# Using the exact Zoeppritz equations in pseudo-linear form: Inversion for density

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## ABSTRACT

A simple method is presented for inverting offset-varying amplitudes to obtain the density contrast across an interface. This method employs no approximations to the Zoeppritz equations, and may be applied to either conventional or converted-wave data. It consists essentially of solving a cubic equation for the density contrast, the coefficients of which are dependent upon amplitudes, velocities, and angles of incidence. These must be known or estimated beforehand. The effect of errors in the input are considered.


## INTRODUCTION

One as yet elusive goal in exploration geophysics is the accurate determination of density contrasts from AVO inversion. Our objective in this paper is to present a new method for extracting densities for amplitude data, which may be of value in improving the accuracy for these estimates.

## THEORY

We begin with the exact Zoeppritz equations for $R_{\mathrm{PP}}$ and $R_{\mathrm{PS}}$ expressed in pseudo-linear form (Ursenbach, 2003a,b). Inspection shows that both sides of these equations are quadratic in $\Delta\rho/\rho$. Thus either expression can be written in the form

$$a + b(\Delta\rho/\rho) + c(\Delta\rho/\rho)^2 = 0 \qquad (1)$$

where $a$, $b$, and $c$ depend on $\beta/\alpha$, $\Delta\alpha/\alpha$, $\Delta\beta/\beta$, $\theta$, and either $R_{\mathrm{PP}}(\theta)$ or $R_{\mathrm{PS}}(\theta)$. Given data at several values of $\theta$, say, $\theta_i = 0°, 1°, 2°, \ldots 30°$, then a least-squares estimate of $\Delta\rho/\rho$ is given by solution of the cubic equation

$$\left[\sum_i a_i b_i\right] + \left[2\sum_i a_i c_i + \sum_i b_i^2\right]\frac{\Delta\rho}{\rho} + 3\left[\sum_i b_i c_i\right]\left(\frac{\Delta\rho}{\rho}\right)^2 + 2\left[\sum_i c_i^2\right]\left(\frac{\Delta\rho}{\rho}\right)^3 = 0\,. \qquad (2)$$

This relation has three roots, at least one of which must be real. In applying this relation we have found that the physical root can be described as the real root with the smallest absolute value.

## RESULTS

We performed calculations on 125 interfaces. A detailed description of this data set is given elsewhere (Ursenbach, 2003a). We present the results in graphical form below by plotting the difference between the value of $\Delta\rho/\rho$ obtained from Equation (2) and the exact value of $\Delta\rho/\rho$ used in generating synthetic values of $R_{\mathrm{PP}}(\theta_i)$ and $R_{\mathrm{PS}}(\theta_i)$, where $\theta_i = 0°, 1°, 2°, \ldots 30°$. This error quantity is plotted against the value of $\Delta\beta/\beta$ used in generating the reflectivities, as in earlier studies errors have tended to correlate with this quantity. We also perform crossplots of the estimated $\Delta\rho/\rho$ against the exact $\Delta\rho/\rho$, and we do this for both Equation (2) and for the three-parameter inversion method used in paper III. Both methods are based on the exact Zoeppritz equations, so it is a very meaningful comparison.

One issue in carrying out these tests is the choosing of values for $\beta/\alpha$, $\Delta\alpha/\alpha$, and $\Delta\beta/\beta$ to use in the *a*, *b*, *c* coefficients. If the exact values are used for these and other inputs, then $\Delta\rho/\rho$ is predicted exactly, to machine accuracy, as it should. To obtain non-trivial results we have performed calculations in which errors are added to one of the velocity ratios, or to the reflectivities, or to the angles of incidence. To aid comparison with results from three-parameter AVO inversion, we employ the same errors as in the previous paper in this volume (Ursenbach, 2003c).

