Continued fraction expansion of the square-root operator

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ABSTRACT

A number of approaches to solving the wave equation require approximations to square-root of (1-x^2). When x is very small, the Taylor series approximations are usually sufficient. When using the first two terms of the series, we get a parabolic equation that leads to the 15-degree approximation to the wave equation. Improved finite-difference solutions to very steep dips are found from rational functions that are often derived from continued fraction expansion.

A very simplified description of continued fraction expansion is presented, starting with real numbers and progressing to functions.

INTRODUCTION

Continuous fraction expansion is an alternate method for describing a rational number or a function. We will start with a rational numbers, and then proceed to the functions and the square-root operator.

Continuous fraction expansion of numbers

Recall some basic definitions and properties:

Integer: a whole number, natural number, (no fractional parts), -1, 6, 0, etc.

Real number: contains a fractional part, may be decimal with floating point, 365.25.

Rational number: a real number that can be expressed as a quotient of integers, 7/3.

Irrational number: a real number that can't be expressed as a quotient of integers, i.e., π,

Consider the expression:

$$A = \frac{p}{q} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{\cdots \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}}, \qquad (1)$$

where A is a rational number, and p and q are integers. When A is a positive-real (not complex) number, a_1, a_2, …, a_n are positive integers. Consider the rational number 8/5,

$$\frac{8}{5}=1+\frac{3}{5}=1+\frac{1}{5/3}=1+\frac{1}{1+2/3}=1+\frac{1}{1+\frac{1}{3/2}}=1+\frac{1}{1+\frac{1}{1+1/2}}=1+\frac{1}{1+\frac{1}{1+\frac{1}{2}}}. \tag{2}$$

There are a number of ways to write this number in a more compact form, such as

$$1+\frac{1}{1+}\frac{1}{1+}\frac{1}{2}, \tag{3}$$

$$[1,1,1,2], \tag{4}$$

or

$$[1:1,1,2]. \tag{5}$$

Note that the first number is the truncated integer value of the rational number, and if the number is less than 1.0, (A < 1.0) then the first number would be 0, i.e.,

$$\frac{8}{11}=[0:1,2,1,2]. \tag{6}$$

The number of fractions can be finite for a rational number that is composed of integers, or can be infinite for numbers such as π, and have cyclical properties for $\sqrt{2}$: i.e.,

$$\pi=[3;7,15,1,292,1,1,1,2,1,3,1,14,2,1,1,2,2,2,2,1,84,2,...], \tag{7}$$

and

$$\sqrt{2}=[1:2,2,2,2,2,2,2,2,2,2,2,...]=[1:2^*]. \tag{8}$$

Let's play with π for a minute, knowing $\pi=3.14159265359...$, and continue the fraction expansion for three terms.

$$\begin{aligned}\pi &\approx 3.1416\\ &=3+0.1416\\ &=3+\frac{1}{10000/1416} \quad =3+\frac{1}{7.0621}\\ &=3+\frac{1}{7+\frac{1}{10000/621}} \quad =3+\frac{1}{7+\frac{1}{16.1031}}\\ &\approx 3+\frac{1}{7+\frac{1}{16}}\end{aligned}. \tag{9}$$

We can now back substitute to get a rational value for π and then its equivalent decimal value.

$$\pi \approx 3+\frac{1}{7+\frac{1}{16}} = 3+\frac{1}{113/16} = 3+\frac{16}{113} = \frac{355}{113} = 3.14159292035\ldots \; . \tag{10}$$

Note that the rational approximation for π of 355/113 = [3;7,16] is a very accurate representation, up to the 7th decimal point.

