Linearized quantities in T.I. media

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ABSTRACT

When describing transversely isotropic (T.I.) media it has become common to use, apart from reference qP and qS_V velocities, α and β, the two parameters, ε and δ, to account for the deviation of the coupled qP and qS_V modes of wave propagation from the isotropic case. The prefix "q" denotes"quasi", and is used to distinguish anisotropic from isotropic media as different inherent concepts, not explicitly obvious in the isotropic case, are required to be addressed when dealing with anisotropic media of any degree of complexity. The dimensionless quantity, ε, is a measure of the ellipticity of the qP wavefront and δ, the "strange" parameter, is employed as a measure of deviation of the qP wavefront or slowness surface from the ellipsoidal and also of the qS_V wavefront or slowness surface from the spherical. As the parameter, δ, has been described in the literature as "conceptually inaccessible", it would seem a logical progression to attempt to determine an alternative parameterization in physically realizable quantities. It is that topic which is dealt with in this note. The linearized qP and qS_V phase (wavefront normal) velocities and the linearized PP and $S_V S_V$ reflection coefficients to first order accuracy at an interface between two T.I. media are examined in this regard. The solution proposed involves a simple reorganizing of terms in the linearized expressions for the two phase velocities and reflection coefficients resulting from the introduction of the parameter σ. This quantity is used in a modified form here compared with that used by other authors, usually when discussing qS_V wave propagation in a T.I. medium.

The parameterization of this media type in (ε,σ) rather than (ε,δ) is transparent from a numerical perspective as little if any changes would be required in any related software.

INTRODUCTION

A number of authors have presented linearized approximations of differing forms to the reflection and transmission coefficients in elastic isotropic media (Bortfield, 1961; Chapman, 1976; Aki and Richards, 1980; Shuey, 1987). Extensions of these approximations to transversely isotropic (T.I.) media have appeared in several papers including, Banik (1987), where a discussion of the physical meaning of δ is given, Thomsen (1993), who presents linearized expressions for the PP and $S_V S_V$ reflection coefficients and the significance of δ within this approximation, and Blangy (1994) where linearized T.I. expressions for reflection coefficients are used in an *AVO* analysis. The formulae presented for the PP and $S_V S_V$ reflection coefficients in a transversely isotropic case employ expressions involving the dimensionless quantities, ε and δ, introduced into the seismic anisotropy literature by Thomsen (1986), and subsequently entering into common usage, to quantify, generally, the deviation of a T.I. medium from the isotropic case. Using this notation a T.I. medium may be fully parameterized in terms of ε and δ, together with the reference qP and qS_V velocities, α and β, which will be formally defined later, and density, ρ. As the linearized qP and qS_V phase velocities (Thomsen, 1986) are relevant to this topic they will be given preliminary consideration.

The object of this note is to comment on a certain aspect of the linearized derivations and to introduce some minor adjustments in the equations presented by the authors using an analysis of the exact expressions for the qP and qS_V phase velocities. This adjustment consists of introducing the dimensionless variable $\sigma = (\delta - \varepsilon)$*. A similar quantity was isolated by Banik (1987), Thomsen (1993) and Tsvankin and Thomsen (1994) among other more recent works by these authors. It is usually defined in their papers as $\sigma = (\alpha/\beta)^2 (\varepsilon - \delta)$** and used most often by these authors in reference to the qS_V propagation mode. In the recent paper of Grechka and Tsvankin (2002) the quantity $\eta = (\varepsilon - \delta)/(1 + 2\delta)$ is defined as the "anellipticity coefficient" which is used to describe the deviation of a T.I. medium from the ellipsoidal. The difference between a similar quantity used here and identified as σ is that η results from a linearized approximation.

The exact and approximations for "mildly" anisotropic eikonal equations (Shoenberg and Helbig, 1996), which will be required here, may be found in the Appendix. As far as it is possible, the notation used by Thomsen (1986) and (1993) will be retained in the expressions for the linearized phase velocities and reflection coefficients.

