SH wave propagation in viscoelastic media

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ABSTRACT

When comparing solutions for the propagation of SH waves in plane parallel layered elastic and viscoelastic (anelastic) media, one of the first things that becomes apparent is that in the elastic case the location of the saddle points required to obtain a high frequency approximation are located on the real p axis. This is true of the branch points also. In a viscoelastic medium this is not the case. The saddle point corresponding to an arrival lies in the first quadrant of the complex p-plane as do the branch points. Additionally, in the elastic case the saddle point and branch points lie on a straight line drawn through the origin (the positive real axis in the complex p-plane), while in the viscoelastic case this is generally not the case and the saddle point and branch points lie in such a manner as to indicate the degree of their complex values.

In this paper, simple *SH* reflected and transmitted particle displacement arrivals due to a point torque source at the surface in a viscoelastic medium composed of a layer over a halfspace will be considered. The path of steepest descent defining the saddle point in the first quadrant will be parameterized in terms of a real variable and the high frequency solutions and intermediate analytic results obtained will be used to formulate more specific constraints and observations regarding saddle point location relative to branch point locations in the complex p-plane.

As saddle point determination for an arrival is, in general, the solution of a non-linear equation in two unknowns (the real and imaginary parts of the complex saddle point p_0), which must be solved numerically, the use of analytical methods for investigating this problem type is somewhat limited.

Numerical experimentation using well documented solution methods, such as Newton's method, was undertaken and some observations were made. Although fairly basic, they did provide for the design of algorithms for the computation of synthetic traces that displayed more efficient convergence and accuracy than those previously employed. This was the primary motivation for this work and the results from the *SH* problem may be used with minimal modifications to address the more complicated subject of coupled $P-SV$ wave propagation in viscoelastic media.

Another reason for revisiting a problem that has received some attention in the literature was to approach it in a fairly comprehensive manner so that a number of specific observations may be made regarding the location of the saddle point in the complex p-plane and to incorporate these into computer software. These have been found to result in more efficient algorithms for the SH wave propagation and a significant enhancement of the comparable software in the $P-SV$ problem.

INTRODUCTION

Approximating the properties constituting the geological structure of the earth as being elastic has long been accepted as a fairly realistic compromise between the actual composition of the earth and what has been deemed able to be modeled numerically in a feasible manner based on both theoretical and numerical considerations. However, it has been generally known, and it has become evident that the inhomogeneity, anisotropy and viscoelasticity (anelasticity) of the material composition of the earth should be considered if accurate numerical simulations of seismic wave propagation are to be undertaken.

Of the three properties mentioned above, which have been the subject of longstanding research, the viscoelastic properties displayed by the earth have presented some of the most interesting, as well as some of the most contentious areas of study. The early works of note into the investigation of the theoretical development of seismic waves propagating in a linearized viscoelastic medium may be found in the SEG reprint series (Toksöz and Johnston, 1981) on this topic.

The publications and results presented in the papers of Le et al. (1994), Hearn and Krebes, (1990), Krebes (1984), Richards (1984), Krebes and Hron (1980), Borcherdt (1977), and Buchen (1971a and 1971b) will be used to determine the accuracy and consistency of the formulae derived here.

An overview of the quantitative theories of the material dispersion process in an absorbing medium may be found in the texts of Aki and Richards (1980) and Toksöz and Johnston (1981) where many references are contained. Those which will be given the most attention here are the works of Futterman (1962) and Azimi et al. (1968) as they provide the mechanism for the explicit mathematical introduction of viscoelasticity into seismic modelling related computations (Zharadnik et al., 2002). It is the intent of this report to present a qualitative treatment of *SH* wave propagation in a viscoelastic medium consisting of a layer over a halfspace, focusing on the reflected and transmitted waves due to incidence of a point torque *SH* source at the surface. This is done in a manner to accommodate most qualitative theories such as plane wave methods (Borchedt, 1977), asymptotic ray theory methods, (Krebes and Hron, 1980), and integral transform methods, Le et al. (1994), Hearn and Krebes (1990), in addition to the related line source problem for viscoelastic media, which was treated in the works of Buchen (1971a and 1971b).

Additionally, the problem will be approached in such a manner that a number of specific observations may be made regarding the location of the saddle point in the complex p-plane. An effort will be made to provide the rationale for the conclusions drawn from these observations. The fundamental reason for pursuing this problem in what may perceived to be a rather pendantic fashion was to translate these observations into more efficient seismic modelling software. These have been found to result in economical algorithms for the SH wave propagation and a significant enhancement of the comparable software in the $P-SV$ problem. Apart from modelling, preliminary numerical tests for incorporation into Born-Kirchhoff type migration algorithms have been initiated.

As the computation of synthetic traces must be carried out in the frequency domain, it is required that the ray tracing equations be calculated at frequency points at which the Fourier amplitude of the source wavelet is nonzero. This wavelet is assumed to be bandlimited of the Gabor or Ricker type.

It is not unreasonable to use as a starting point the stress-strain relation for generalized loading set down by Boltzman in 1876 for a viscoelastic medium, as it has withstood over a century of scrutiny with minimal criticism.

THEORETICAL DEVELOPMENT

The viscoelastic (anelastic) loading as proposed by Boltzman in 1876 is that the generalized stress-strain relation has the form (Aki and Richards, 1980)

$$\mu_e\,\varepsilon(t)=\sigma(t)+\int_{-\infty}^{t}\sigma(\tau)\,\dot{\phi}(t-\tau)\,d\tau=\sigma(t)+\sigma(t)*\dot{\phi}(t) \qquad (1)$$

where μ_e is the unrelaxed elastic modulus associated with shear wave propagation in an isotropic elastic medium, $\varepsilon(t)$, the time dependent strain, $\sigma(t)$, the time dependent stress and $\dot{\phi}\equiv\partial\phi/\partial t$, where $\dot{\phi}$ is the creep function which introduces absorption into an otherwise elastic medium. The operator "$*$" indicates convolution.

It is assumed that some instantaneous loading occurs or some force is applied at $t=0$ which is much greater in magnitude than anything that has occurred, and may still be exciting the medium at that point in time. The medium is assumed to be in macroscopic equilibrium within the time frame occupied by the imposed loading and the subsequent recording of the disturbance. As a result of this, equation (1) may be rewritten as

$$\sigma^{\mathrm{v}}(t)=\sigma(t)+\int_{0}^{t}\sigma(\tau)\,\dot{\phi}(t-\tau)\,d\tau \qquad (2)$$

$$\sigma^{\mathrm{v}}(t)=\mu_e\,\varepsilon(t)=\sigma(t)+\sigma(t)*\dot{\phi}(t)\ , \qquad (3)$$

where the operator "$*$" is as before and the superscript "v" denotes viscoelastic.

In cylindrical coordinates, (r,θ,z), the equation describing *SH* wave propagation in a radially symmetric viscoelastic halfspace, due to a point torque source at the free surface, $(r=0,z=0)$, may be written in terms of shear stress vector components as

$$\frac{\partial}{\partial r}\left(\sigma^{v}_{r\theta}\right)+\frac{2}{r}\sigma^{v}_{r\theta}+\frac{\partial}{\partial z}\left(\sigma^{v}_{z\theta}\right)-\varrho\frac{\partial^2 u}{\partial t^2}=0 \qquad (4)$$

(Mikhailenko (1985)), subject to the torque source boundary condition

$$\sigma^{v}_{z\theta}\Big|_{z=0} = \frac{1}{4\pi}\frac{d}{dr}\left(\frac{\delta(r)}{r}\right)L(t) \ , \tag{5}$$

where δ is the Dirac delta function, ϱ - the density of the medium, $L(t)$ - the time dependence of the source pulse and $u(r,z,t)$ - a scalar displacement. Thus the particle displacement vector may defined in terms of the scalar particle displacement, $u(r,z,t)$, and a unit vector, $\mathbf{e}_\theta$, which is perpendicular to the plane of incidence, such that $\mathbf{u}(r,z,t) = u(r,z,t)\mathbf{e}_\theta$.

The initial conditions required to fully specify the problem are

$$u\big|_{t=0} = \frac{\partial u}{\partial t}\bigg|_{t=0} = 0 . \tag{6}$$

Upon applying a Fourier time transform and a Hankel transform of order one with respect to the spatial variable *r*, defined by

$$u(p,z,\omega) = \omega^2 \int_{-\infty}^{\infty} e^{i\omega t} dt \int_{0}^{\infty} u(r,z,t) J_0(\omega pr) r dr , \tag{7}$$

with p being the horizontal or radial slowness, equation (4) becomes

$$\mu(\omega)\left(\frac{d^2u}{dz^2} - \omega^2 p^2 u\right) + \varrho\omega^2 u = 0 . \tag{8}$$

The quantity $\mu(\omega) = \mu_e\left(1+\dot{\phi}(\omega)\right)$, where $\dot{\phi}(\omega)$ is the Fourier time transform of the creep function and it is to be recalled that μ_e is the unrelaxed elastic modulus. The transformed particle displacement is given by $u = u(p,z,\omega)$. There is no indication of the time and spatial transforms having been introduced in the notation for particle displacement. If there are circumstances where this may cause uncertainty an explanatory note will be added in the text.

Applying the time and radial spatial transforms to the shear stress boundary condition in equation (5) the following is obtained:

$$\sigma^{v}_{z\theta}\Big|_{z=0} = -\frac{\omega p L(\omega)}{4\pi} , \tag{9}$$

$L(\omega)$ being the Fourier time transformed source wavelet.

