halfspace with compressional and shear wave velocities $\alpha = 1.0\, WL/T$ and $\beta = 0.5\, WL/T$ respectively. A line of buried receivers is at 3.0 WL below the free interface. The progression of p_0 towards the branch point at α^{-1} is quite rapid so that from an offset of about 6.0 WL onwards the zero order saddle point approximation is a questionable solution due to the proximity of the saddle point p_0 and branch point $p_\alpha = \alpha^{-1}$.

S* ARRIVAL

There is another solution of equation (16) which lies in the semi-infinite strip $\left(p_\alpha = \alpha^{-1} < p_* < \beta^{-1} = p_\beta : 0 < p_* < i\infty\right)$ and produces the non-geometrical or inhomogeneous arrival, which has been designated as the S^* arrival in the literature (Mikhailenko and Hron, 1981). This event is associated with a point P wave source located close to the free surface. In a halfspace the expected arrivals for a buried receiver would be, in addition to the Rayleigh wave, the direct P and the reflected PP and PS_V arrivals. However, numerical experiments (Hron and Mikhailenko, 1981) have shown that if the P wave source is of the order of less than half a wavelength[1] from the vacuum/solid interface, the saddle point indicated above may make a significant contribution to the recorded disturbance. This has been extended to a P wave source at depth in the proximity of an interface of moderate to large changes in elastic parameters (Daley and Hron, 1983b). The vertical component of the S^* disturbance is given by the zero order saddle point solution of

$$u_z\left(r,z,\omega\right) = -A\left(\omega\right)\frac{\beta}{\alpha}\sqrt{\frac{\omega}{2\pi r}}e^{-i\pi/4}\int_{\Omega_*} R_{PS}\left(p\right)\exp\left[i\omega\left(pr+\eta h+\xi z\right)\right]\frac{p^{3/2}\,dp}{\eta} \quad (27)$$

For this part of the contour the solution of the equation defining the saddle point, p_*, requires

$$\left.\frac{d\tau}{dp}\right|_{p=p_*} = \left[r - p\left(\frac{h}{\eta}+\frac{z}{\xi}\right)\right]_{p=p_*} = 0, \ \left(p_\alpha < p_* < p_\beta : 0 < p_* < i\infty\right) \quad (28)$$

and must also be obtained numerically. However in this instance, the quantity being solved for is complex, requiring a method for simultaneously solving for both the real and imaginary parts of $p_* = p_*^R + ip_*^I$, with both p_*^R and p_*^I positive. Newton's method (Press et al., 1997) is one possibility for accomplishing this.

The parameterization of the saddle point contour used here is similar to that used for the geometrical reflected PS_V, equation (15), and in terms of the real parameter y may be written as

$$\left(\beta^{-2} - p^2\right)^{1/2} = \left(\beta^{-2} - p_*^2\right)^{1/2} - y\,e^{-i\pi/4+i\theta_*} \quad , \ -\infty < y < \infty \quad (29)$$

(Figure 4). Here, p_* is generally complex and will be written as $\left|p_*\right|e^{i\theta_*}$ or $p_* = p_*^R + i\,p_*^I$ with θ_* being the positive acute angle between the real p axis and a line drawn from the origin to the point p_* in the first quadrant of the complex p-plane. The quantities p_*^R and p_*^I are real and positive.

[1] A wavelength *(WL)* which will be used as a measure of distance, for this problem is defined in terms of the P wave velocity in the halfspace, α, and the predominant frequency of the source wavelet, f_0, as $\lambda = \alpha/f_0$. The measure of time will be a period, *(T)*, and defined as $T = 1/f_0$.