Rational approximations to π may be formed by considering only a few terms of the continued fraction expansion of (7). If we start with the first, and then include each additional term we get the first eight rational approximations

$$\frac{3}{1},\quad \frac{22}{7},\quad \frac{333}{106},\quad \frac{355}{113},\quad \frac{103993}{33102},\quad \frac{104384}{33215},\quad \frac{208341}{66317},\quad \text{and} \quad \frac{312689}{99532}. \tag{11}$$

Now consider the quadratic equation

$$x^2 - bx - 1 = 0\,, \tag{12}$$

that can be rewritten as

$$x = b + \frac{1}{x}. \tag{13}$$

This is an interesting form for a solution of *x* and we will return to it later. We can expand the equation into a continued fraction form by substituting the right side of the equation into the right side value of *x*, i.e.,

$$\begin{aligned} x &\approx b + \frac{1}{b+\ldots} \\ &\approx \ldots \\ &\approx b + \cfrac{1}{b+\cfrac{1}{b+\cfrac{1}{\ldots}}} \end{aligned} \; . \tag{14}$$

We know the exact positive solution of Equation (12) is given by

$$x = \frac{b+\sqrt{b^2+4}}{2}\,, \tag{15}$$

and when b = 1, we get the “golden mean” number g that is

$$g = \frac{\sqrt{5}+1}{2}\,, \tag{16}$$

which has the elegant continued fraction expansion of

$$g = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}}} \quad . \tag{17}$$

$$g = [1:1,1,1,1,1,\dots]$$

Approximations for g with increasing coefficients are:

$$g \approx 1, \quad 2, \quad \frac{3}{2}, \quad \frac{5}{3}, \quad \frac{8}{5}, \quad \frac{13}{8}, \quad \frac{21}{13}, \quad \frac{34}{21}, \quad \dots . \tag{18}$$

CONTINUED FRACTIONAL EXPANSION OF RATIONAL FUNCTIONS

Consider the expansion of

$$f(x) = a_0 + \cfrac{h}{a_1 + \cfrac{h}{a_2 + \cfrac{h}{a_3 + \cfrac{h}{\dots}}}} \tag{19}$$

where the a's are the fractional expansion coefficients, and h is a variable whose power may match that of an equivalent power series expansion of $f(x)$, such as x, x^2, etc. We define $f_0(x) = f(x)$ and write Equation (19) as a number of different equations that define other functions of x, i.e. $f_1(x)$, $f_2(x)$, etc., as

$$\begin{aligned}
f_0(x) &= a_0 + \frac{h}{f_1(x)} \\
f_0(x) &= a_0 + \cfrac{h}{a_1 + \cfrac{h}{f_2(x)}} \\
f_0(x) &= a_0 + \cfrac{h}{a_1 + \cfrac{h}{a_2 + \cfrac{h}{f_3(x)}}} \\
f_0(x) &= \dots
\end{aligned} \quad . \tag{20}$$

We now desire to evaluate the coefficients a_i. Starting with the first line in Equation (20), we set $x = 0$, then $h = 0$, (it is x, or x^2 etc.) and assuming $f_1(x)$ is not zero, we then get a definition for a_0,

$$a_0 = f_0(x)\big|_{x\to 0}. \tag{21}$$

From the first and second lines in Equation (20) we observe that

$$f_1(x) = a_1 + \frac{h}{f_2(x)}, \tag{22}$$

and when $x = 0$, we get

$$a_1 = f_1(x)\big|_{x\to 0}. \tag{23}$$

Rewriting the first line in Equation (20), we define $f_1(x)$ as

$$f_1(x) = \frac{h}{f_0(x) - a_0}, \tag{24}$$

giving a definition for a_1 as

$$a_1 = \left.\frac{h}{f_0(x) - a_0}\right|_{x\to 0}. \tag{25}$$

Note in Equation (25) that a_0 is a fixed value and that $f_0(x) \to a_0$, tending to give a zero in the denominator. That is OK as we also tend to zero in the numerator. We must therefore be careful when evaluating this equation for a_1.

The derivation of a_1 forms a foundation for computing the remaining coefficient of a_i. We can observe from the other lines in Equation (20) that Equations (22) and (24) will be similar for expressions of $f_2(x)$ and $f_3(x)$; i.e.,

$$\begin{aligned} f_2(x) &= a_2 + \frac{h}{f_3(x)} \\ f_2(x) &= \frac{h}{f_1(x) - a_1} \end{aligned}, \tag{26}$$

and

$$\begin{aligned} f_3(x) &= a_3 + \frac{h}{f_4(x)} \\ f_3(x) &= \frac{h}{f_2(x) - a_2} \end{aligned}, \tag{27}$$

giving solutions for a_2 and a_3 when proceeding as above.