The frequency domain solution of (8) and (9) subject to the initial conditions (6) in a viscoelastic halfspace, $(r \geq 0, z \geq 0)$ is

$$u(r,z,\omega)=\frac{\omega\beta(\omega)L(\omega)}{i}\int_0^\infty J_1(\omega pr)\exp\left(i\omega\eta|z|\right)\frac{p^2dp}{\eta}, \tag{10}$$

with $J_1(\xi)$ the order one Bessel function, $\eta=\left(\beta^{-2}(\omega)-p^2\right)^{1/2}$, $\operatorname{Im}(\eta)\geq 0$ and $\beta^2(\omega)=\mu(\omega)/\varrho$.

Before proceeding further, some observations on, and clarifications of, notation involving the complex valued quantities involved in this problem should be given. As the solution must satisfy radiation conditions for large *r* or *z*, the complex frequency dependent integration variable, *p* (horizontal slowness) must lie in the first quadrant, inclusive of the real axis, such that for all *p* considered, $p=p_{\mathrm{Re}}+ip_{\mathrm{Im}}$. The real (Re) and imaginary (Im) components, p_{Re} and p_{Im}, are required to be real and positive or zero. The complex velocity, which defines the viscoelastic modulus $\mu(\omega)$, is related to the complex velocity as $1/\beta(\omega)=\left[\left[1/\beta(\omega)\right]_{\mathrm{Re}}+i\left[1/\beta(\omega)\right]_{\mathrm{Im}}\right]$ so that with a frequency independent density, $\mu(\omega)=\varrho\beta^2(\omega)$.

At times it will be convenient to write a complex quantity in polar notation, $\kappa=|\kappa|e^{i\psi}$ where ψ is a positive angle measured counter-clockwise from the positive real p axis. A specific point in the complex $p-$plane will be denoted simply as "p_i" and a general point in this plane as "p". In addition, all square roots of complex quantities will be assumed to be the principal value square root.

If a plane boundary is introduced at a depth *h* below the free surface, the time transformed reflection $R_{SH}(p,\omega)$, and transmission, $T_{SH}(p,\omega)$, coefficients may be obtained by requiring the continuity of particle displacement, *u*, and shear stress, $\sigma_{r\theta}$, at the solid/solid boundary (Aki and Richards, 1980). As a result, the reflected wave and transmitted wave vector components of particle displacement, which are perpendicular to the plane of incidence, may be written in the following form

$$u_r(r,z,\omega)=\frac{\omega L(\omega)}{2ip_1}\int_{-\infty}^{\infty}R_{SH}(p,\omega)H_1^{(1)}(\omega pr)\exp\left(i\omega\eta_1(h+h_1)\right)\frac{p^2dp}{\eta_1} \tag{11}$$

and

$$u_t(r,z,\omega)=\frac{\omega L(\omega)}{2ip_1}\int_{-\infty}^{\infty}T_{SH}(p,\omega)H_1^{(1)}(\omega pr)\exp\left(i\omega(\eta_1h+h_2\eta_2)\right)\frac{p^2dp}{\eta_1}, \tag{12}$$

respectively. In the above, $p_1=\beta_1^{-1}(\omega)$, $\eta_i=\left(\beta_i^{-2}-p^2\right)^{1/2},(i=1,2)$, h_1 and h_2 are positive real values and are the distances from the layer-halfspace boundary and lie in the layer in the case of the reflected wave at h_1 and in the halfspace at h_2 in the case of the transmitted wave (Figure (1)). The integration path has been modified utilizing a relation between $J_1(\xi)$ and $H_1^{(1)}(\xi)$ (Abramowitz and Stegun (1980)). Other terms which need

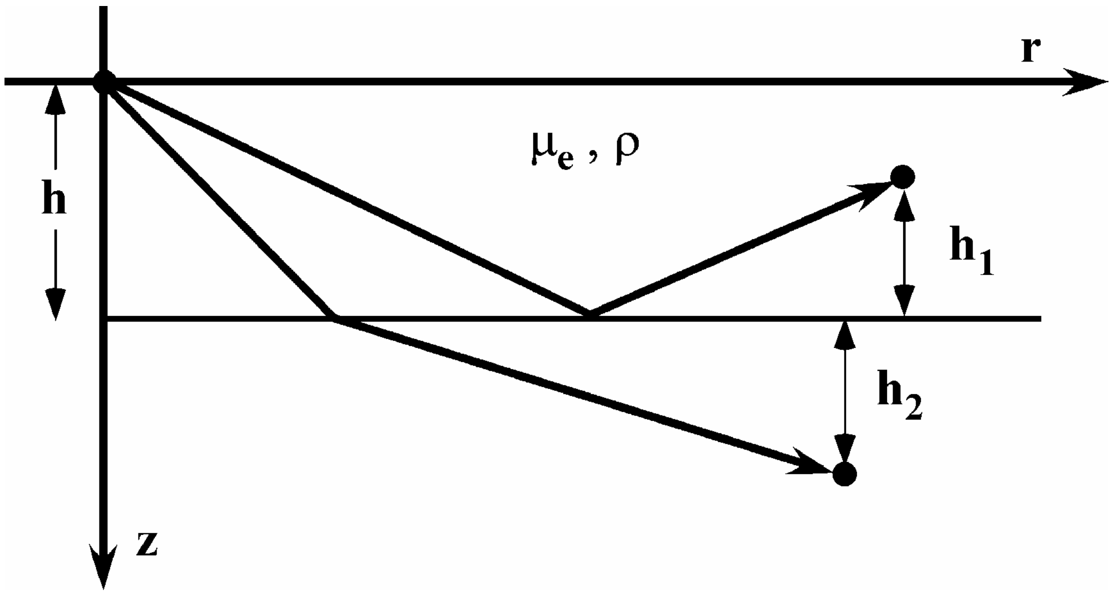

FIG 1. Source-receiver geometry for the *SH* wave propagation problem considered. The paths for the reflected and transmitted rays are shown. Both the layer of thickness *h* and the halfspace are assumed to be anelastic. The elastic parameters in the layer and the halfspace are defined by the complex velocities, $(\beta_j, j=1,2)$ and real densities, $(\rho_j, j=1,2)$, respectively.

definition in (11) and (12) are the reflection and transmission coefficients, which have the form (Borcherdt, 1977)

$$R_{SH}(p,\omega)=\left[\mu_1(\omega)\eta_1(p,\omega)-\mu_2(\omega)\eta_2(p,\omega)\right]/D(p,\omega) \tag{13}$$

$$T_{SH}(p,\omega)=2\mu_1(\omega)\eta_1(p,\omega)/D(p,\omega) \tag{14}$$

where

$$D(p,\omega)=\mu_1(\omega)\eta_1(p,\omega)+\mu_2(\omega)\eta_2(p,\omega) \tag{15}$$

and as before

$$\eta_i(p,\omega)=\left(\beta_i^{-2}-p^2\right)^{1/2},\ i=1,2 \text{ with } \mathrm{Im}(\eta_i)\geq 0, \tag{16}$$

so that the radiation conditions are satisfied. The subscripts "1" and "2" refer to the layer and the halfspace respectively. From this point on, the suffixes "(ω)" and "(p,ω)" will be dropped and not used unless ambiguity might arise from their omission.

Upon examination of (11) and (12), one would expect to record a reflected arrival within the layer and a possible critically refracted (head) wave at an arbitrary point(r,h_1). In the halfspace, a refracted arrival as well as the possibility of a critically refracted wave due to a branch cut integral could be recorded at the point (r, h_2) (Cerveny and Ravindra, 1970).

Applying standard saddle point approximation procedures (Marcuvitz and Felsen 1973), it is found that the saddle point, p_0, is generally complex and lies in the first quadrant of the complex p-plane (inclusive of the real axis and exclusive of the imaginary axis). For either a reflected or transmitted ray a steepest descent contour may be defined in terms of some real parameter y as,

$$\left(\beta_1^{-2} - p^2\right)^{1/2} = \left(\beta_1^{-2} - p_0^2\right)^{1/2} - y\exp\left(-i\pi/4 + i\phi_0\right), \quad -\infty < y < \infty, \quad 0 \le \phi_0 < \pi/4 \quad (17)$$

or equivalently,

$$\eta_1 = \tilde{\eta}_1 - y\exp\left(-i\pi/4 + i\phi_0\right). \quad (18)$$

(Figure 2). This steepest descent contour is a generalization of the method described in Cerveny and Ravindra (1970) and is dealt with in more detail in the Appendix.

The tilde above η_1 indicates that it is evaluated at the saddle point. From the above equations, the following relation,

$$\frac{p\,dp}{\eta_1} = dy\ \exp\left(-i\pi/4 + i\phi_0\right), \quad (19)$$

is obtained. The quantity ϕ_0 is the positive angle between the real positive p axis and the line connecting the origin with the saddle point p_0, which is yet to be determined.