We have carried out calculations for six types of input errors:

1. Gaussian noise added to reflectivities
2. A constant term added to $\beta/\alpha$
3. A constant term added to $\Delta\alpha/\alpha$
4. A constant term added to $\Delta\beta/\beta$
5. Gaussian noise added to angles of incidence
6. A systematic error linearly scaling the angles of incidence

The results are shown in Figures 1-6 below.

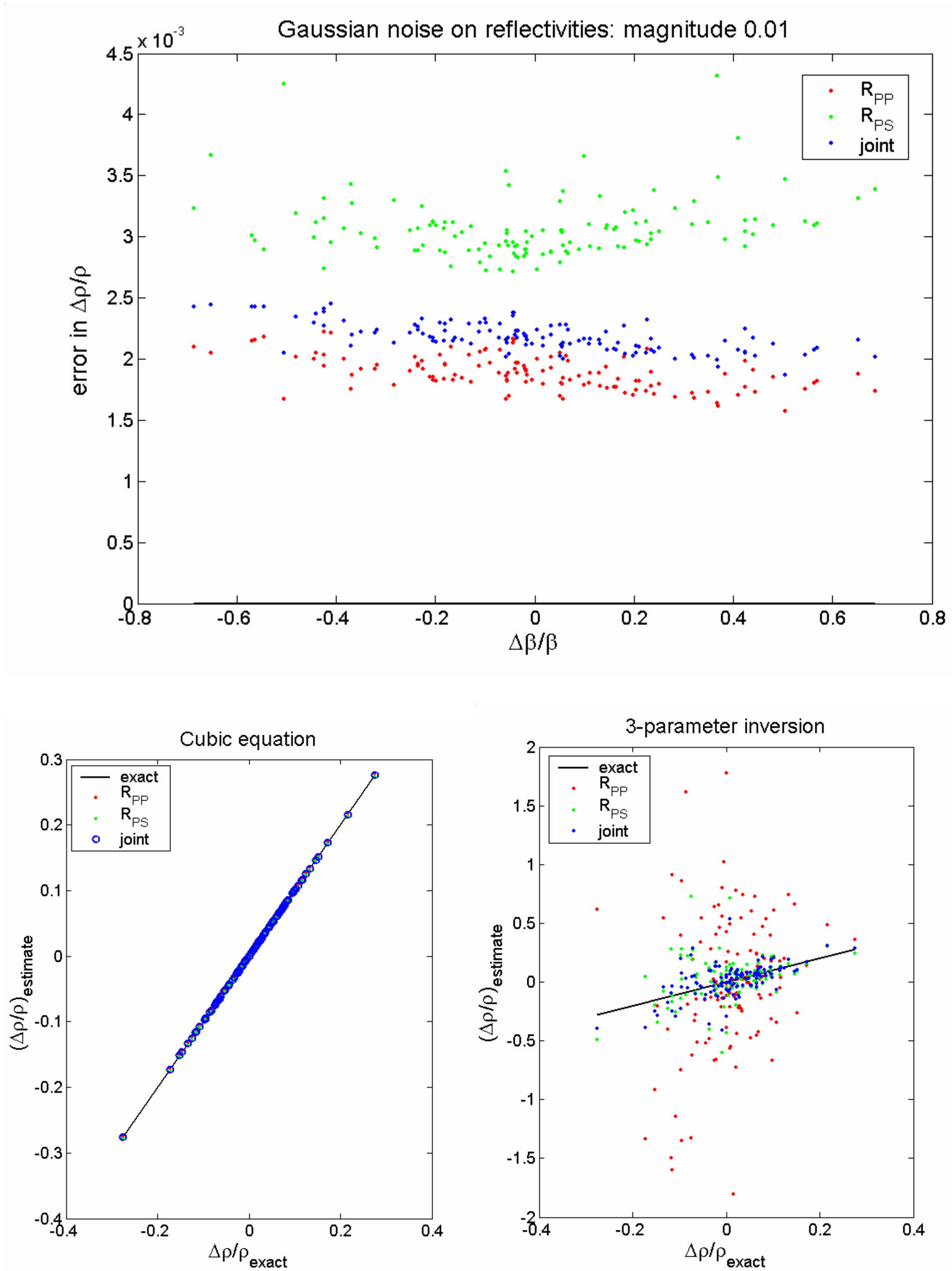


FIG. 1. The error of $\Delta\rho/\rho$ as predicted by Equation (2) is given in the upper panel. Random noise has been added to the data amplitudes. The data is replotted in the lower left panel and compared with the three-parameter inversion method (lower right panel; Ursenbach, 2003c).