As previously stated the two anisotropic parameters, ε and δ, and the velocities, α and β, together with density, ρ, defining a T.I. medium may be expressed in terms of the traditional anisotropic coefficients $C_{ij} = \rho A_{ij}$, A_{ij} having the dimensions of velocity squared. The relevant formulae relating the two types of media specifications will be presented and discussed before proceeding further. The first of these, ε, defines, for the compressional wavefront case, the difference between a spherical wavefront and an ellipsoid of revolution wavefront which is symmetric about the vertical. It is the measure of the ratio of the horizontal and vertical velocities of the ellipsoid of revolution, the first step in progressing from a spherical to a general T.I. wavefront type. (A qS_V wavefront at this stage would still be spherical.) The quantity ε is defined as (Thomsen, 1986).

$$\varepsilon = \frac{A_{11} - A_{33}}{2A_{33}}, \qquad (1)$$

where $\sqrt{A_{11}} = \alpha_{\pi/2}$ is the qP wave velocity in the horizontal direction, i.e., parallel to an interface, assuming that the axes of anisotropy in the medium are aligned with the interface. In a similar manner $\sqrt{A_{33}} = \alpha$ is the velocity normal to the interface or the normal incidence velocity of the qP wavefront. The shear wave velocity along the

*This definition of σ is not related to that found in Shuey (1985).

** In Banik's (1987) notation, $\varepsilon_S = \sigma/4$.

meridional axes in a T.I. media is defined as $\sqrt{A_{44}} = \beta$. Utilizing these definitions, the following additional quantities have the form

$$A_{11} = A_{33}\left(1+2\varepsilon\right) \approx A_{33}\left(1+\varepsilon\right)^2 . \tag{2}$$

The approximate relation is made under the assumption the that $\varepsilon << 1.0$, which results in

$$\alpha_{\pi/2} \approx \left(1+\varepsilon\right)\alpha . \tag{3}$$

As shown in the Appendix, if the quantity δ is identically equal to ε in a T.I. medium, the well-known condition of the medium degenerating to the ellipsoidal case occurs. The definition of δ involves, among other anisotropic coefficients, the coefficient A_{13}, itself conceptually vague from a physical point of view, and is given by the relation

$$\delta = \frac{\left(A_{13}+A_{55}\right)^2 - \left(A_{33}-A_{55}\right)^2}{2A_{33}\left(A_{33}-A_{55}\right)^2} . \tag{4}$$

Thomsen, in his 1993 paper, comments on δ in the following manner:

"The parameter δ is intuitively inaccessible, and is tedious to measure in the laboratory, but Thomsen (1988) showed that δ is much more important in most exploration contexts than is the more familiar anisotropy parameter ε."

It is to be taken from the above statement that the deviation of the T.I. wavefront from the ellipsoidal is more important in seismic applications than is the dependence on ε and that in any approximation to a quantity involving these two parameters should reflect this. However, as a consequence of the phrase, "… The parameter δ is intuitively inaccessible … ", the same of which may be said of the quantity A_{13}, it would seem that some other parameter which is more understandable from a physical point of view may be chosen to describe a medium of this type and it would be desirable in practice to have such a quantity which could be referred to in a fairly definitive manner as the deviation of the T.I. medium from the degenerate ellipsoidal case.

The discussion presented here will be limited to a relatively small region of incident angles about normal incidence where a significant part of the actual seismic acquisition is done. As a consequence only terms up to first order in the linearized approximations will be considered, as this serves the purpose of the motivation for this work. A discussion of higher order terms and their usefulness may be found in Blangy (1994).

LINEARIZED qP AND qS_V PHASE VELOCITIES IN T.I. MEDIA

As a point of reference, it might be instructive to first present the linearized expression for the qP phase (wavefront normal) velocity in the ellipsoidal case of transverse isotropy. With θ being the phase angle, the exact equation for this velocity is

$$v_{qP}(\theta) = \left(A_{11}\sin^2\theta + A_{33}\cos^2\theta\right)^{1/2} . \tag{5}$$

Expanding equation (5) in a power series, retaining only the zero- and first-order terms and introducing α and ε as defined in the previous section has

$$v_{qP}(\theta) \approx \alpha\left(1+\varepsilon\sin^2\theta\right), \tag{6}$$

which displays a first-order dependence on ε.

The linearized phase velocity of the quasi-compressional, qP, velocity as derived by Thomsen (1986) is specified by the relation

$$v_{qP}(\theta) = \alpha\left(1+\delta\sin^2\theta\cos^2\theta+\varepsilon\sin^4\theta\right) \tag{7}$$

and contains an implicit first order dependence on ε in the ellipsoidal limit, $\delta \to \varepsilon$. The above approximation was obtained by expanding the exact expression for the qP phase velocity in a Taylor series expansion in ε and δ^* with δ^* having a slightly different definition than δ, given by

$$\delta^* = \frac{1}{2A_{33}^2}\left[2\left(A_{13}+A_{55}\right)^2 - \left(A_{33}-A_{55}\right)\left(A_{11}+A_{33}-2A_{55}\right)\right]. \tag{8}$$