Introducing the asymptotic form of the Hankel function (Abramowitz and Stegun, 1980) and retaining only the leading term, together with the change of variable given by (19) leads to the following expression for the reflected arrival when the zero order saddle point approximation is used

$$u_r\left(r, z, \omega\right) = -\frac{L(\omega)}{i}\left(\frac{p_0^3\tilde{\eta}_1}{r\left(h + h_1\right)p_1^4}\right)^{1/2} R_{SH}\left(p_0, \omega\right)\exp\left(i\omega\tau_r - \omega\kappa_r\right). \quad (20)$$

The traveltime of the reflected disturbance is given by $\tau_r = \mathrm{Re}\left[f_r\left(p_0\right)\right]$ and the damping factor introduced by the medium being viscoelastic is $\kappa_r = \mathrm{Im}\left[f_r\left(p_0\right)\right]$. In the above, $f_r\left(p_0\right) = rp_0 + \left(h + h_1\right)\tilde{\eta}_1$ and p_0 is the solution of

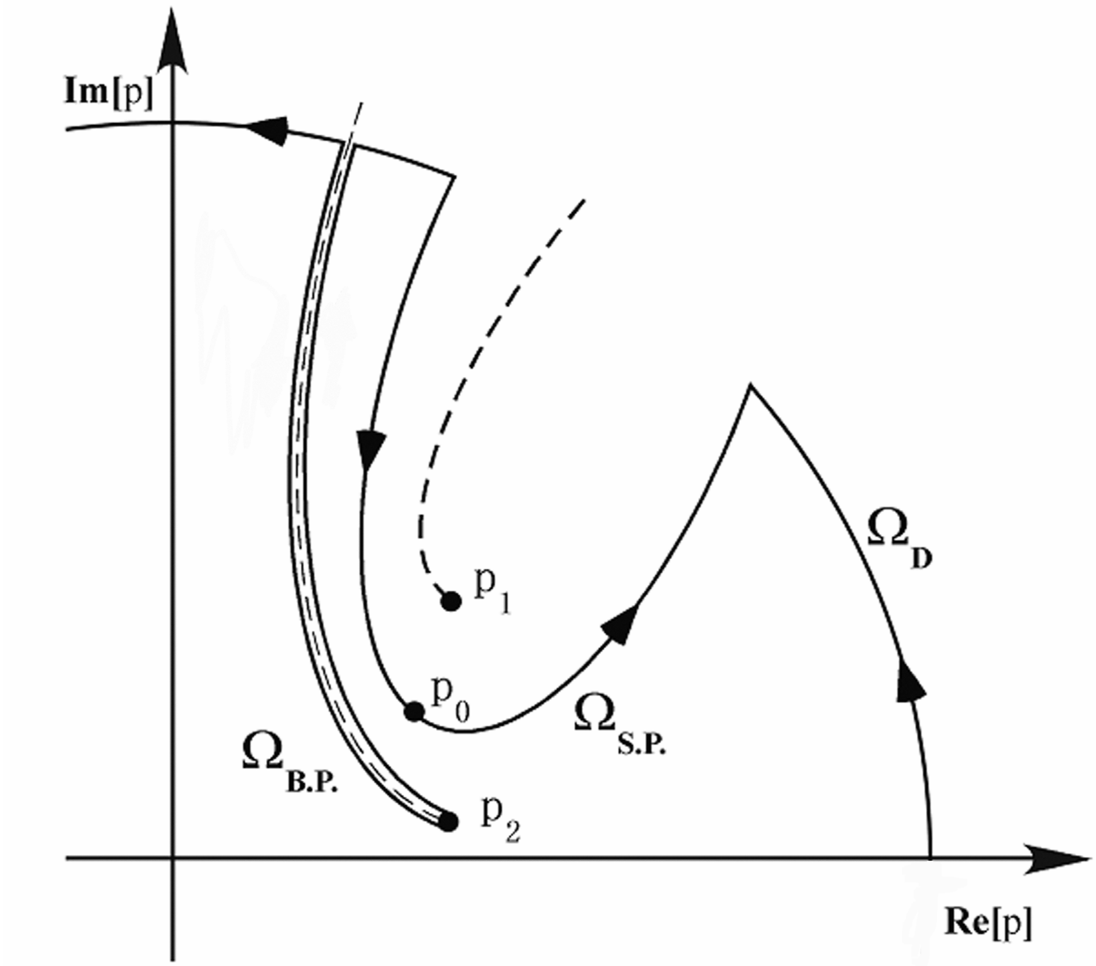

FIG 2. Schematic of the steepest descent (saddle point) contour, $\Omega_{S.P.}$, for the reflected *SH* arrival in the surface layer. Hypothetical values for the complex values of the slowness in the layer, p_1, the halfspace, p_2, and the saddle point, p_0, were chosen to illustrate the procedure. The branch cut integral along $\Omega_{B.P.}$ is not considered here but was included in the figure for completeness.

$$\left.\frac{df_r}{dp}\right|_{p=p_0} = r - \frac{p_0\left(h+h_1\right)}{\tilde{\eta}_1} = 0 \tag{21}$$

The complex quantities p_0, p_1 and $\tilde{\eta}_1$, the value of η_1 at the saddle point, used in equation (20) have the following form in polar notation: $p_1 = \left|p_1\right| e^{i\phi_1} = \rho_1 e^{i\phi_1}$, $p_0 = \left|p_0\right| e^{i\phi_0} = \rho_0 e^{i\phi_0}$ and $\tilde{\eta}_1 = \left|\tilde{\eta}_1\right| e^{i\psi_0} = \tilde{n}_1\, e^{i\psi_0}$, respectively. As will be shown in a later section, (18) may be simplified considerably.

At this point it should be noted that the source wavelet, $L(t)$ with its associated Fourier time source transform, $L(\omega)$ is assumed to be bandlimited. As the velocities have also been assumed to be frequency dependent the saddle point is required to be computed at each frequency point at which the Fourier spectrum of the source wavelet is non-zero.

The expression for the transmitted arrival obtained using the zero order saddle point approximation is

$$u_t(r,z,\omega) = -\frac{L(\omega)}{i}\left(\frac{p_0^3}{r\,p_1^2\tilde{\eta}_1^2\tilde{a}}\right)^{1/2} T_{SH}(p_0,\omega)\exp(i\omega\tau_t - \omega\kappa_t)\,, \tag{22}$$

where $f_t(p_0) = rp_0 + h\,\tilde{\eta}_1 + h_2\,\tilde{\eta}_2$ and p_0 is the solution of

$$\left.\frac{df_t}{dp}\right|_{p=p_0} = r - p_0\left(\frac{h}{\tilde{\eta}_1} + \frac{h_2}{\tilde{\eta}_2}\right) = 0 \tag{23}$$

and

$$\tilde{a} = \frac{hp_1^2}{\tilde{\eta}_1^3} + \frac{h_2 p_2^2}{\tilde{\eta}_2^3}\,. \tag{24}$$

As in the previous case, $\tau_t = \mathrm{Re}\left[f_t(p_0)\right]$ is the traveltime of the transmitted ray while $\kappa_t = \mathrm{Im}\left[f_t(p_0)\right]$ is the damping factor. The function $f_\ell(p_0)$ and $f_\ell'(p_0)$ $(\ell = r,t)$ will be discussed in more detail in a later section.

DISCUSSION OF SOLUTION

Snell's Law

The first matter which should be clarified is the notation used in this paper compared with earlier mentioned works in this subject area (Borcherdt (1977), Krebes and Hron (1980b) and to present the basis for the viscoelastic equivalent of Snell's Law at the interface between two such media for *SH* waves. To accomplish this, a brief overview of the presentation of the theory found in the earlier papers will be given, with only *type II S-waves* being considered (Borcherdt (1977). Writing the displacement in terms of potentials (Aki and Richards, 1980), the Fourier time transformed field equation for *SH* wave propagation in a viscoelastic medium results in the Helmholtz equation,

$$\nabla^2\psi + k^2\psi = 0\,, \tag{25}$$

where $k^2 = \omega^2/\beta^2(\omega)$, $\beta^2(\omega) = \mu(\omega)/\varrho$. The plane wave solution of (25) is given by

$$\psi(r,z,\omega) = C\exp\left[i\left(\mathbf{P}+i\mathbf{A}\right)\cdot\mathbf{r}\right] = C\exp\left(i\mathbf{k}\cdot\mathbf{r}\right). \tag{26}$$

In (26), C is an arbitrary constant amplitude term and $\mathbf{k} = \mathbf{P} + i\,\mathbf{A}$ is the complex wavenumber vector. The vectors **P** and **A** are the propagation and attenuation vectors, respectively. **P** is perpendicular to the planes of constant phase such that $\mathbf{P}\cdot\mathbf{r} = \text{constant}$ and **A** is perpendicular to planes of constant amplitude, requiring $\mathbf{A}\cdot\mathbf{r} = \text{constant}$.

The phase velocity is defined as

$$\mathbf{v} = \frac{\omega\mathbf{P}}{\mathbf{P}\cdot\mathbf{P}}. \tag{27}$$

Further,

$$k^2 = \mathbf{k}\cdot\mathbf{k} = \mathbf{P}\cdot\mathbf{P} - \mathbf{A}\cdot\mathbf{A} + 2i\,\mathbf{P}\cdot\mathbf{A} \tag{28}$$

or

$$k^2 = \mathbf{k}\cdot\mathbf{k} = \left|\mathbf{P}\right|^2 - \left|\mathbf{A}\right|^2 + 2i\left|\mathbf{P}\right|\left|\mathbf{A}\right|\cos\gamma\,, \tag{29}$$

where γ is called the *attenuation angle* and is the angle between the propagation vector $\mathbf{P}$ and the attenuation vector, $\mathbf{A}$. If the angle γ is equal to zero, i.e., if $\mathbf{P}$ is parallel to $\mathbf{A}$ the wave is called *homogeneous*. Otherwise, the wave is referred to as *inhomogeneous.* The restriction on γ for the purpose of satisfying the radiation condition is

$$0 \le \left|\gamma\right| < \pi/2\,. \tag{30}$$

Equating the real and imaginary parts of equation (29) results in a set of two equations in the unknowns $\left|\mathbf{P}\right|$ and $\left|\mathbf{A}\right|$ which may be solved to obtain

$$\left|\mathbf{P}\right| = \left(\frac{1}{2}\left(\operatorname{Re}\left[k^2\right] + \left\{\left(\operatorname{Re}\left[k^2\right]\right)^2 + \frac{\left(\operatorname{Im}\left[k^2\right]\right)^2}{\cos^2\gamma}\right\}^{1/2}\right)\right)^{1/2} \tag{31}$$

$$\left|\mathbf{A}\right| = \left(\frac{1}{2}\left(-\operatorname{Re}\left[k^2\right] + \left\{\left(\operatorname{Re}\left[k^2\right]\right)^2 + \frac{\left(\operatorname{Im}\left[k^2\right]\right)^2}{\cos^2\gamma}\right\}^{1/2}\right)\right)^{1/2}. \tag{32}$$