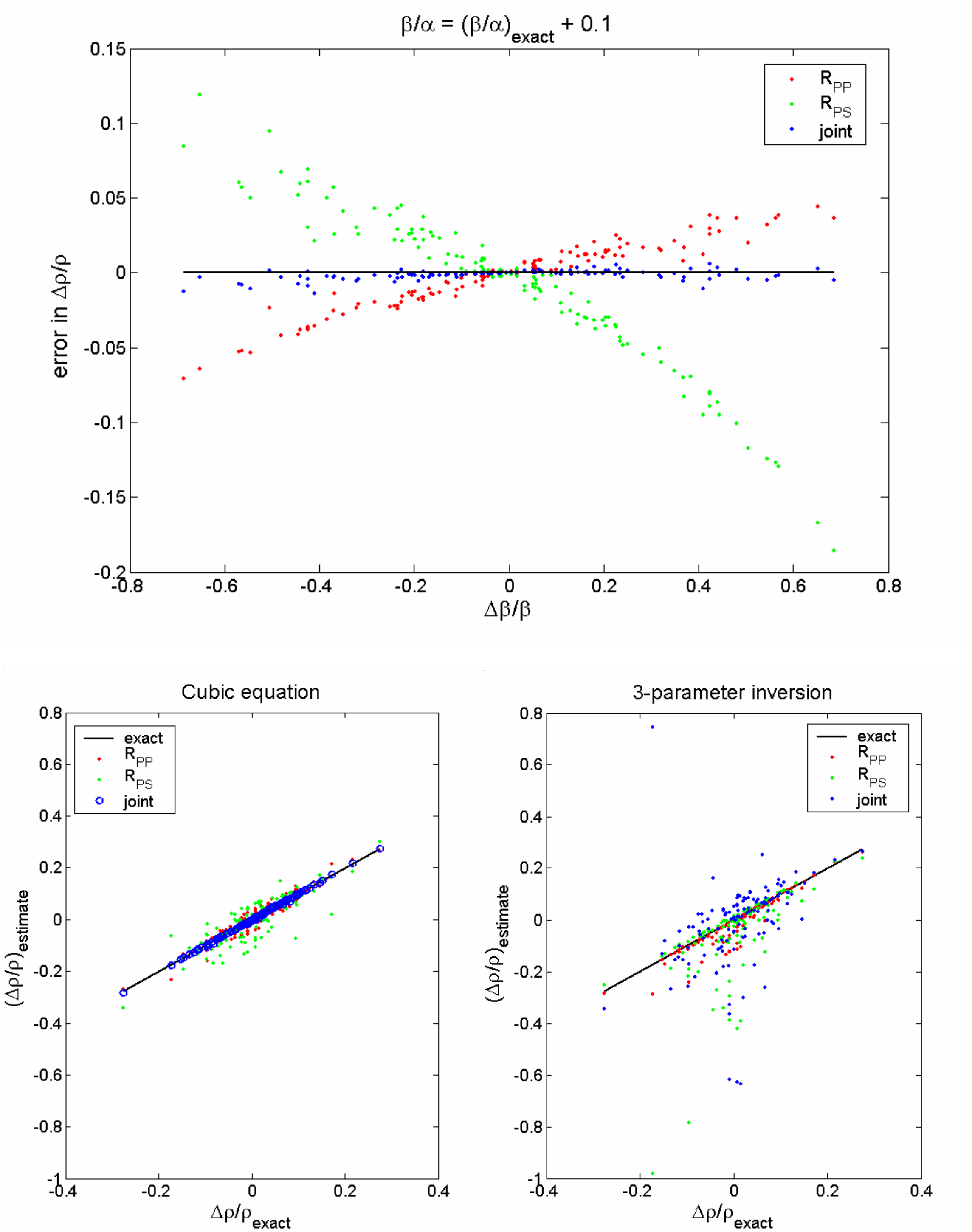


FIG. 2. The error of $\Delta\rho/\rho$ as predicted by Equation (2) is given in the upper panel. An error term has been added to $\beta/\alpha$. The data is replotted in the lower left panel and compared with the three-parameter inversion method (lower right panel; Ursenbach, 2003c).