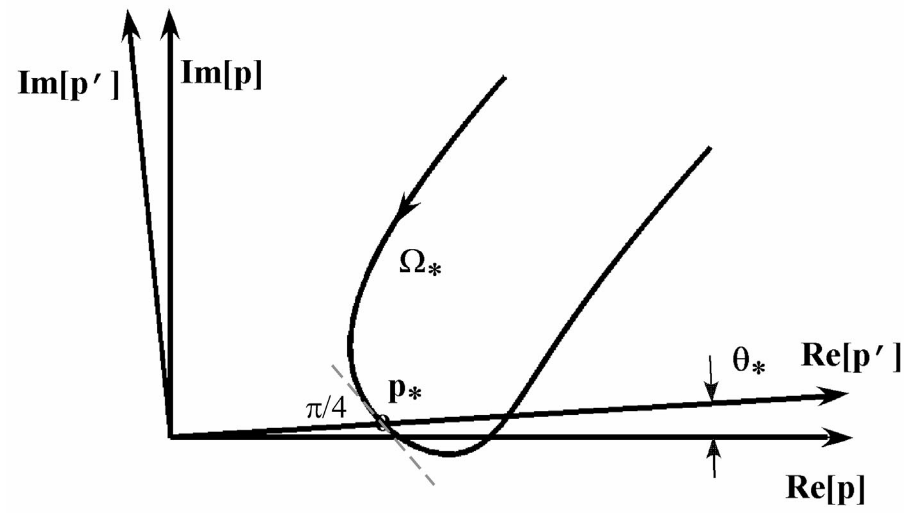

FIG. 4. Location in the first quadrant of the complex *p*-plane of the saddle point p_* associated with the S^* arrival. The angle θ_* determined by the saddle point location defines an orthonormal transformation from the model (unprimed) coordinate system to the primed one in which the saddle point p'_* is real.

The introduction of the angle θ_* into the expression for the contour parameterization has the effect of an orthonormal rotation of the $\mathbf{p}$ plane into the $\mathbf{p}'$ plane, which is such that the real axis of the $\mathbf{p}'$ plane passes through the complex valued saddle point, p_*, so that p'_* is a real quantity in the rotated Cartesian system.

The branch cut originating at $p = \beta^{-1}$ is moved to the first quadrant through the variable change

$$\left(\beta^{-2} - p^2\right)^{1/2} = y\, e^{i\pi/4} \quad , \; 0 < y < \infty \tag{30}$$

where again y is a real valued quantity and in an analogous manner to the saddle point parameterization, $y = 0$ corresponds to $p = \beta^{-1}$. All quantities subscripted with "*" will assumed to be evaluated at p_*.

From equation (29) the relationship

$$\frac{p\,dp}{\xi} = dy\, e^{-i\pi/4 + i\theta_*} \qquad \left(\frac{dp}{dy} = \frac{\xi}{p} e^{-i\pi/4 + i\theta_*}\right) \tag{31}$$

is obtained, where as before, $\xi = \left(\beta^{-2} - p^2\right)^{1/2}$. In a manner analogous to the reflected geometrical ray PS_V the exponential term, $\tau(r, z, p)$, is expanded in a Taylor series. In this instance it is about $p = p_* \left(y = 0 \right)$, with $\tau\left(r, z, p\right)$ as in the previous section being given by

$$\tau\left(r, z, p\right) = r\,p + h\eta + z\xi \tag{32}$$

Using equation (31), the truncated Taylor series, which again retains terms up to y^2, and using $d^2\tau/dp^2$ derived in the previous section the following expansion is obtained

$$\tau\left(r, z, p\right) \approx \tau\left(r, z, p_*\right) + \frac{1}{2}\left[\frac{d^2\tau}{dp^2}\left(\frac{dp}{dy}\right)^2\right]_{p=p_*} y^2 + \cdots \ . \tag{33}$$

Substituting quantities defined earlier into the expansion of the exponential and denoting that a parameter is evaluated at p_* with the subscript "*", results in the term in y^2 being given by

$$\begin{aligned} a_*^2 &= \omega\left(\frac{h}{\alpha^2\eta_*^3} + \frac{z}{\beta^2\xi_*^3}\right)\left(\frac{\xi_*^2}{p_*^2}\right)e^{2i\theta_*} \\ &= \omega\,\tilde{a}_*^2\left(\frac{\xi_*^2}{p_*^2}\right)e^{2i\theta_*} \\ &= \omega\,\hat{a}_*^2 \ . \end{aligned} \tag{34}$$