The viscoelastic equivalent of Snell's Law at the interface between two viscoelastic media for an incident *SH* wave at the interface which produces reflected, and transmitted waves is that the horizontal wave numbers, or equivalently, the horizontal slownesses, of the 3 waves involved are equal. Denoting the incident, reflected and transmitted with the subscripts, "i", "r", and "t" respectively, then the generally complex horizontal wavenumbers are given as

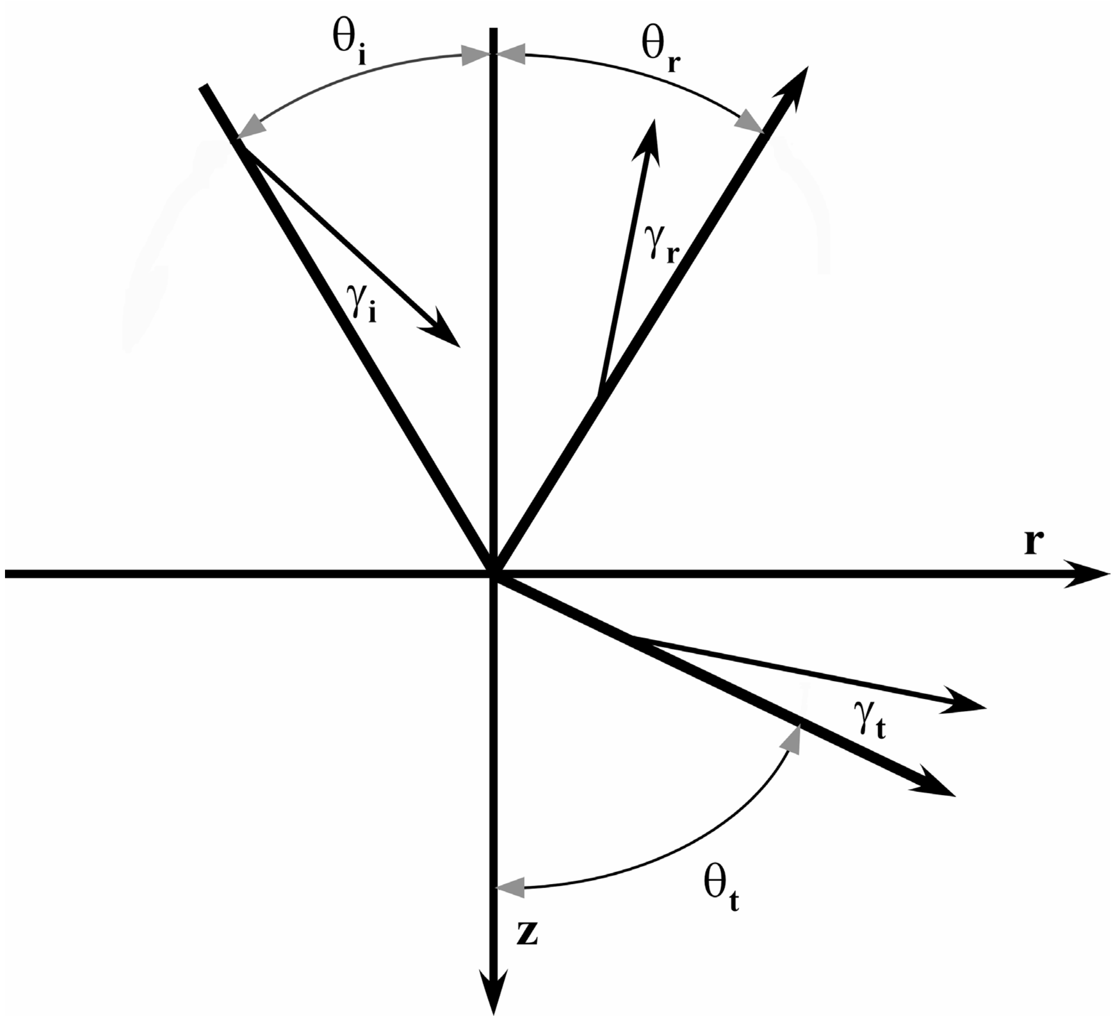

FIG. 4. Geometry of incidence of a plane *SH* wave at a solid – solid interface between two viscoelastic media. The propagation vectors $\mathbf{P}_j$ $\left(j=i,r,t\right)$ are perpendicular to planes of constant phase while $\mathbf{A}_j$ $\left(j=i,r,t\right)$ are perpendicular to planes of constant amplitude. The angular difference between the two, γ_j, is referred to as the attenuation angle.

$$k^{(x)}=k_i^{(x)}=k_r^{(x)}=k_t^{(x)} \tag{33}$$

which after referring to Figure 4 take the forms

$$\begin{aligned} k^{(x)} &= \left|\mathbf{P}_j\right|\sin\theta_j + i\left|\mathbf{A}_j\right|\sin\left(\theta_j-\gamma_j\right) \quad \left(j=i,r,t\right). \\ &= k_{\mathrm{Re}}^{(x)} + i\,k_{\mathrm{Im}}^{(x)} \end{aligned} \tag{34}$$

With "Re" and "Im" indicating real and imaginary parts, together with the fact that they must independently be equal, leads to

$$k_{\mathrm{Re}}^{(x)} = \left|\mathbf{P}_i\right|\sin\theta_i = \left|\mathbf{P}_r\right|\sin\theta_r = \left|\mathbf{P}_t\right|\sin\theta_t \tag{35}$$

and

$$k_{\mathrm{Im}}^{(x)} = \left|\mathbf{A}_i\right|\sin\left(\theta_i - \gamma_i\right) = \left|\mathbf{A}_r\right|\sin\left(\theta_r - \gamma_r\right) = \left|\mathbf{A}_t\right|\sin\left(\theta_t - \gamma_t\right). \tag{36}$$

It should be noted that the horizontal wave number, $k^{(x)}$, is related to the generally complex horizontal slowness, p, through the identity $k^{(x)} = \omega p$, so that once p_0 is computed the vectors $\mathbf{P}_j$ and $\mathbf{A}_j$, together with the θ_j and γ_j, are uniquely determined for each layer through which a specific ray propagates.

Reflected Wave Saddle Point Solution

Returning to the saddle point approximations for the reflected and transmitted rays obtained for viscoelastic media, the first matter to be considered is the consequence of determining the saddle points for these two arrival types. In the case of the reflected arrival equation (21) defines the saddle point location and it may be recast as

$$\frac{r}{\left(z + h_1\right)} = \frac{p_0}{\tilde{\eta}_1}. \tag{37}$$

As the left hand side of equation (37) is real, the right hand side must also be real. This requires that the phases of the generally complex quantities p_0 and $\tilde{\eta}_1$ be equal, so that the saddle point, p_0, must lie on the straight line joining the origin and the point $p_1 = \beta_1^{-1}$. This condition not only determines the angle at which the ray, along which the energy propagates, leaves the source point, but also the initial angle of the attenuation vector which, in this case, is the same as the propagation vector; i.e., $\gamma = 0$. As previously mentioned this is referred to as a homogeneous wave (Borcherdt, 1977). The result of this argument is that the expression for the reflected wave, equation (20), simplifies to

$$u_r\left(r, z, \omega\right) = \frac{L(\omega)}{i}\left(\frac{\rho_0^3 \tilde{n}_1}{r\left(z + h\right)}\right)^{1/2} \frac{R_{SH}\left(p_0, \omega\right)}{\rho_1^2} \exp\left(i\omega\tau_r - \omega\kappa_r\right), \tag{38}$$

where $\rho_0 = \left|p_0\right|$, $\rho_1 = \left|p_1\right|$, $\tilde{n}_1 = \left|\tilde{\eta}_1\right|$ and as before, $\tau_r = \mathrm{Re}\left[f_r\left(p_0\right)\right]$ and $\kappa_r = \mathrm{Im}\left[f_r\left(p_0\right)\right]$.

Transmitted Wave Saddle Point Solution

The treatment of the transmitted wave is not as simple as the reflected wave in a single layer, unless the phase of the complex quantities $p_2 = \beta_2^{-1}$ and $p_1 = \beta_1^{-1}$ are equal, which is a very restricted case. In this instance, the wave is homogeneous, with p_0, p_1 and p_2 all situated on a straight line passing through the origin. As this is a homogeneous wave, the propagation vector and attenuation vector are parallel $(\gamma_i = 0,\ i = 1,2.)$ along the total ray length.