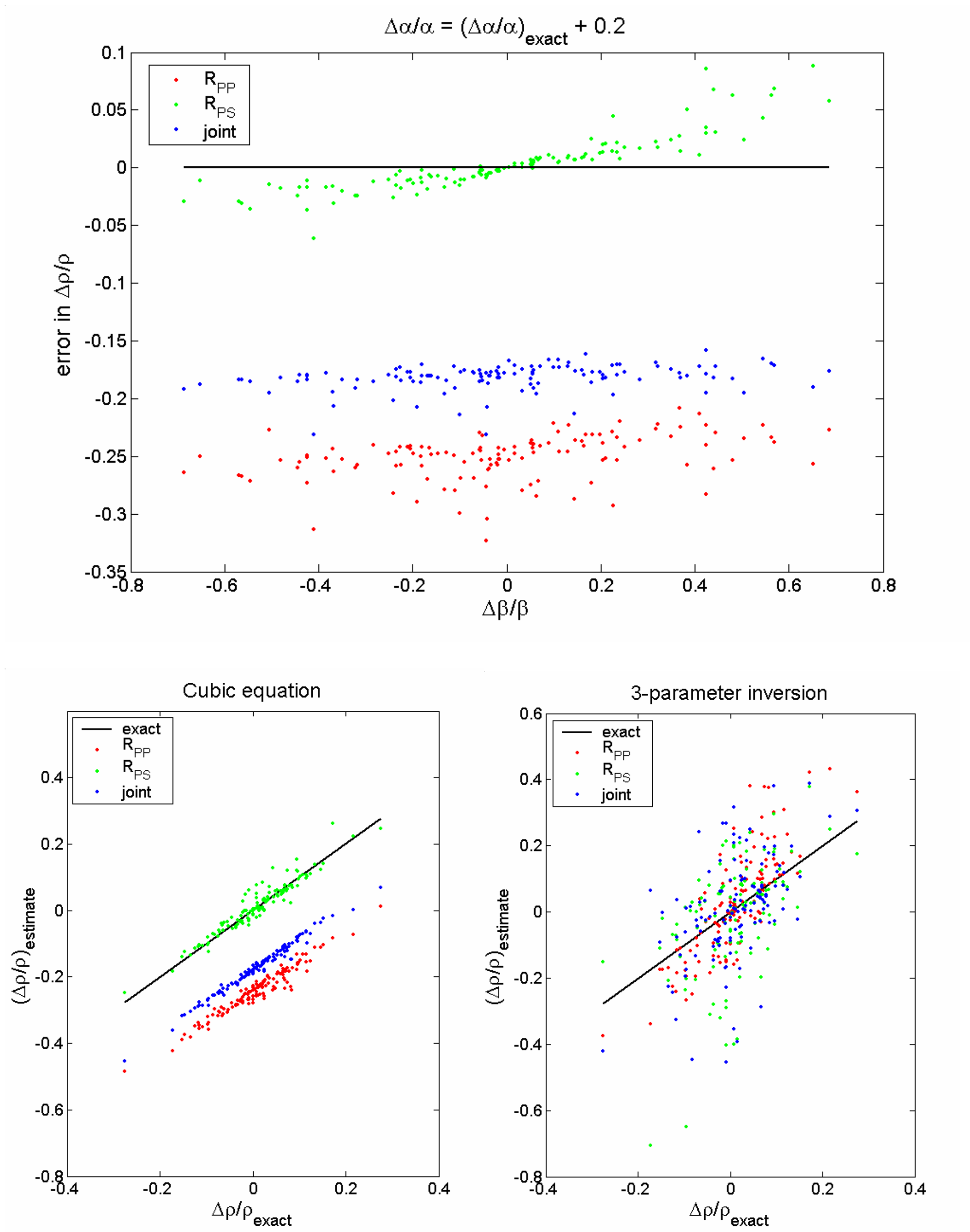


FIG. 3. The error of $\Delta\rho/\rho$ as predicted by Equation (2) is given in the upper panel. An error term has been added to $\Delta\alpha/\alpha$. The data is replotted in the lower left panel and compared with the three-parameter inversion method (lower right panel; Ursenbach, 2003c).