Implementing the expressions derived above the expression for the vertical component of particle displacement of the S^* disturbance due to the saddle point at $p = p_*$ may then be written in the intermediate form

$$u_z\left(r, z, \omega\right) = A\left(\omega\right)\frac{i\beta}{\alpha}\sqrt{\frac{\omega p_*}{2\pi r}}\,\frac{\xi_*}{\eta_*}R_{PS}\left(p_*\right)e^{i\omega\tau_*}\int_{-\infty}^{\infty}\exp\left[-a_*^2 y^2/2\right]dy \tag{35}$$

where the expression for the "travel time" term, $\tau_*\left(r, z, p_*\right)$, is now a complex quantity consisting of the actual travel time of the disturbance, τ_*, together with a term which introduces an exponential decay factor, γ_*, whose damping effect is dependent upon the distance of the source from the free surface interface and the relative size of the imaginary part of η_*. Analytic expressions for τ_* and γ_* may not be obtained as they are the result of the numerical solution of equation (27) for p_* substituted into equation (32) and the real and imaginary parts of $\tau\left(r, z, p_*\right)$ taken to yield these two values. The travel time and exponential decay factor are written as

$$\begin{aligned}\tau_*(r,z,p_0) &= \tau_*^R(r,z,p_*) + i\tau_*^I(r,z,p_*) \\ &= \tau_* + i\gamma_*\end{aligned} \quad . \tag{36}$$

Continuing with the evaluation of the saddle point approximation solution in equation (35) results in the vertical component of particle displacement of the S^* arrival being given by

$$u_z(r,z,\omega) = A(\omega)\frac{i\beta}{\alpha}\sqrt{\frac{p_*}{r_*}}\frac{R_{PS}(p_*)}{\tilde{a}_*}\frac{p_*}{\eta_*}e^{i\omega\tau_* - \omega\gamma_*} \tag{37}$$

The corresponding horizontal displacement component of the S^* has the form

$$u_r(r,z,\omega) = A(\omega)\frac{i\beta}{\alpha}\sqrt{\frac{p_*}{r}}\frac{R_{PS}(p_*)}{\tilde{a}_*}\frac{\xi_*}{\eta_*}e^{i\omega\tau_* - \omega\gamma} \,. \tag{38}$$

These solutions will be discussed in more detail in the next two sections.

COMPARISON OF SOLUTIONS

The following quote from paper 1 will be discussed here:

"The existence of the S^* arrival is restricted to that part of the elastic halfspace which includes the outer domain of a conical surface, symmetric about the vertical axis passing through the source. The angle Θ^* satisfies the relation $\Theta^* = 2\sin^{-1}(\beta/\alpha)$ where α and β are, respectively, the phase velocities of the compressional (P) and shear (S_V) waves in the halfspace."

In the general case of the saddle point approximation for the S^* arrival, discussed in this report, the real part of p_* is constrained to lie in the range $\alpha^{-1} < p_* < \beta^{-1}$. This holds for the simplified case discussed in paper 1 with the exception that there, p_* was obtained as a result of the simplification of the saddle point approximation, which forced it to be real and thus lie on the real p axis in the range $\alpha^{-1} < p_* < \beta^{-1}$. The delimiter "$<$" is used rather than "$\leq$" as "$<$" indicates that the ordinary saddle point method used is not valid at (or near) the end points of the real axis range. The end points, α^{-1} and β^{-1}, are both branch points. At these points alternative, more complex approximations to the Sommerfeld type integral are required for accurate results (Felsen and Marcuvitz, 1973).

Given the phase function, which is to be expanded about the saddle point, as

$$\tau(r,z,p) = r\,p + h\eta + z\,\xi \tag{39}$$

with h being the depth of the point P source below the free interface, z the depth of the receivers, r the offset or horizontal distance from the source to receiver, and the previously defined quantities

$$\eta = \left(\alpha^{-2} - p^2\right)^{1/2} \quad \mathrm{Im}(\eta) \geq 0 \tag{40}$$

$$\xi = \left(\beta^{-2} - p^2\right)^{1/2} \quad \mathrm{Im}(\xi) \geq 0\,. \tag{41}$$

For h small the term exponential term $e^{i\omega\tau(r,z,p)}$ was approximated in paper 1 as

$$\exp\left[i\omega\tau(r,z,p)\right] \approx \exp\left[-\eta_* h\right]\exp\left[i\omega(rp + \xi z)\right]. \tag{42}$$