In general, however, such is not the case and the more complicated relation, obtained from equation (23),

$$f(p) = r - p_0\left(\frac{h}{\tilde{\eta}_1} + \frac{h_2}{\tilde{\eta}_2}\right) = 0\,, \tag{39}$$

must be solved. All quantities except *r*, *z*, and h_2 are complex so that equation (39) is written as the coupled set of equations

$$r - \mathrm{Re}\left\{p_0\left(\frac{h}{\tilde{\eta}_1} + \frac{h_2}{\tilde{\eta}_2}\right)\right\} = 0 \tag{40}$$

$$\mathrm{Im}\left\{p_0\left(\frac{h}{\tilde{\eta}_1} + \frac{h_2}{\tilde{\eta}_2}\right)\right\} = 0 \tag{41}$$

This is a set of two equations for the real and imaginary parts of the complex horizontal slowness at the saddle point, p_0. The determination of these two values makes possible the specification of the complex horizontal slownesses for both the layer and the halfspace allowing for the unique determination of $(\theta_i, i = 1,2)$ and $(\gamma_i, i = 1,2)$ using Snell's Law for a viscoelastic media from equations (33) – (36). This is referred to as an inhomogeneous wave (Borcherdt, 1977). The progression of the complex saddle point, p_0, in the complex *p*-plane for increasing offsets of the transmitted ray has three possible cases characterized by the phases of the complex quantities $p_1 = \rho_1 e^{i\phi_1}$ and $p_2 = \rho_2 e^{i\phi_2}$: $\phi_2 < \phi_1$, $\phi_2 = \phi_1$ and $\phi_2 > \phi_1$. This aspect of the problem will be dealt with in more detail in a subsequent section.

Numerical Implementation of Viscoelasticity – Introducing Q

After a significant amount of numerical testing and consultation with the literature related to this matter and with several academic and industry researchers, the theory presented by Futterman (1962) and Azimi et al. (1968) were deemed to be the most useful and accurate when used together with the high frequency geometrical optics solution method of computing synthetic traces (Zharadnik. et al., 2002). An assumption used in this discussion of seismic wave propagation in a viscoelastic medium is that the *quality factor* defining a medium's degree of absorption is such that $Q(\omega) > 1$ where $Q(\omega)$ is to be defined in terms of the complex frequency dependent velocity.

The attenuating mechanism discussed in Futterman (1962) and Azimi et al. (1968) both assume that at some reference circular frequency, $\omega_R = 2\pi f_R$ a reference absorption (quality) factor, $Q_R = Q(\omega_R)$, and a real reference *SH*-wave velocity, $V_{SH}(\omega_R)$, are known. These quantities are real valued. At some other frequency, ω, the values of $Q(\omega)$ and $V_{SH}(\omega)$ are given by the relations

$$Q(\omega) = Q(\omega_R)\left[1.0 - \frac{1.0}{\pi Q(\omega_R)}\ln\left(\frac{\omega}{\omega_R}\right)\right] \tag{42}$$

and

$$V_{SH}(\omega) = V_{SH}(\omega_R)\frac{Q(\omega_R)}{Q(\omega)}. \tag{43}$$

The two values $Q(\omega)$ and $V_{SH}(\omega)$ obtained above are also real. Viscoelasticity or absorption is introduced into a medium through a complex velocity obtained by an analysis of the attenuating mechanism. The high frequency expression for this complex velocity, $\beta(\omega)$, in terms of the real parameters $Q(\omega)$ and $V(\omega)$ used in this paper is defined by

$$\frac{1}{\beta(\omega)} = \frac{1}{V(\omega)}\left[1 + \frac{i}{2Q(\omega)}\right]. \tag{44}$$

Equations (42)-(44) are the same as those presented in Zharadnik. et al. (2002).

General Case – Many Layers and Ray Segments

A generalization of what has been presented in the previous sections will now be considered and some proposals submitted. A viscoelastic medium composed of N plane parallel layers overlying a halfspace will be assumed. An arbitrary ray with a known number of ray segments in each layer propagates from a point source located on the surface through the viscoelastic layers and is reflected from the J^{th} interface, $(1 \le J \le N)$, and returns to some point in the K^{th} layer $(1 \le K \le J)$ within the medium where its disturbance is recorded. It is assumed that the ray segments in the J layers form a geometrical ray path. The complex phase or real traveltime and exponential damping at some specified frequency is given by the formula

$$f(p) = rp + \sum_{j=1}^{J} m_j h_j \eta_j. \tag{45}$$

The saddle point is defined by solving for the complex quantity, p_0, at the specified frequency in the equation

$$\left.\frac{df}{dp}\right|_{p=p_0} = r - p_0 \sum_{j=1}^{J} \frac{m_j h_j}{\tilde{\eta}_j} = 0 \tag{46}$$

where the source-receiver offset is r, m_j is the number of ray segments in the j^{th} layer and $\eta_j = \left(\beta_j^{-2} - p^2\right)^{1/2}$, $\tilde{\eta}_j$ being the value of η_j at p_0. The value of p_0 is obtained by writing equation (46) in a manner similar to that used in equations (40) and (41) and solving the two resulting non-linear equations in the two unknowns – the real and imaginary parts of p_0. One of the more numerically reliable methods of accomplishing this is by using the Newton-Raphson Method (Press et al., 1997) or some variation thereon. An initial guess at the complex quantity p_0 may be obtained by using Brent's Method or the Regula Falsi Method (Press et al., 1997). The intermediate real values of the layer velocities at a given frequency (equation (33)) are used to obtain the real part of the p_0 estimate and the related $Q_j(\omega)$ to estimate the imaginary part. As p_0 is to be computed at all nonzero frequency points of the amplitude spectrum of the Fourier transform of the source wavelet for a specific ray, the second and subsequent frequency points use the result of the previous frequency point as an initial guess and the Brent's Method step is bypassed. After considerable testing and comparison with other algorithms this sequence of solution steps appears to be an efficient and stable manner of solving this problem.

The complex quantities $p_j = \beta_j^{-1} = \rho_j e^{i\phi_j}$ and $\tilde{\eta}_j = \left(\beta_j^{-2} - p_0^2\right)^{1/2} = \tilde{n}_j e^{i\psi_j}$ are known in each of the J layers. Let $\rho_{\min}$ be the minimum of ρ_j, $(1 \le j \le J)$ and that there also exists some p_{Mx} and p_{Mn}, $(1 \le Mx \le J)$ and $(1 \le Mn \le J)$, such that ϕ_{Mn} is the minimum phase and ϕ_{Mx} is the maximum phase when compared to the phases of all of the complex quantities p_j, $(1 \le j \le J)$.

Consider some radical η_j, associated with the j^{th} layer and defined at or near the generally complex saddle point p_0 as

$$\tilde{\eta}_j = \left(\beta_j^{-2} - p_0^2\right)^{1/2} = \left(p_j^2 - p_0^2\right)^{1/2}. \tag{47}$$

For the sign notation used in this paper it is required that $\mathrm{Re}\left[\tilde{\eta}_j\right] > 0$ for causality and $\mathrm{Im}\left[\tilde{\eta}_j\right] \ge 0$ to satisfy the radiation condition. Let $p_j = \rho_j e^{i\phi_j}$ and $p_0 = \rho_0 e^{i\phi_0}$ where as before $\rho_j = \left|p_j\right|$ and $\rho_0 = \left|p_0\right|$. The radical $\tilde{\eta}_j$ may then be rewritten as

$$\tilde{\eta}_j = e^{i\phi_j}\left(\rho_j^2 - \rho_0^2 e^{-2i\left(\phi_j - \phi_0\right)}\right)^{1/2}. \tag{48}$$

It follows that

$$\begin{aligned}\tilde{\eta}_j &= e^{i\phi_j}\left[\rho_j^2 - \rho_0^2\cos 2\left(\phi_j - \phi_0\right) + i\rho_0^2 \sin 2\left(\phi_j - \phi_0\right)\right]^{1/2} \\ &= \tilde{n}_j e^{i\left(\phi_j + \zeta_j\right)}\end{aligned}. \tag{49}$$

The definitions of $\tilde{n}_j$ and ζ_j are

$$\tilde{n}_j = \left[\left(\rho_j^2 - \rho_0^2\cos 2\left(\phi_0 - \phi_j\right)\right)^2 + \left(\rho_0^2 \sin 2\left(\phi_0 - \phi_j\right)\right)^2\right]^{1/4} \tag{50}$$

and

$$\zeta_j = \frac{1}{2}\tan^{-1}\left[\frac{\rho_0^2 \sin 2\left(\phi_j - \phi_0\right)}{\rho_j^2 - \rho_0^2 \cos 2\left(\phi_j - \phi_0\right)}\right]. \tag{51}$$

To simplify notation, let

$$\chi_j = 2\left(\phi_j - \phi_0\right) \tag{52}$$

so that equations (50) and (51) may be written as

$$\tilde{n}_j = \left[\left(\rho_j^2 - \rho_0^2\cos\chi_j\right)^2 + \left(\rho_0^2 \sin\chi_j\right)^2\right]^{1/4} \tag{53}$$

and

$$\zeta_j = \frac{1}{2}\tan^{-1}\left[\frac{\rho_0^2 \sin\chi_j}{\rho_j^2 - \rho_0^2 \cos\chi_j}\right]. \tag{54}$$

As a consequence of requiring that the quality factors $Q_j\left(\omega\right)$ be constrained such that $Q_j\left(\omega\right) > 1$, the range of values that ϕ_j may have is $0 \le \phi_j < \pi/4$. This value for the minimum value which $Q_j\left(\omega\right)$ may attain is highly unrealistic. A minimum value of $Q_j\left(\omega\right) > 10$, although still low, would change the upper bound on ϕ_j so that $0 \le \phi_j < \pi/30$. However, the value $Q_j\left(\omega\right) > 1$ $\left(\phi_j < \pi/4\right)$ will be retained in what follows.