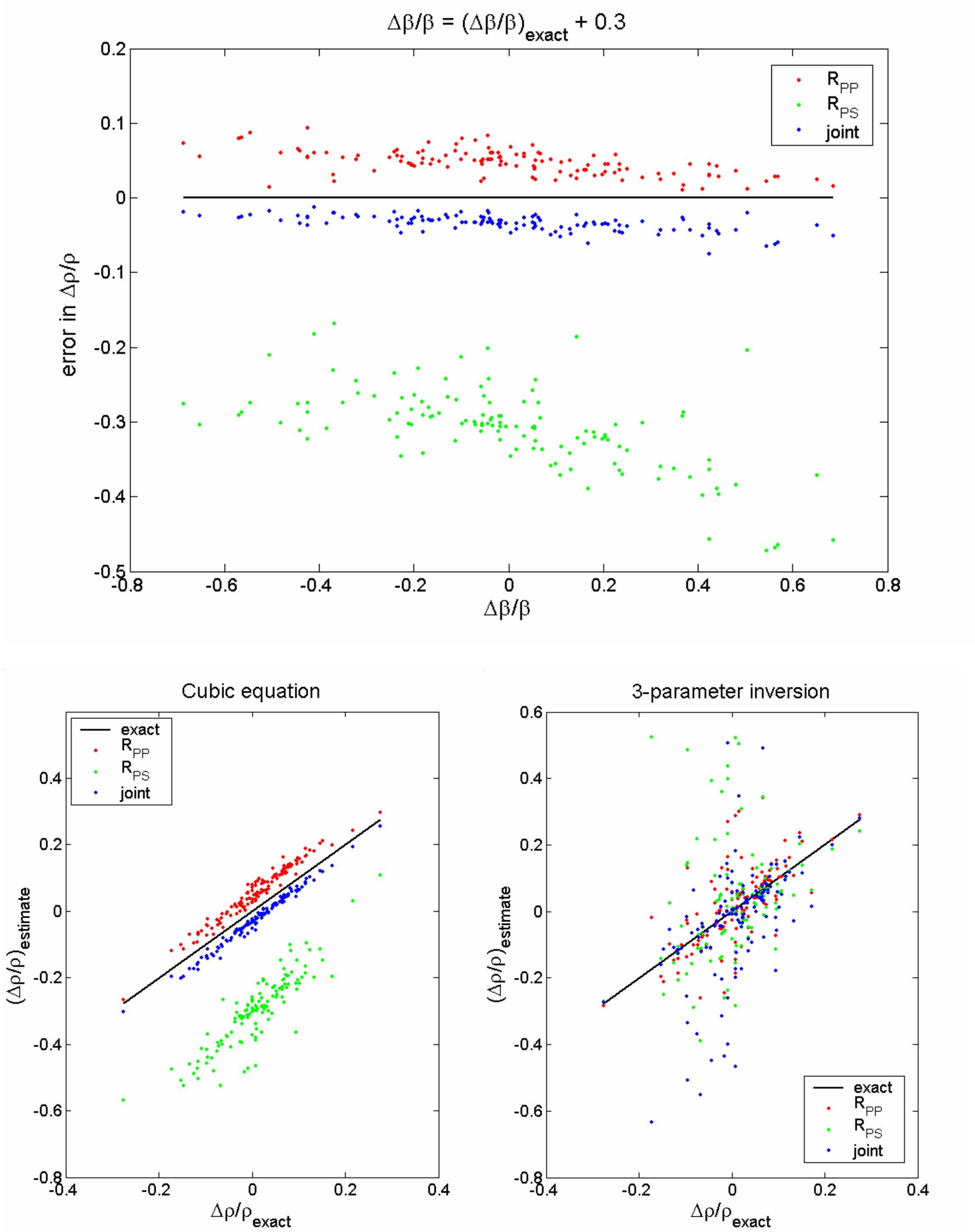


FIG. 4. The error of $\Delta\rho/\rho$ as predicted by Equation (2) is given in the upper panel. An error term has been added to $\Delta\beta/\beta$. The data is replotted in the lower left panel and compared with the three-parameter inversion method (lower right panel; Ursenbach, 2003c).