Adding and subtracting the quantity "$\varepsilon\sin^2\theta$" from (7) and using the variable quantifying the variation of the *T.I.* wavefront from the ellipsoidal case discussed in the Appendix, $\sigma = \left(\delta-\varepsilon\right)$, results in

$$v_{qP}(\theta) = \alpha\left(1+\left(\sigma+\varepsilon\right)\sin^2\theta-\sigma\sin^4\theta\right). \tag{9}$$

As the first-order approximation will be used in the reflection coefficients in the next section the first-order approximation of equation (9) is

$$v_{qP}(\theta) = \alpha\left(1+\left(\sigma+\varepsilon\right)\sin^2\theta\right) . \tag{10}$$

This equation is consistent with the ellipsoidal case, having a first-order dependence on ε and in agreement with the fact that the qP wavefront is dependent on a two-parameter specification, ε and σ.

The qS_V linearized phase velocity derived by Thomsen (1986) is defined as

$$v_{qS_V}(\theta) = \beta\left(1+\left(\alpha/\beta\right)^2\left(\varepsilon-\delta\right)\sin^2\theta\cos^2\theta\right). \tag{11}$$

In terms of the introduced parameter, σ, the above equation becomes

$$v_{qS_V}(\theta)=\beta\left(1-(\alpha/\beta)^2\sigma\sin^2\theta+(\alpha/\beta)^2\sigma\sin^4\theta\right), \tag{12}$$

where clearly both the first and second order terms in the linearized approximation are dependent only on σ, the deviation from the ellipsoidal. The lack of an explicit dependence on ε is what would be expected as the velocities along the meridional axes of the qS_V wavefront are $\beta=\sqrt{A_{55}}$. The altered expression for equation (12), accurate to order one is

$$v_{qS_V}(\theta)=\beta\left(1-(\alpha/\beta)^2\sigma\sin^2\theta\right). \tag{13}$$

The phase velocities of first-order accuracy have been given here as the linearized reflection coefficients used in the next section are of order one accuracy and it is cautiously predict at this point that the first order linearized reflection coefficients considered will display the same type of dependence on the parameters ε and σ as do the phase velocities.

LINEARIZED PP AND S_VS_V REFLECTION COEFFICIENTS IN *T.I.* MEDIA

To further illustrate the possible usefulness of the $(\varepsilon,\sigma(\varepsilon,\delta))$ parameterization over that in (ε,δ), the linearized PP and S_VS_V reflection coefficients at an interface between two *T.I.* media (upper = 1 and lower = 2) given in Thomsen (1993) will be considered. Some definitions of the quantities involved and constraints on them should first be addressed. Variables prefixed with "Δ" indicate the difference of the quantity between media, i.e., $\Delta\xi=\xi_{(2)}-\xi_{(1)}$, while those with an overscore refer to the mean of the quantity, $\bar{\xi}=\left(\xi_{(2)}+\xi_{(1)}\right)/2$. Further, weak anisotropy is assumed so that (Thomsen, 1993)

$$\left|\frac{\Delta\xi}{\bar{\xi}}\right|<<1, \quad \xi=\alpha,\beta,\rho. \tag{14}$$

ρ being density and

$$\left|\xi_{(i)}\right|<<1, \quad \xi_{(i)}=\varepsilon_{(i)},\delta_{(i)},\sigma_{(i)}, \quad i=1,2. \tag{15}$$

The linearized, accurate to order one, PP reflection coefficient between two *T.I.* media, in Thomsen's 1993 work *(Equation (10))* is introduced as

$$R_P(\theta)=\frac{1}{2}\left[\frac{\Delta Z}{\bar{Z}}\right]+\frac{1}{2}\left[\frac{\Delta\alpha}{\bar{\alpha}}-\left(\frac{2\bar{\beta}}{\bar{\alpha}}\right)^2+\left(\delta_{(2)}-\delta_{(1)}\right)\right]\sin^2\theta, \tag{16}$$

where $\bar{Z}=\bar{\rho}\bar{\alpha}$ is the average compressional velocity impedance and $\bar{G}=\bar{\rho}\bar{\beta}^2$ the shear modulus impedance. Rewriting (16) in terms of (ε,σ), which in a manner similar to the phase velocity involves adding and subtracting the two terms "$\varepsilon_{(2)}\sin^2\theta/2$" and "$\varepsilon_{(1)}\sin^2\theta/2$", results in

$$R_P(\theta)=\frac{1}{2}\left[\frac{\Delta Z}{\bar{Z}}\right]+\frac{1}{2}\left[\frac{\Delta\alpha}{\bar{\alpha}}-\left(\frac{2\bar{\beta}}{\bar{\alpha}}\right)^2+\Delta\sigma+\Delta\varepsilon\right]\sin^2\theta . \tag{17}$$