From this recursive nature we then generalize the solution for a_i with

$$f_i(x) = a_n + \frac{h}{f_{i+1}(x)}$$
$$f_i(x) = \frac{h}{f_{i-1}(x) - a_{i-1}}, \tag{28}$$

giving

$$\boxed{a_i = \left.\frac{h}{f_{i-1}(x) - a_{i-1}}\right|_{x\to 0}}. \tag{29}$$

I again point out that solving for a_i is not trivial as we always have a zero/zero condition. This form is not useful for digital computation but is, of course, very useful for analytic evaluations of the coefficients.

Example of a function

We will demonstrate the process by choosing the square-root function that is used in obtaining the finite difference solution to the wave-equation, i.e.,

$$f(x) = \sqrt{1-x^2}\ . \tag{30}$$

The power series is defined by

$$(1-x^2)^{\frac{1}{2}} = 1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} - \frac{5x^8}{128}\ldots \qquad \left[x^2 < 1\right]. \tag{31}$$

This power series expansion is in even powers of x, therefore we will use $h = x^2$, giving the form of the continued fraction expansion as

$$f(x) = \sqrt{1-x^2} = a_0 + \cfrac{x^2}{a_1 + \cfrac{x^2}{a_2 + \cfrac{x^2}{a_3 + \cfrac{x^2}{\ldots}}}}. \tag{32}$$

From Equation (21), we get the first coefficient a_0,

$$a_0 = f_0(x) = f(x=0) = 1. \tag{33}$$

We now use the recursion relationship to get the second coefficient a_1 from $f_1(x)$,

$$a_1 = \left. f_1(x)\right|_{x\to 0} = \left.\frac{x^2}{f_0(x) - a_0}\right|_{x\to 0} = ?\frac{x^2}{1-1}. \tag{34}$$

Wow, that looks nasty as we have a potential divide by zero if we replace $f_0(x)$ with a_0. But the numerator also tends to zero giving the zero/zero problem that was mentioned earlier. Therefore we go back to Equation (34) to get

$$f_1(x)\Big|_{x\to 0} = \frac{x^2}{\sqrt{1-x^2}-1}. \tag{35}$$

When x is very small, we can replace the square-root by the first two terms of the power series to get

$$a_1 = f_1(x)\Big|_{x\to 0} \approx \frac{x^2}{\left(1-\frac{x^2}{2}\right)-1} = -2. \tag{36}$$

We now proceed to get the second coefficient a_2 from $f_2(x)$,

$$f_2(x) = \frac{x^2}{f_1(x)-a_1}, \tag{37}$$

then

$$f_2(x) = \frac{x^2}{\frac{x^2}{f_0(x)-a_0}-a_1}, \tag{38}$$

$$f_2(x) = \frac{x^2}{\frac{x^2}{\sqrt{1-x^2}-a_0}-a_1}, \tag{39}$$

$$f_2(x) = \frac{x^2}{\frac{x^2}{\sqrt{1-x^2}-1}+2}. \tag{40}$$

We will try the same approximation,

$$f_2(x)\Big|_{x\to 0} = \frac{x^2}{\frac{x^2}{\left(1-\frac{x^2}{2}\right)-1}+2} \Rightarrow \frac{x^2}{-2+2} \Rightarrow \frac{0}{0}. \tag{41}$$

with poor results. We will try a higher order substitution for the square-root

$$f_2\left(x\right)\Big|_{x\to 0} = \frac{x^2}{\dfrac{x^2}{\left(1-\dfrac{x^2}{2}-\dfrac{x^4}{8}\right)-1}+2}, \tag{42}$$

$$f_2\left(x\right)\Big|_{x\to 0} = \frac{x^2}{\dfrac{x^2}{-\dfrac{x^2}{2}-\dfrac{x^4}{8}}+2}, \tag{43}$$