If $0 \le \phi_j < \pi/4$ it is ensured that the radiation condition is satisfied as the phase of $\tilde{\eta}_j$ is then given by $0 \le \phi_j + \zeta_j < \pi/2$. It is required only to show that this inequality in ζ_j is valid only at the upper bound of the restriction on ϕ_j, i.e., $\phi_j < \pi/4$. This requires that it be shown that $-\pi/4 < \zeta_j \le \pi/4$.

Due to the imposed constraint $Q_j(\omega) > 1$, all $\phi_j < \pi/4\,(1 \le j \le J)$, it will be assumed for the present and shown later that there exists a ϕ_{Mn} and ϕ_{Mx} such that $\phi_{Mn} \le \phi_0 \le \phi_{Mx}$ $(Mn, Mx \in [1, 2, \ldots J])$. Thus $\phi_j - \phi_0$ must be less than $\pi/4$ and as a result,

$$-\frac{\pi}{2} < \chi_j = 2(\phi_j - \phi_0) < \frac{\pi}{2}. \tag{55}$$

If the above relation is satisfied then

$$-\frac{\pi}{4} < \zeta_j < \frac{\pi}{4} \tag{56}$$

and the radiation condition is satisfied.

Further, what may be inferred from the above is that using equation (55) in equation (54) that the following inequality must hold for all j:

$$\rho_j^2 - \rho_0^2 \cos 2(\phi_j - \phi_0) > 0. \tag{57}$$

Equation (57) is the viscoelastic equivalent of the elastic case requirement that $\rho_0 < \rho_j$, or more specifically $\rho_0 < \rho_{\min}$. In the elastic case, as $\rho_0 \to \rho_{\min}$ the offset, r tends to infinity $(r \to \infty)$. Another way of stating this for the anelastic case is that as $r \to \infty$ then for some $k, (1 \le k \le J)$, $\rho_0 \to \rho_k \big/ \sqrt{\cos 2(\phi_k - \phi_0)}$ which apart from indicating that $\rho_0 \to \rho_k$ as $r \to \infty$ also requires that $\phi_0 \to \phi_k$. However, these two conditions must be satisfied in such a manner that all quantities associated with the j^{th} layer $(1 \le j \le J)$ satisfy both causality and radiation conditions. A more correct definition of the reference k^{th} (*maximum velocity*) layer is that layer $(1 \le k \le J)$ that has the minimum value of $\mathrm{Re}[p_j]$. Equation (57) has the effect of using the coordinate system defined by the saddle point, $p_0 = \rho_0 e^{i\phi_0}$ as the reference system rather than the real and imaginary axis of the complex *p*-plane. As long as realistic values of $Q(\omega)$ are used, the above analysis is valid in almost all possible situations of this problem. A more general and complicated pursuit of this is considered unwarranted as it would apply only in areas where previous assumptions in this paper make its application invalid.

An example of this is shown in Figure (5) where the magnitude of the complex slownesses in both layers of a two layer model are the same, $\rho_1^{-1} = \rho_2^{-1} = 3.0 km/s$. The thickness of each of the layers is $1 km$ and the saddle points are computed for the range of offsets $0 \le r < \infty$, in km for the reflected *SH* primary. Different $Q_j(\omega)$ values are used in the two layers with $Q_1(\omega) = 25.0$ and $Q_2(\omega) = 10.0$ at a reference frequency $\omega_r = 30 Hz$. The saddle points were computed for a frequency of $\omega = 50 Hz$. Two other examples of the saddle point progression associated with the same offset range $0 \le r < \infty$ km are presented in Figures (6) and (7) where in the first layer 1 is the *fast* layer

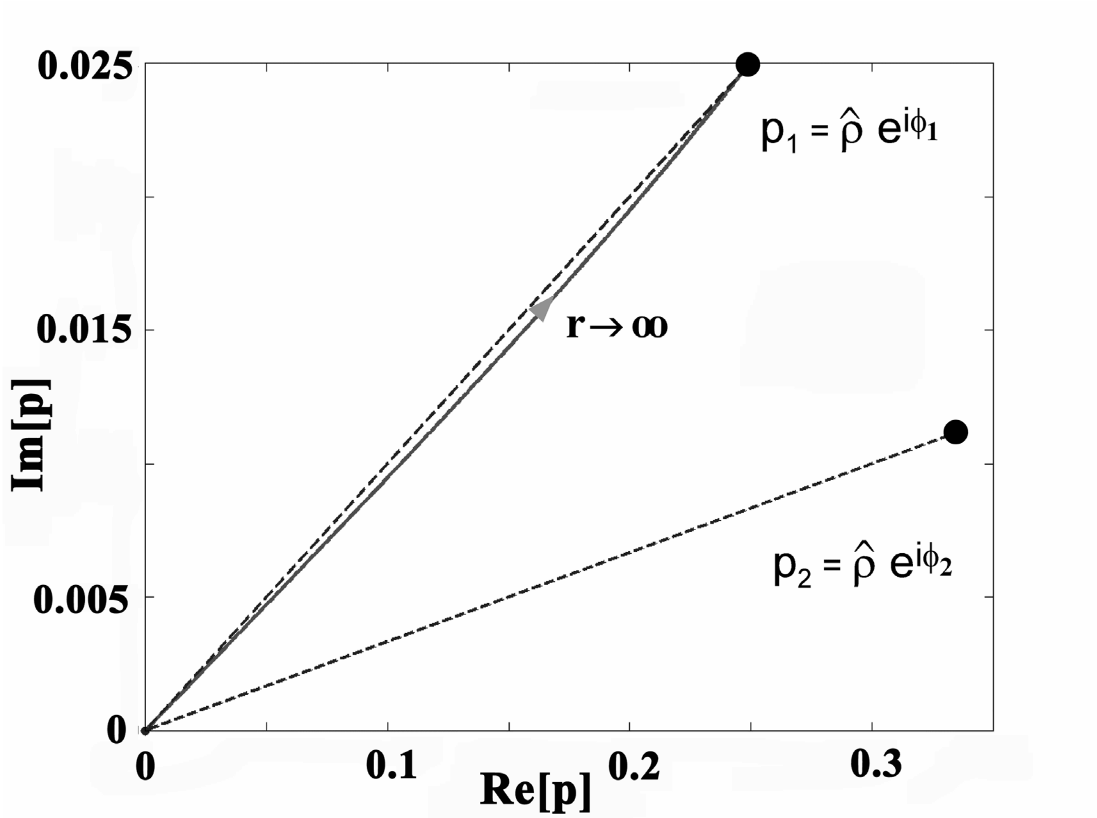

FIG. 5. The complex saddle point path in the complex $p-$plane for the range of offsets $0 \le r < \infty$ km. The two layered model employed here assumes that the absolute values of the velocities in both media are equal. The layers differ only in the quality factor, $Q_j(\omega)$, which is 25 for medium 1 $(j=1)$ and 10 for medium two $(j=2)$. Other media parameters are given in the text. The angles ϕ_1 and ϕ_2 are 5.75° and 4.0°.

$\left(\rho_1^{-1} = 4.0\,km/s : \rho_2^{-1} = 3.0\,km/s\right)$ and in the second, the layers have the this properties $\left(\rho_1^{-1} = 3.0\,km/s : \rho_2^{-1} = 4.0\,km/s\right)$. All other quantities are the same as in Figure (5). As different scales are required to be used for the real and imaginary axis of the complex *p*-plane the results will appear somewhat out of proportion.

The simplest case of a ray propagating in a plane layered viscoelastic medium consisting of J layers with at least one ray segment in each layer and the total ray comprising a geometrical ray path will be now considered. Let all of the $\phi_j \left(1 \le j \le J\right)$ be equal to some angle, say $\hat{\phi}$. All of the branch points, $p_j \left(1 \le j \le J\right)$, then lie on the straight line from the origin at the angle $\hat{\phi}$ measured counter clockwise from the real p axis. An orthonormal rotation of the $p-$plane into the $p'-$plane through the angle $\hat{\phi}$

reduces this problem to the elastic case in the $p'-\text{plane}$. That is to say, the saddle point p_0 or p_0' lies along the line corresponding to the real axis in the $p'-\text{plane}$.