Equations (16) and (17) are identical with the exception that in equation (16), (ε,δ) are used as the anisotropic indicators while in (17), (ε,σ) have been introduced. As in the case of the phase velocity discussed in the previous section, the first order term in the linearized expansion of the R_P reflection coefficient should contain both the quantity controlling the deviation of the ellipsoidal case from a T.I. media, σ, and the ellipticity factor, ε. This is not immediately obvious from equation (16). The anisotropic parameters used in equation (17) both have physical meaning, are physically realizable and, given a reasonable acquisition geometry, both ε and σ may be at least grossly estimated from traveltime measurements (Gassmann, 1964).

The linearized $S_V S_V$ reflection coefficients at the interface of two T.I. media approximated to first order is given by Thomsen (1993), with a similar result to be found in Banik (1987), as

$$\begin{aligned}R_{S_V}(\theta)=&-\frac{1}{2}\left[\frac{\Delta\rho}{\bar{\rho}}+\frac{\Delta\beta}{\bar{\beta}}\right]-\frac{1}{2}\frac{\Delta\beta}{\bar{\beta}}\tan^2\theta+2\left[\frac{\Delta\rho}{\bar{\rho}}+2\frac{\Delta\beta}{\bar{\beta}}\right]\sin^2\theta\\&-\left[\left(\frac{\bar{\alpha}}{\bar{\beta}}\right)^2\left(\delta_{(2)}-\delta_{(1)}-\varepsilon_{(2)}-\varepsilon_{(1)}\right)\right]\sin^2\theta\end{aligned} . \tag{18}$$

Introducing σ through the definition of $\Delta\sigma$ results in equation (16) becoming

$$\begin{aligned}R_{S_V}(\theta)=&-\frac{1}{2}\left[\frac{\Delta\rho}{\bar{\rho}}+\frac{\Delta\beta}{\bar{\beta}}\right]-\frac{1}{2}\frac{\Delta\beta}{\bar{\beta}}\tan^2\theta+2\left[\frac{\Delta\rho}{\bar{\rho}}+2\frac{\Delta\beta}{\bar{\beta}}\right]\sin^2\theta\\&-\left[\left(\frac{\bar{\alpha}}{\bar{\beta}}\right)^2\Delta\sigma\right]\sin^2\theta\end{aligned} . \tag{19}$$

As in the R_P case, the dependence of R_{S_V} on ε and σ is related to the qS_V phase velocity of order one accuracy dependence on these quantities. The dependence of R_{S_V} on the single parameter describing deviation from the ellipsoidal case is indicated, as the linearized qS_V phase velocity is a function of just this anisotropic quantity. It should be noted that equations (16), (17), (18), and (19) display only first order accuracy to terms in $\sin^2\theta$. The comparison of linearized phase velocity dependencies on ε and σ to other linearized reflection and transmission coefficients with the same dependencies shows a fairly consistent, if not predictable, pattern.

CONCLUSIONS

A slightly different manner of specifying anisotropy in transversely isotropic media, within the framework of the widely used (ε,δ) notation, has been presented without altering the outcome of the linearized equations for qP and qS_V phase velocities and the PP and $S_V S_V$ reflection coefficients at an interface between two T.I. media. A previously defined parameter has been introduced in a slightly altered form to act as the indicator of the deviation of a T.I. medium from the ellipsoidal. This parameter, $\sigma=(\delta-\varepsilon)$ is used to replace δ in this capacity, as σ is a physically realizable and measurable quantity, where δ has been described as "intuitively inaccessible", or its physical significance analyzed in terms of formulae based on weak anisotropic linearized approximations (Banik (1987). In fact, as seen in the above relation involving ε, δ and σ, it may be conjectured that physical meaning can only be assigned to δ when it is measured relative to another parameter with an identifiable physical meaning, which in this instance is ε.

Finally, as proposed by many authors, linearized approximations should be used almost exclusively in efforts to obtain an understanding of the physics of the problem at hand. For all other applications, with specific exceptions, exact formulae are highly encouraged to minimize the possibility of errors in numerical accuracy and loss of concepts inherent in the "exact" theory.