$$f_2\left(x\right)\Big|_{x\to 0} = \frac{x^2}{\dfrac{x^2-2\left(\dfrac{x^2}{2}+\dfrac{x^4}{8}\right)}{-\dfrac{x^2}{2}-\dfrac{x^4}{8}}} = \frac{x^2}{\dfrac{\dfrac{x^4}{4}}{-\dfrac{x^2}{2}-\dfrac{x^4}{8}}} = \frac{x^2}{\dfrac{\dfrac{x^2}{4}}{-\dfrac{1}{2}-\dfrac{x^2}{8}}} = \frac{x^2}{\dfrac{-x^2}{2}} = -2. \tag{44}$$

We therefore have

$$a_2 = f_2\left(x\right)\Big|_{x\to 0} = -2, \tag{45}$$

giving the first three terms of the continued fraction expansion

$$\sqrt{1-x^2} \approx \left[1:-2,-2\right]\Big|_{x^2}. \tag{46}$$

It appears that solving for the higher order values for a_i will require progressively more algebra, and one may guess that the solution will probably be quite straightforward; indeed, the solution is

$$\sqrt{1-x^2} \approx \left[1:-2,-2,-2,-2,-2,-2,...\right]\Big|_{x^2}, \tag{47}$$

which is more predictable that the power series expansion of Equation (31).

Using the first three terms, the expansion gives:

$$\sqrt{1-x^2} \approx 1-\frac{x^2}{2-\dfrac{x^2}{2-\dfrac{x^2}{2}}} = 1-\frac{x^2}{2-\dfrac{2x^2}{4-x^2}} = 1-\frac{4x^2-x^4}{8-4x^2} = \frac{8-8x^2+x^4}{8-4x^2}, \tag{48}$$

which leads to the 60-degree finite-difference solution to the wave-equation.

The rational Equation (48) may be expanded by division into the following power series:

$$\frac{8-8x^2+x^4}{8-4x^2}=1-\frac{1}{2}x^2-\frac{1}{8}x^4-\frac{1}{16}x^6-\frac{1}{32}x^8-\frac{1}{64}x^{10}-\ldots, \tag{49}$$

which may be compared to the original Taylor series expansion of the square-root Equation (31) . The first four coefficients are identical, with the next few coefficients reasonably close.

A second approach

A second method by Lee and Suh (1985), that is similar to the recursive form described in Equation (13), uses variables Y and S in the equation

$$Y^2+2Y+1=\left(Y+1\right)^2=1+S\,. \tag{50}$$

Removing the unit values we get

$$Y^2+2Y=Y\left(Y+2\right)=S\,, \tag{51}$$

or

$$Y=\frac{S}{2+Y}\,. \tag{52}$$

This basic form may be expanded into a continued fraction expansion form by replacing the right hand value of Y by the entire left-hand side producing:

$$Y=\cfrac{S}{2+\cfrac{S}{2+Y}}=\cfrac{S}{2+\cfrac{S}{2+\cfrac{S}{2+\cfrac{S}{2+Y}}}}\,. \tag{53}$$

From Equation (50),

$$Y+1=\left(1+S\right)^{1/2}, \tag{54}$$

and then taking the first two terms of the Taylor series expansion of the square-root, we get an approximation for Y as

$$Y+1\approx 1+\frac{S}{2} \tag{55}$$

or

$$Y\approx\frac{S}{2}\,. \tag{56}$$

Inserting this value in the recursive form of Equation (53) we see that the expansion can be defined to any accuracy. However we will return to Equation (52) and express it as a recursive formula,

$$Y_{n+1} = \frac{S}{2+Y_n}, \tag{57}$$

where we choose Y_0 to be zero. We can then see that a value for n = 2 becomes

$$Y_3 = \frac{S}{2+\dfrac{S}{2+\dfrac{S}{2}}}, \tag{58}$$

which is the same form of Equation (48). All we need to do is replace the variables Y and S with