Here $\eta_* = \left(p_*^2 - \alpha^{-2}\right)^{1/2}$ with the value of p at the S^* saddle point location, p_*, obtained from the simplified equation

$$\left.\frac{d\left(rp + \xi z\right)}{dp}\right|_{p=p_*} = 0 \tag{43}$$

the solution being

$$p_* = \frac{r}{\beta\left(r^2 + z^2\right)^{1/2}} \tag{44}$$

As all quantities in equation (44) are real, then as previously mentioned, the saddle point p_* is required to lie on the real p axis in the range $\alpha^{-1} < p_* < \beta^{-1}$. The lower bound corresponds to

$$p = \frac{1}{\alpha} = \left.\frac{\sin\theta_P}{\alpha}\right|_{\theta_P = \pi/2}. \tag{45}$$

It follows from Snell's Law that

$$\Theta^* = 2\sin^{-1}\left(\beta/\alpha\right) = 2\chi^*. \tag{46}$$

Thus, according to this analysis, if the receiver line is located at a depth z, the first offset, r_F, at which the S^* arrival can be seen is

$$r_F = z\tan\chi^* \tag{47}$$

The other limit for p_* is $p_* = \beta^{-1}$ resulting in the following condition being satisfied

$$1 = \frac{r}{\left(r^2 + z^2\right)^{1/2}}\,. \tag{48}$$

This requires that $r \to \infty$ which would be expected.

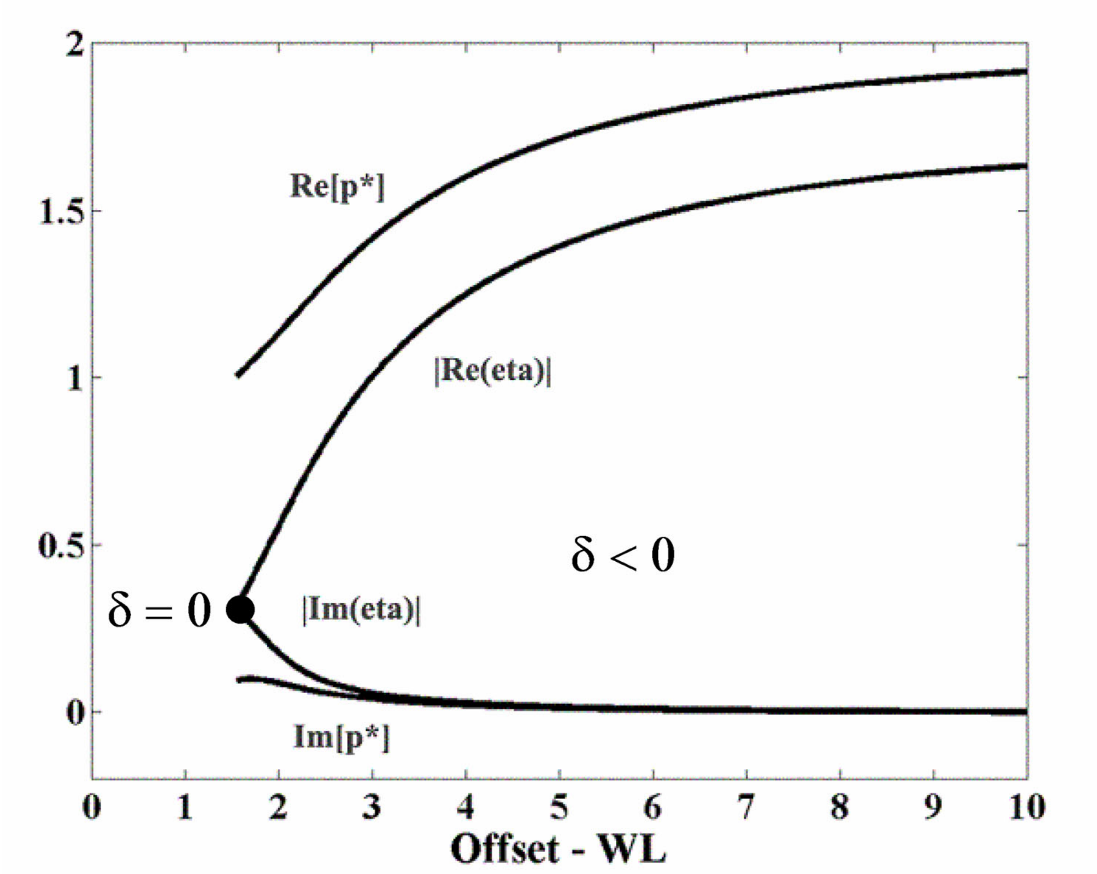

FIG. 6. The real and imaginary parts of p_* and the absolute values of the real and imaginary parts of η_* plotted against the offset of the receivers at a depth of 3.0 *WL*. The plotting of the values is stopped when the quantity $\delta = \alpha^{-2} - \left(p_{R_*}^2 - p_{I_*}^2\right)$ has a zero crossing. After that point the saddle point derived does not satisfy radiation conditions leading to a non-physical arrival. This is because the only solution to the saddle point equation, apart from the geometrical reflected PS_V disturbance, only exists on a non-physical Riemann sheet.