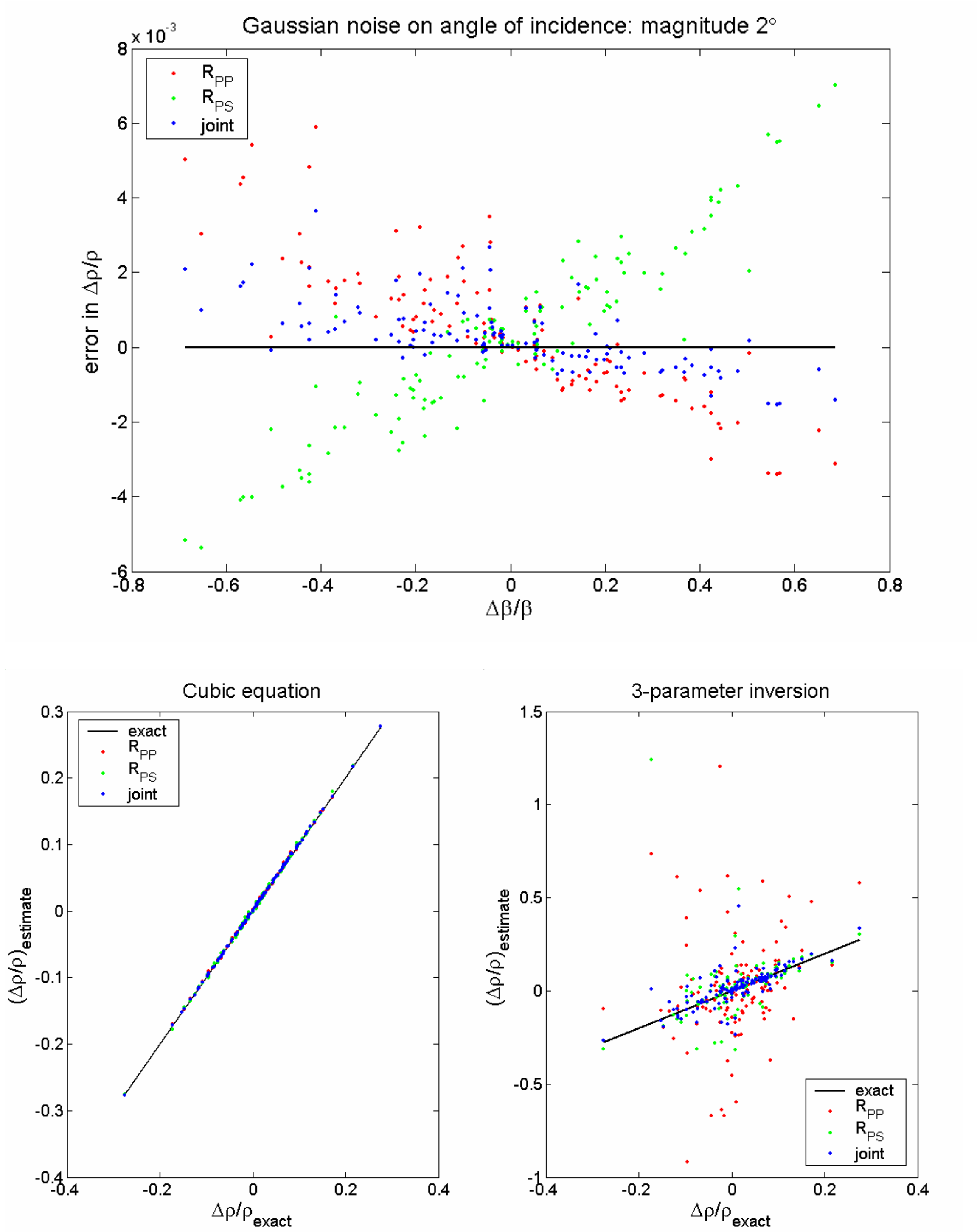


FIG. 5. The error of $\Delta\rho/\rho$ as predicted by Equation (2) is given in the upper panel. Gaussian noise has been added to the angle of incidence. The data is replotted in the lower left panel and compared with the three-parameter inversion method (lower right panel; Ursenbach, 2003c).