$$Y+1 = \sqrt{1-x^2} \quad \text{and} \quad S = -x^2 \tag{59}$$

to be identical to Equation (48), i.e.,

$$\sqrt{1-x^2} \approx \frac{8-8x^2+x^4}{8-4x^2}. \tag{60}$$

A third approach

The coefficients in the rational expression of Equation (48) could also have been found by replacing the numerical values in Equation (49) with variables, and equating the coefficients in the resulting power series of Equation (49),

$$\frac{1+ax^2+bx^4}{1+cx^2} = 1+(a-c)x^2+[b-(a-c)]x^4-c[b-(a-c)]x^6+c^2[b-(a-c)]x^8\ldots, \tag{61}$$

with those of the Taylor series in Equation (31), i.e.,

$$\begin{aligned} &1+(a-c)x^2+[b-(a-c)]x^4-c[b-(a-c)]x^6+c^2[b-(a-c)]x^8\ldots \\ &\approx 1 \quad -\frac{1}{2}x^2 \qquad -\frac{1}{8}x^4 \qquad -\frac{1}{16}x^6 \qquad -\frac{5}{128}x^8\ldots \end{aligned}. \tag{62}$$

Solving for a, b, and c, from the first three terms of the series we get, $a=-1$, $b=\frac{1}{8}$ and $c=\frac{-1}{2}$, giving

$$\sqrt{1-x^2} \approx \frac{1-x^2+\dfrac{x^4}{8}}{1-\dfrac{x^2}{2}}, \tag{63}$$

which is identical to Equation (48).

Improvements to equalize the error

The rational approximations to the square-root can be improved with one more step. The error in the approximation of Equation (48) increases with increasing *x*. It is possible to modify these coefficients so that the error is constrained over a chosen interval of *x*, say $0 < x < 0.5$. Two approaches are possible. The first method is presented in the paper by Lee and Suh (1985), and uses a least-squares approximation method. Results from this work are included in Yilmaz (2001).

A second method approximates a truncated polynomial with Chebychev polynomials and is discussed in Canahan et al. (1969).

The standard two-term or 45-degree approximation may be defined from two continued fraction expansions,

$$(1-x^2)^{\frac{1}{2}} \approx \frac{4-3x^2}{4-x^2} = \frac{1-.75x^2}{1-.25x^2}. \tag{64}$$

This equation can be improved substantially to a 60-degree approximation by using the following coefficients,

$$(1-x^2)^{\frac{1}{2}} \approx \frac{2.484-2.163x^2}{2.488-x^2} = \frac{0.998-0.869x^2}{1.0-0.402x^2}. \tag{65}$$

Improvements to other approximations allow, for example, a 60-degree approximation to achieve an accuracy equivalent to 80 degrees.

Historical glimpse of continued fraction expansion (from Barrow's web site below)

Continued fractions first appeared in the works of the Indian mathematician Aryabhata in the 6th century. He used them to solve linear equations. They re-emerged in Europe in the 15th and 16th centuries and Fibonacci attempted to define them in a general way. The term "continued fraction" first appeared in 1653 in an edition of the book Arithmetica Infinitorum by the Oxford mathematician, John Wallis. Their properties were also much studied by one of Wallis's English contemporaries, William Brouncker, who along with Wallis, was one of the founders of the Royal Society. At about the same time, the famous Dutch mathematical physicist, Christiaan Huygens made practical use of continued fractions in building scientific instruments. Later, in the eighteenth and early nineteenth centuries, Gauss and Euler explored many of their deep properties.

CONCLUSIONS

A simplified description of continued fraction expansion was presented as background for forming finite-difference solutions to the wave equation.

REFERENCES

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Lee, M. W., and Suh, S. Y., 1985, Optimization of one-way wave equations: Geophysics, **50**, 1634–1637.
Yilmaz, O., 2001, Seismic Data Analysis, Vol. 1: SEG.

The following two web sites provided information used in this paper and are excellent sources of information.

http://home.att.net/~numericana/answer/fractions.htm#continued G. P. Michon

http://pass.maths.org.uk/issue11/features/cfractions J. D. Barrow