A comparison will now be made of the above and the solution for the saddle point location in the semi-infinite strip $\left(\alpha^{-1} < \mathrm{Re}\left[p_*\right] < \beta^{-1}; 0 \le \mathrm{Im}\left[p_*\right] < \infty\right)$. In this case, where the saddle point case is complex, the distinct ray, analogous to $\sin\theta_P = \pi/2$ in the instance previously considered, occurs when

$$\delta = \alpha^{-2} - \left(p_*^R\right)^2 + \left(p_*^I\right)^2 = 0 \qquad (49)$$

It is at this point where the sign of the imaginary part of η_* changes leading to an unphysical radiation condition. As $\delta \to 0^-$

$$\lim(\eta)\Big|_{\delta\to 0^-} = \left(-i2p_R p_I\right)^{1/2} = e^{-i\pi/4}\left(2p_R p_I\right)^{1/2} = (1-i)\left(p_R p_I\right)^{1/2} \tag{50}$$

Using numerical methods the algorithm at this point will attempt to locate a solution and most often will accomplish this. However, the solution will not satisfy the radiation condition as it is located on a non-physical Riemann sheet.

Apart from the above, it is in this area where

$$\left.\frac{d^2\tau}{dp^2}\right|_{p=p_0} = 0 \tag{51}$$

introducing a further complication as higher order terms in the Taylor series expansion of $\tau(r,z,p)$ must be calculated before an approximation to the saddle point integral can be obtained.

The solution obtained in this report will now be compared with that computed in paper 1 using a numerical example for this purpose. The *P* and S_V velocities in the halfspace are $1.0\,WL/T$ and $0.5\,WL/T$, respectively, a line of receivers located at a depth of $3.0\,WL$ and a point source of *P* waves at $(r,z) = (0.0\,WL, 0.125\,WL)$. Using equation (46) results in the S^* onset angle, χ_*, offset r_F and ray parameter, p_F given as

$$\chi_* = \sin^{-1}(\beta/\alpha) = 30.0° \tag{52}$$

$$r_F = z\tan\chi_* = 1.732\,WL \tag{53}$$

and consistent with the definition of

$$p_F = \frac{\sin\chi_*}{\beta} = 1.0\,T/WL\,. \tag{54}$$

Using the solution method presented in this report the following values of the offset r_F, and complex ray parameter, p_F may be obtained numerically as

$$r_F \;=\; 1.5490\;WL \tag{55}$$

$$p_F \;=\; (1.0046 + i\,0.0959)\;T/WL \tag{56}$$

The related angle χ_* that the onset or distinct S^* ray makes with the vertical is complex in this instance, defined as

$$\chi_* = \sin^{-1}(\beta\, p_F) \tag{57}$$

where in equation (57) $\beta = 0.5\,WL/L$ and p_F is given by equation (56). This results in the real part of χ_* being approximately equal to $27.31°$ which is slightly less than that predicted by the earlier method, as is the value of r_F.

The fact that both of these high frequency approximations to the S^* arrival region of existence predict that there is a cutoff angle defining an area where the wave does not exist is questionable. It is highly probable that what is occurring here is similar to passing from the illuminated to the shadow region in a diffraction type problem and that this arrival is present in some form and for some angular distance into the conical region defined by Θ^*. The interference of the reflected PS_V and the S^* events in this near offset area in the "exact" synthetics (Figure 7) hinders speculation into what is actually occurring.

Preliminary investigations using analytical continuation of the solution into the "region of non-existence" and boundary layer methods have been somewhat impeded by the presence of both exponential and nonexponential terms in the integrand containing the radical $\eta = \left(\alpha^{-2} - p^2\right)^{1/2}$ which must be dealt with in a proper manner. This suggests that a higher order approximation to the geometrical reflected PS_V in the vicinity of a branch point should first be obtained and what is learned from that exercise applied to a higher order solution for the S^* problem.