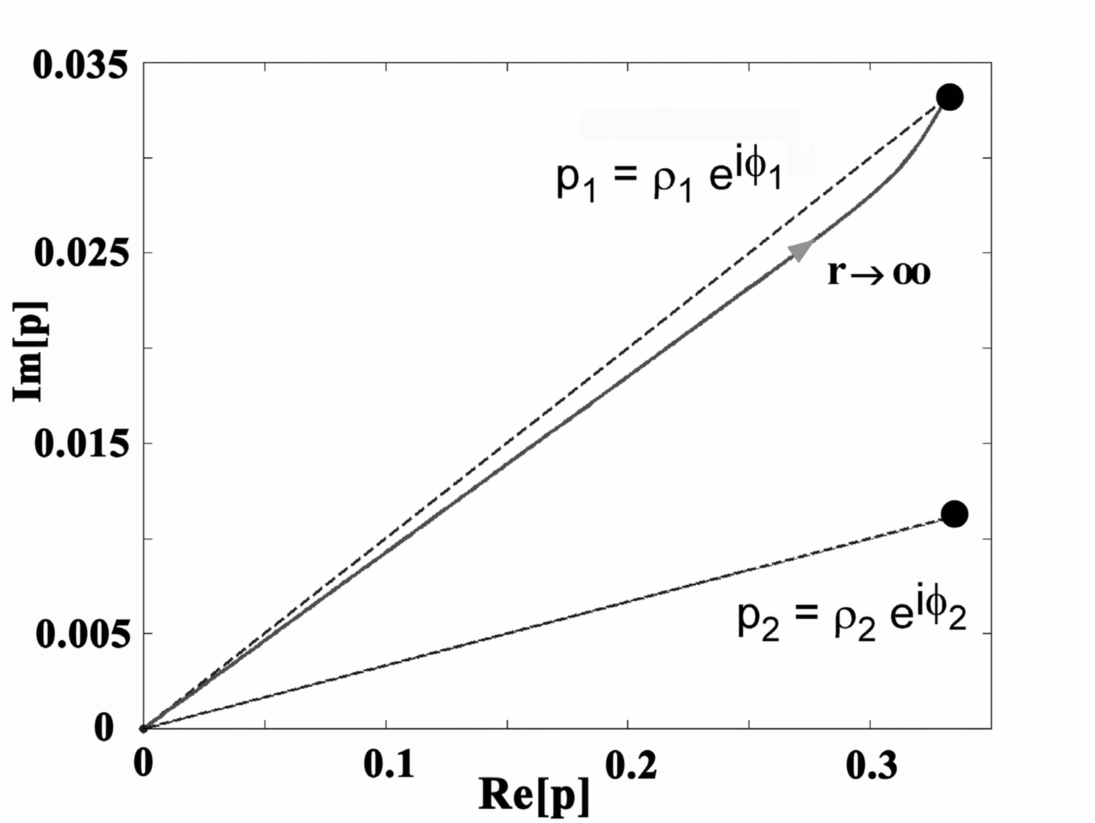

FIG. 6. An example of the saddle point progression associated with the offset range $0 \le r < \infty \;\; km$ where the first layer 1 is the *fast* layer $\left(\rho_1^{-1} = 4.0\, km/s : \rho_2^{-1} = 3.0\, km/s\right)$.. The values of ϕ_1 and ϕ_2 are 5.75° and 4.0°.

Following the analysis of the problem in a manner similar to elastic case, the maximum value that ρ_0 may attain for a geometrical ray path is dictated by the minimum value of $\rho_j \left(1 \le j \le J\right)$ indicated by $\rho_{\min}$, i.e., $0 \le \rho_0 < \rho_{\min} \;\left(p_0 = \rho_0 e^{i\hat{\phi}}\right)$. (The values of the subscripted ρ have the same values in both coordinate systems as the orthonormal rotational transformation preserves length.) This is due to the fact that the saddle point is a solution of a modified form of equation (36) in the text

$$\frac{df\left(p_0\right)}{dp} = r - \frac{p_0\, m_{\min} h_{\min}}{\tilde{\eta}_{\min}} - \sum_{j=1;\, j\neq \min}^{J} \frac{p_0\, m_j h_j}{\tilde{\eta}_j} = 0 \qquad (58)$$

or equivalently in the rotated system

$$\frac{df\left(p_0'\right)}{dp} = r - \frac{p_0' \, m_{\min} h_{\min}}{\tilde{\eta}_{\min}'} - \sum_{j=1;\, j \neq \min}^{J} \frac{p_0' \, m_j h_j}{\tilde{\eta}_j'} = 0 . \tag{59}$$

As $\rho_0 \to \rho_{\min}$, $\tilde{\eta}_{\min} \to 0$ so that $r \to \infty$. Thus there can be no geometrical arrival in the region $\rho_0 \geq \rho_{\min}$. The only arrival corresponds to the saddle point in the region

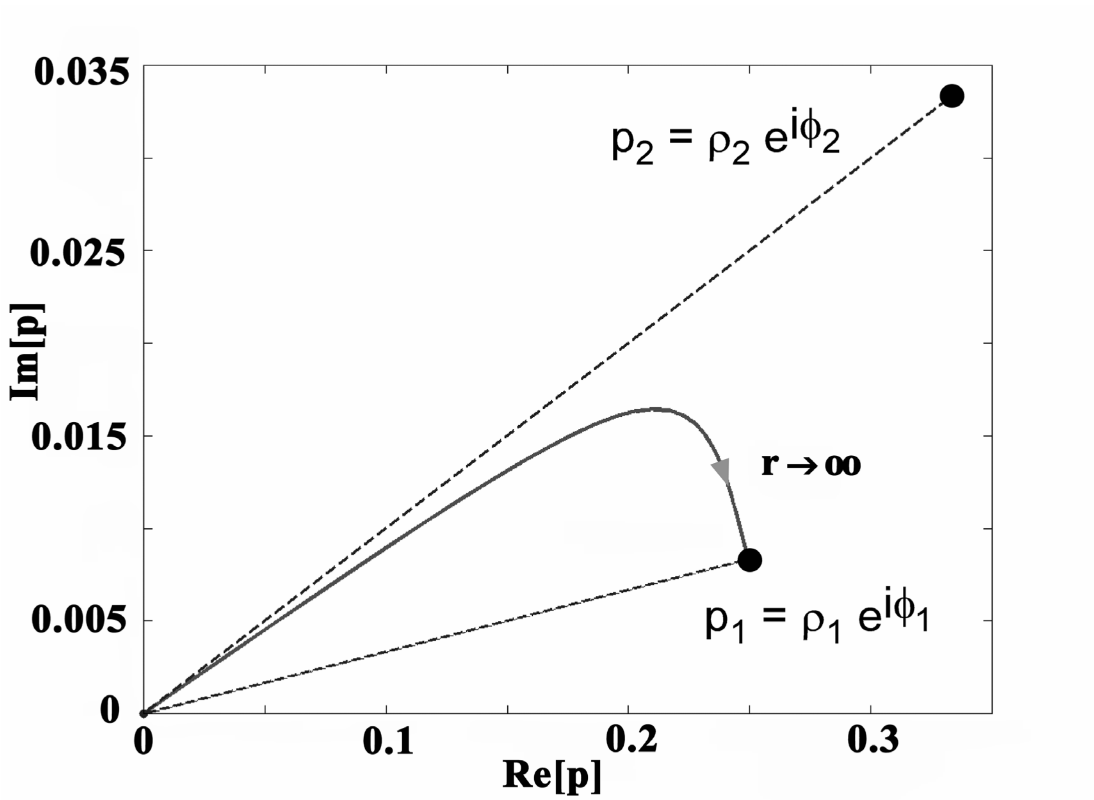

FIG. 7. A third example of saddle point progression associated with the offset range $0 \leq r < \infty$ km. Here layer 2 is the *fast* layer $\left(\rho_1^{-1} = 3.0\, km/s : \rho_2^{-1} = 4.0\, km/s\right)$. The values of ϕ_1 and ϕ_2 are 4.0° and 5.75°.

$\left(0 \leq \rho_0 < \rho_{\min} ; \phi_0 = \hat{\phi}\right)$ implying that both $p_0/\tilde{\eta}_j$ and $p_0/\tilde{\eta}_j'$ are real for all *j*. In equations (58) and (59), m_j and h_j are the number of ray segments in the j^{th} layer and its thickness, respectively.

Returning to the two layered media problem, equation (36) may be written as

$$r - p_0 \frac{m_1 h_1}{\tilde{\eta}_1} - p_0 \frac{m_2 h_2}{\tilde{\eta}_2} = 0 . \tag{60}$$

It is again convenient to introduce another coordinate system rotated by the angle ϕ_0 with respect to the original system. This transformation is again orthonormal and in the rotated or primed coordinate system equation (59) for this two layered example becomes

$$r - p_0' \left(\frac{m_1 h_1}{\tilde{\eta}_1'} + \frac{m_2 h_2}{\tilde{\eta}_2'} \right) = 0\,, \tag{61}$$

where p_0' is a real number. Assume that both ϕ_1' and ϕ_2' are greater than zero. The imaginary part of equation (61) is then

$$\left(\frac{m_1 h_1 \sin\left(\phi_1' + \zeta_1'\right)}{\left|\tilde{n}_1'\right|} + \frac{m_2 h_2 \sin\left(\phi_2' + \zeta_2'\right)}{\left|\tilde{n}_2'\right|} \right) = 0\,, \tag{62}$$

where ϕ_j' and ζ_j' $\left(j = 1,2\right)$ are all positive. This indicates that equation (58) has no solution as all other quantities in the equation are real and positive and $\sin\left(\phi_j' + \zeta_j'\right) > 0 \left(j = 1,2\right)$ by a previous argument. As a consequence, the saddle point p_0' must lie in a position such that one of the $\left(\phi_j',\, j = 1,2\right)$ is positive and the other negative. In the unprimed system this requires that either

$$\phi_1 > \phi_0 > \phi_2 \tag{63}$$

which indicates that the quality factors in the two layers are related as $Q_2\left(\omega\right) > Q_1\left(\omega\right)$, or

$$\phi_2 > \phi_0 > \phi_1 \tag{64}$$

from which it may be inferred that $Q_1\left(\omega\right) > Q_2\left(\omega\right)$. A similar result is obtained if it is initially assumed that ϕ_1' and ϕ_2' are both less than zero.

Using a special case of the above to show the more general result of a ray propagating in J layers $\left(J > 2\right)$, ϕ_1 may be chosen as $\phi_{\max}$ and ϕ_2 as $\phi_{\min}$, where it has been previously stated in the text that $\phi_{\max}$ and $\phi_{\min}$ are the maximum and minimum phases of all the complex quantities p_j, $\left(1 \le j \le J\right)$. If this condition holds then all other layers must obey the condition that

$$\phi_{\min} < \phi_j < \phi_{\max} \quad \left(j = 3, 4, \dots J\right). \tag{65}$$

The more general statement of this is that for a ray propagating in J layers there exist a $\phi_{\min} < \phi_0$ and a $\phi_{\max} < \phi_0$, such that for all $\left(j : 1 \le j \le J : j \ne j_{\min},\, j \ne j_{\max}\right)$ then $\phi_{\min} < \phi_j < \phi_{\max}$, as in the argument leading to equation (65).