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APPENDIX: PHASE (WAVEFRONT NORMAL) VELOCITIES IN T.I. MEDIA

The exact expressions for the qP and qS_V phase (wavefront normal) velocities, with the prefix "q" denoting "quasi" as the rays and wavefront normals in this medium type display subtly different qualities than the P and S_V propagation modes in an isotropic medium, may be written in the modified forms (Gassmann, 1964) as

$$V_N^{(qP)}(\theta) = \left[\left(A_{11} \sin^2 \theta + A_{33} \cos^2 \theta \right) + \frac{A_\alpha}{2} \left[\left(1 + 4\kappa_D \right)^{1/2} - 1 \right] \right]^{1/2}, \tag{A.1}$$

$$V_N^{(qS_V)}(\theta) = \left[A_{55} - \frac{A_\alpha}{2} \left[\left(1 + 4\kappa_D \right)^{1/2} - 1 \right] \right]^{1/2}. \tag{A.2}$$

The quantities requiring definition in the above are

$$\begin{aligned} A_\alpha &= \left(A_{11} - A_{55} \right) \sin^2 \theta + \left(A_{33} - A_{55} \right) \cos^2 \theta \\ &= A_{11} \sin^2 \theta + A_{33} \cos^2 \theta - A_{55} \end{aligned}, \tag{A.3}$$

$$\kappa_D = \frac{A_D \sin^2 \theta \cos^2 \theta}{A_\alpha^2}. \tag{A.4}$$

Approximating the radical in κ_D in (A.1) and (A.2) by retaining only the first two terms in the binomial expansion has the effect of smoothing the slowness surfaces (eikonals) and consequently the phase velocities. The results termed “mild” anisotropy (Shoenberg and Helbig, 1996) may be written as

$$V_N^{(qP)}(\theta) = \left[\left(A_{11} \sin^2 \theta + A_{33} \cos^2 \theta \right) + \frac{A_D \sin^2 \theta \cos^2 \theta}{A_\alpha^2} \right]^{1/2}, \tag{A.5}$$

$$V_N^{(qS_V)}(\theta) = \left[A_{55} - \frac{A_D \sin^2 \theta \cos^2 \theta}{A_\alpha^2} \right]^{1/2}, \tag{A.6}$$

$$A_D = \left(A_{13} + A_{55} \right)^2 - \left(A_{11} - A_{55} \right) \left(A_{33} - A_{55} \right), \tag{A.7}$$

or in terms of Thomsen’s (ε, δ) notation

$$A_D = 2 A_{33} \left(A_{33} - A_{55} \right) \left(\delta - \varepsilon \right) = 2 \alpha_0^2 \left(\alpha^2 - \beta^2 \right) \left(\delta - \varepsilon \right). \tag{A.8}$$

As may be apparent from viewing the phase velocity equations given in (A.5) and (A.6), when $A_D \equiv 0$, or equivalently, $\delta = \varepsilon$, the transversely isotropic medium being considered degenerates to the ellipsoidal case, indicating than A_D may be considered a measure of deviation of a transversely isotropic medium from the ellipsoidal case. It seems not an unwise move then to parameterize the anisotropy within a transversely isotropic medium in terms of ε - the deviation of the ellipsoidal from the spherical and A_D - the deviation of the transversely isotropic medium from the ellipsoidal case. As ε is a dimensionless quantity and A_D has the dimensions of velocity to the fourth power, a dimensionless equivalent of this is sought. It should be further noted that this deviation term is the same for both qP and qS_V phase velocity expressions, differing only in sign and that σ may be positive, negative or zero. Although the definition of the quantity σ was obtained from an approximation to the eikonal equations its use in the exact eikonals is valid.

As mentioned in the text, the term, σ, is discussed in Banik (1987), Thomsen (1993) and Tsvankin and Thomsen (1994) and defined in those papers to be $\sigma = (\alpha/\beta)^2(\varepsilon - \delta)$, to indicate a quantity dealing only with the measure of qS_V anisotropy in a medium and $\eta = (\varepsilon - \delta)/(1+2\delta)$ the "anellipticity coefficient" that is used as a measure of deviation of a generally transversely isotropic medium from the ellipsoidal case. In this report, σ will be defined as

$$\sigma = (\delta - \varepsilon). \tag{A.9}$$

The motivation for this parameter change is that σ, at least in theory, is a measurable quantity with a physical significance.