One matter that has received the expected clarification when the second method is used is that the appropriate time delay is introduced into the travel time of the S^* wave. In the first approximation the onset time of the S^* arrival was the time taken for a ray to travel from the point O^* to the receiver where $t = 0$ corresponded to the P wave source excitation at O. This delay is approximately the time taken for the P wave to travel from O to O^*.

DISCUSSION

Two figures from the paper of Hron and Mikhailenko (1981) showing their original numerically computed synthetic traces together with the source wavelet used which, in part, lead to the search for a mathematical explanation of what has become to be referred to as the S^* arrival are displayed in Figure (7). The inclusion of figures displaying fairly contentious findings from a previously refereed paper was done to minimize any preliminary defence of the results and focus on resolving the issues introduced.

Apart from the presence of the nongeometrical S^* arrival in the synthetic traces other presumably anomalous behavior may be seen in Figure 7. The first of these is a nonzero PS_V reflected arrival in the vertical component of particle displacement at zero offset. Studies that consisted of taking higher order terms in the Sommerfeld integral to obtain modified geometrical optics (asymptotic ray theory) approximations showed that the numerical results obtained by Hron and Mikhailenko (1981) were correct. The nonzero PS_V reflected arrival, evident in the vertical component of particle displacement, was treated using this approximate method by Daley and Hron (1987) and their findings were subsequently confirmed by Psencik (1988).

A second area where close examination of the synthetics reveals apparent inconsistencies is at far offsets. Neglecting the PP reflected arrival due its interference