CONCLUSIONS

A zero order saddle point solution for ray theoretical amplitudes in a viscoelastic medium composed of J plane layers has been presented. In an effort to enhance computational efficiency, a fairly rigorous treatment of the saddle point contour together with the relationship between the branch points and saddle point, which for the sign convention used here, lie in the first quadrant of the complex p-plane. It was determined that the saddle point must lie in the angular segment $\phi_{\min} \le \phi_0 \le \phi_{\max}$ with $\phi_{\min}, \phi_{\max} \in (\phi_1, \phi_2, \ldots \phi_J)$. Further, the layer of focus is that layer, say, k, in which the real part of p_k is a minimum when compared to all other $p_j \in (p_1, p_2, \ldots p_J)$. It is in this layer, as the offset r tends to infinity, $\rho_0 \to \rho_k$ and $\phi_0 \to \phi_k$.

The mathematical mechanism to introduce viscoelasticity into a medium, based on the earlier works of Futterman (1962) and Azimi et al. (1968), and presented most recently in the literature by Zharadnik et al. (2002) was used here as the basis of introducing a frequency dependent quality factor, $Q(\omega)$, into the numerical computations. The mathematical analysis is somewhat biased by this formulation but can, in general, be taken as valid for any frequency dependent quality factor specification.

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APPENDIX: THE INTEGRATION CONTOUR

The integral in the variable p over the range $\left(-\infty < p < \infty\right)$ is replaced by the integral around the semicircle Ω_D with its radius $R \to \infty$ plus the saddle point integral contour, $\Omega_{S.P.}$, together with any branch cut integral contours, $\Omega_{B.P.}$, which may contribute to the total wavefield. Any poles due to singularities of the integrand are not considered, so that,

$$\Omega = \Omega_{p=-\infty}^{p=\infty} + \Omega_D + \Omega_{S.P.} + \sum_j \Omega_{B.P.}^j = 0. \qquad \text{(A.1)}$$

This is shown schematically in Figure (2) for the case of a layer over a halfspace being considered here. The total integral around the contour Ω should in reality be equal to the sum of the residues of the poles of the integrand rather than "0" as indicated in equation (A.1). However, for this problem, the introduction of any possible poles is not critical to the discussion and could tend to unduly complicate matters.

The radiation conditions requires that the integral around $\Omega_D = 0$ as its radius $R \to \infty$, resulting in

$$\Omega_{p=-\infty}^{p=\infty} = -\Omega_{S.P.} - \sum_j \Omega_{B.P.}^j. \qquad \text{(A.2)}$$

The branch point contributions will be given only cursory treatment, as the focus of this paper is body waves associated with the saddle point contour integrations.

For a saddle point lying on the real axis in the complex p-plane, a contour which passes through the saddle point at the angle $-\pi/4$ has been presented and discussed in Cerveny and Ravindra (1970), which contains a number of references to this contour choice for certain problems which arise in geophysics. This contour, parameterized by the real variable y, for a saddle point on the real p axis, is given as

$$\left(\beta_M^{-2} - p^2\right)^{1/2} = \left(\beta_M^{-2} - p_0^2\right)^{1/2} - y\exp\left(-i\pi/4\right), \qquad -\infty < y < \infty. \qquad \text{(A.3)}$$

The saddle point, p_0 lies on the real p axis if the medium is isotropic and homogeneous. In the above case the layer velocity, β_M, a purely real quantity, is arbitrarily chosen. It may be taken as the shear wave velocity in the maximum velocity layer through which the ray traverses in its path from source to receiver, the velocity in the surface layer, or the velocity in the deepest layer through which the ray travels.

The generalization to a saddle point which lies in the first quadrant of the p-plane, not on the real p axis, is achieved by the addition to the parameterization term in y on the *R.H.S.* of (A.3) of $e^{i\phi_0}$ resulting in the new contour,

$$\left(\beta_M^{-2} - p^2\right)^{1/2} = \left(\beta_M^{-2} - p_0^2\right)^{1/2} - y\exp\left(-i\pi/4 + i\phi_0\right), \quad -\infty < y < \infty, \quad 0 \le \phi_0 < \pi/4 \,. \text{(A.4)}$$

This has the effect of an orthonormal rotation of the p-plane to the p'-plane, where the saddle point p_0 lies on the real axis of the p'-plane. If an analysis of the problem is done using the theory associated with saddle point approximations the results are similar, as in the p'-plane the saddle point contour passes through the saddle point, $p_0\left(p_0'\right)$ at an angle of $\pi/4$ relative to the real p' axis (Figure 3).

The sense of the saddle point portion of the contour is opposite to that which would be expected. This is done to accommodate the choice of parameterization of this part of the total contour, which is defined above. It is not difficult to show, once the saddle point approximation of the integral has been developed to a certain point that the integrals in real variable y from $-\infty$ to ∞ and ∞ to $-\infty$ are equal if the change of variable $y \to -y$ is made.

The Taylor series expansion of $f\left(p\right)$ near the saddle point in terms of the real variable y, where f may refer to any complex phase function, is given by

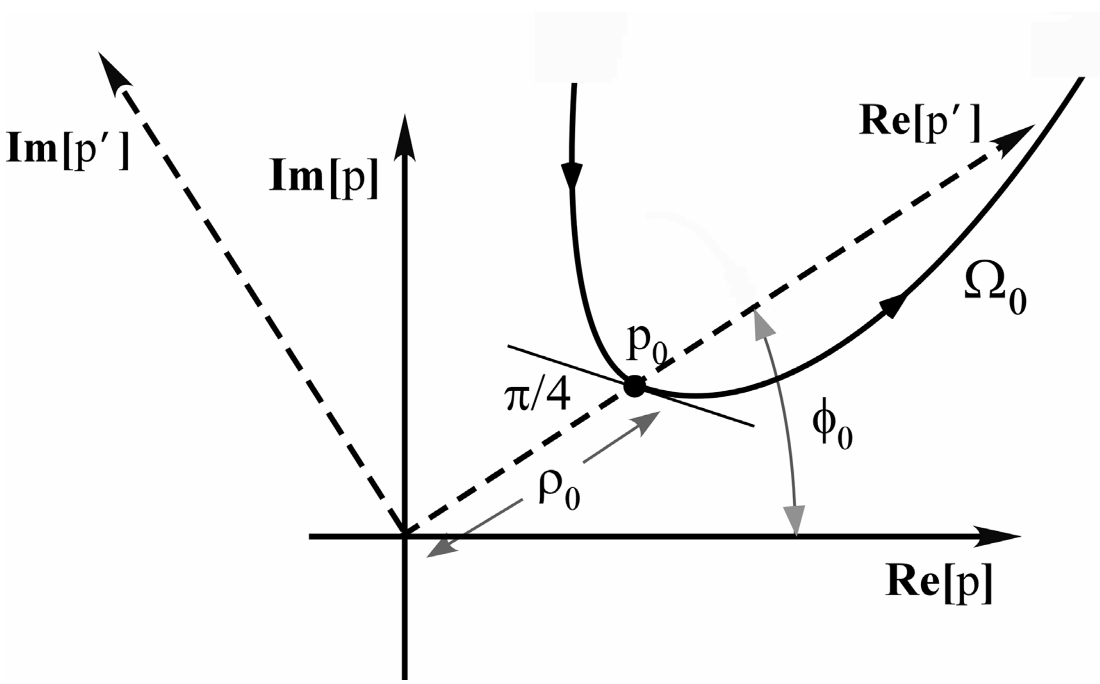

FIG. 3. A schematic illustrating the introduction of an orthonormal rotation from the **p**-plane to the $\mathbf{p}'$ - plane. In the $\mathbf{p}'$ - plane the saddle point p_0' lies on the real axis of this coordinate system. As the transformation orthonormal, $\rho_0 = \rho_0'$, i.e., length is preserved.

$$f(p) \approx f(p_0) + \left(\frac{df}{dp}\frac{dp}{dy}\right)_{p_0} y + \frac{1}{2}\left[\frac{d^2 f}{dp^2}\left(\frac{dp}{dy}\right)^2 + \frac{df}{dp}\frac{d^2 p}{dy^2}\right]_{p_0} y^2 + \cdots \qquad \text{(A.5)}$$

and as a consequence of $df/dp = 0$, at the saddle point, p_0, the following holds

$$f(p) \approx f(p_0) + \frac{1}{2}\left[\frac{d^2 f}{dp^2}\left(\frac{dp}{dy}\right)^2\right]_{p=p_0\,(y=0)} y^2 + \cdots \qquad \text{(A.6)}$$

As there is considerable latitude in choosing the branch cuts, to be consistent with that contour chosen for saddle point contribution the following parameterization is used

$$\left(p_{j-1}^2 - p^2\right)^{1/2} = \left(p_{j-1}^2 - p_j^2\right)^{1/2} - y\exp\left(-i\pi/4 + i\phi_j\right), \qquad 0 < y < \infty, \quad 0 \le \phi_j < \pi/4 \text{ .(A.7)}$$

Here the reference layer chosen for the branch point corresponding to the j^{th} layer was the $\left(j-1\right)^{th}$. However, this is arbitrary and any other layer could have been used. A branch cut contribution to the total wave field results from the integration around the branch cut with the major contribution to the integral being in the vicinity of the branch point, p_j, $\left(y=0\right)$.