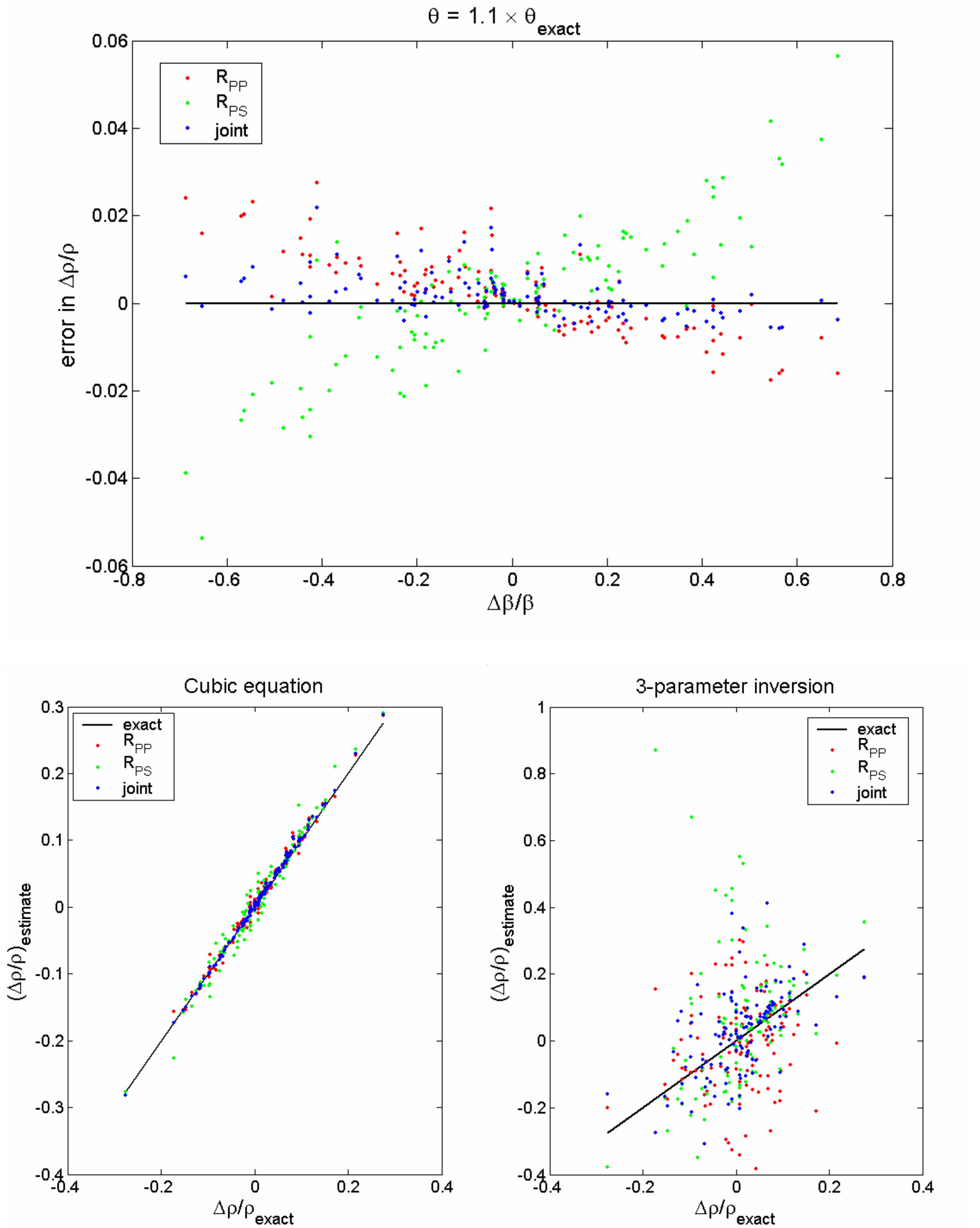


FIG. 6. The error of $\Delta\rho/\rho$ as predicted by Equation (2) is given in the upper panel. The angle of incidence has been scaled by a factor of 1.1. The data is replotted in the lower left panel and compared with the three-parameter inversion method (lower right panel; Ursenbach, 2003c).

## DISCUSSION

In the figures above the method of Equation (2) is compared to a three-parameter inversion using an identical level of theory and error inputs. In most cases the new method gives a clear improvement. For errors in $\beta/\alpha$, $\Delta\alpha/\alpha$, and $\Delta\beta/\beta$ the results depend sensitively on the type of data being used: $\beta/\alpha$ errors are best handled by joint inversion, $\Delta\alpha/\alpha$ errors by P-S inversion, $\Delta\beta/\beta$ errors by P-P inversion. The behaviour of the errors is distinctly different than for those obtained through three-parameter inversion. The greatest difference between the two methods though appears to be in their handling of random errors. While, as noted in Paper III, three-parameter inversion results are very sensitive to random noise, especially in amplitudes, their propagation appears to be controlled very tightly in Equation (2). This is obviously a point worthy of further study.