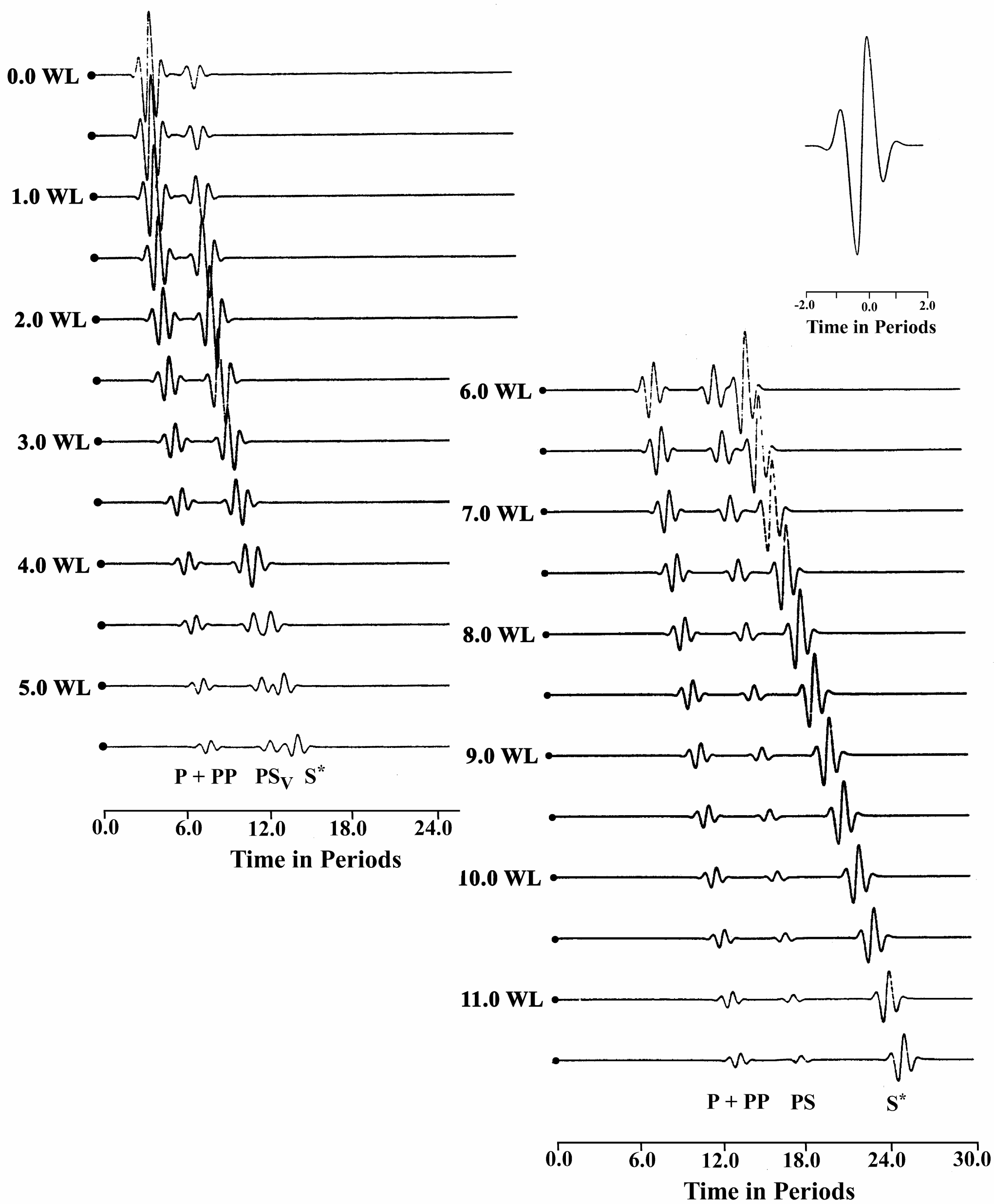

FIG. 7. A collection of figures from the paper of Hron and Mikhailenko (1981) showing synthetic traces of the vertical component of particle displacement. The source wavelet used is located in the upper right corner. The values of the compressional and shear wave velocities are $1.0\,WL/T$ and $0.5\,WL/T$, respectively. The *P* wave source is at $0.0\,WL$ offset and at a depth of $0.125\,WL$ below the free surface. A line of receivers is located at a depth of $3.0\,WL$. The S^* arrival is quite prominent at large offsets and the unexpected (from a zero order geometrical optics perspective) PS_V reflected event at zero offset on the vertical component of particle displacement and the phase mismatch of approximately $\pi/2$ at far offsets for the same event. The scaling of the traces differs between the two panels.

with the direct P ray and considering the PS_V event an unexpected (in the zero order geometrical optics solution sense) deviation almost totally in terms of phase distortion from what one would expect is seen. From the insert in the figure containing the source wavelet it would appear that the arrival is out of phase by $\pi/2$ if zero order asymptotic ray theory were used as an indicator. This apparent inconsistency is due to the saddle point corresponding to the PS_V reflected event being in the proximity of the branch point at $p=\alpha^{-1}$ (Figure 5). A higher order saddle point solution in this vicinity is more complicated than problems of this type discussed in the literature (Felsen and Markuvitz, 1974) in that the quantity that varies most rapidly here is the radical $\eta=\left(\alpha^{-2}-p^{2}\right)^{1/2}$ and it is present in both the exponential and non-exponential integrand functions in the Sommerfeld type integral exact solution. A canonical problem for an S_H seismic wave propagation problem with this type of functional dependence on the radical η is considered in Daley and Hron (1988) where a modified higher order saddle point solution is derived and compares favourably with a numerically exact finite difference - finite integral transform results.

CONCLUSIONS

The saddle point approximation to the non-geometrical S^* arrival discussed in a paper written 2 decades ago was re-examined and the basis for one of the several characteristic properties, which were enumerated there, has been analyzed in a more mathematically correct manner. It has been shown that the restriction of the existence of the S^* wave due to the forcing of p_* to lie on the real p axis producing what was termed the distinct ray also occurs when p_* is a complex quantity. The existence of this apparent sharp boundary is still questionable and requires further investigation.

The introduction of the more mathematically correct zero order saddle point solution does not greatly improve the numerical results, but by its implementation provides a better insight into the existence of this non-geometrical arrival including a time lag for the disturbance to travel from the point source of P waves at O to the apparent source of the S^* wave at O^*.

The more mathematically correct method of evaluating the zero order saddle point contribution associated with the S^* arrival has raised several additional questions of mathematical if not seismological interest and should be addressed by considering the use of modified higher order saddle point solution techniques.

The P wave source considered here has been assumed located in the vicinity of the free surface. The case of the same type of source adjacent to an interface at depth of high velocity and density contrast has been shown to behave in a similar manner (Daley and Hron, 1983b), but has not been addressed in this report.

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