In earlier papers (Ursenbach, 2003a,b), we have presented two new approximations, namely, the pseudo-linear and pseudo-quadratic approximations, and discussed them in connection with the well-known Aki-Richards approximation. The approach that has been employed in developing Equation (2) may be applied to any of these as well. For the pseudo-quadratic theory, it again yields a cubic equation of the form of Equation (2), but with simpler expressions for the coefficients. For the Aki-Richards and pseudo-linear methods it yields a linear equation, i.e., Equation (2) with $c_i = 0$. All three of these methods have been investigated in addition to the exact theory presented here, and all appear to improve upon their three-parameter counterparts.

## CONCLUSIONS

We have shown that the density contrast may be straightforwardly predicted from the exact Zoeppritz equation by solution of a cubic polynomial. We have carried out some preliminary numerical tests which suggest that this method is more effective in reducing the propagation of input errors, and especially random errors, than is the traditional three-parameter inversion. We believe this method merits further investigation.

## REFERENCES

Ursenbach, C.P., 2003a, Testing pseudo-linear Zoeppritz approximations: P-wave AVO inversion, CREWES 2003 Research Report.

Ursenbach, C.P., 2003b, Testing pseudo-linear Zoeppritz approximations: Multicomponent and joint AVO inversion, CREWES 2003 Research Report.

Ursenbach, C.P., 2003c, Using the exact Zoeppritz equations in pseudo-linear form: Isolating the effects of input errors, CREWES 2003 Research Report.