Quasi-compressional group velocity approximations in a weakly anisotropic orthorhombic medium

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ABSTRACT

Using modifications to the standard linearized approximation of the phase velocity for quasi-compressional *(qP)* wave propagation in a weakly anisotropic orthorhombic medium, two approximate eikonal equations are constructed. Corresponding expressions for the group velocities are then derived. In the first approximation, the degenerate (ellipsoidal) case of *qP* wave propagation in an orthorhombic medium is examined and an exact group velocity expression obtained, together with the exact expressions for the slowness vector components, for this simple case. This ellipsoidal group velocity is taken as the reference or background velocity surface and is employed as a trial solution in the first approximate eikonal equation, where the resultant group velocity surface is shown to be a perturbed ellipsoid. All formulae are in terms of angles related to the group velocity vector. As in the solution method used for the first approximate *qP* eikonal equation, the method of characteristics is employed in obtaining a group velocity approximate expression using another related linearized eikonal equation. The result is a more complex expression for the group velocity vector components. For completeness, analytic expressions for the exact components of the group velocity vector are presented. The group velocity expressions, approximate versus exact, are numerically compared for two orthorhombic anisotropic models that may be classified as weakly anisotropic or, possibly more accurately, weakly anellipsoidal, as the background group velocity is an ellipsoid. The extension of what is presented here to more complex anisotropic structures can be achieved in a similar manner.

INTRODUCTION

In the recent literature on wave propagation in anisotropic media, specifically quasi-compressional *(qP)* waves in a medium displaying orthorhombic symmetry (for example; Song and Every 2000, Song et al. 2001), a number of approximate techniques, usually based on perturbation theory, have been used to advance the understanding of wave propagation in these complex anisotropic structures. The motivation for this is that the exact analytical expressions for quantities such as eikonal equations, phase and group velocities and polarization vectors are so complex that they usually reveal inadequate information when attempting to determine their significance. The more general linearized anisotropic problem is dealt with in Jech and Pšenčík (1989), Pšenčík and Gajewski (1998) and Every and Sachse (1992) and other cited references. Explicit expressions for *qP* ray tracing, yielding linearized group velocity approximations, in the general as well as subset media types, including orthorhombic, may be found in Pšenčík and Farra (2005). In that paper, an isotropic background medium is assumed. As the exact solution for the linearized problem may be determined for an ellipsoid, it is this that will be used as a reference velocity surface or background medium. The statement in Mensch and Farra (1999), " Examples obtained in a homogeneous orthorhombic medium show that a reference media with ellipsoidal anisotropy is a better choice to develop the perturbation approach than an isotropic reference medium," gives further indication that that this is a reasonable manner in which to proceed.

A rewriting of the linearized eikonal equation describing quasi-compressional *(qP)* wave propagation in an orthorhombic medium is presented, from which approximations of the group velocity in that medium type are derived. The method of characteristics (Courant and Hilbert, 1962) is employed, if only in part, in the solution process to obtain *qP* group velocity approximate expressions.

To establish the accuracy of the approximations, the exact expressions for the group velocity vector components in a general orthorhombic medium, the exact *qP* phase velocity expression and hence eikonal ((Every, 1980 and Schoenberg and Helbig, 1997) is employed. For simplicity, but without much loss of generality, the medium of propagation is assumed to be homogeneous, i.e., the anisotropic elastic parameters are independent of the spatial coordinates. The indication of spatial dependence will be retained in what follows except in those instances where inclusion would be incorrect or misleading.

Often in the literature where the topic of orthorhombic anisotropy is addressed, comparisons of approximations to phase velocities, slowness surfaces, group velocities and polarization vectors with exact expressions are done in symmetry planes. This essentially amounts to reducing the problem to wave propagation in a transversely isotropic medium for which comprehensible exact expressions may be found, for example, Gassmann (1965). There is merit in doing this, since, if the approximate and exact formulae do not reasonably coincide in this instance, it is highly probable that at some arbitrary azimuth, removed from a symmetry plane, the fit between the two will deteriorate past some acceptable level. For that reason the focus here will be on checking the group velocity approximations with the exact formulae both in and out of symmetry planes.

THEORETICAL PRELIMINARIES

Determining the linearized phase velocity in an anisotropic medium of orthorhombic symmetry for a quasi-compressional *(qP)* wave using the method given in Backus (1965) yields the following first order linearized approximation in a weakly anisotropic medium, which is the standard that appears in the literature for a range of disciplines of study

$$\begin{aligned} v_{qP}^2(n_k) = {} & A_{11}n_1^4 + A_{22}n_2^4 + A_{33}n_3^4 + \\ & 2(A_{12}+2A_{66})n_1^2n_2^2 + 2(A_{13}+2A_{55})n_1^2n_3^2 + 2(A_{23}+2A_{44})n_2^2n_3^2 . \end{aligned} \tag{1}$$

In Voigt notation, the density normalized elastic anisotropic parameters, A_{ij}, have the dimensions of velocity squared and **n** is a unit 3D vector in the direction of the wave front normal or equivalently the phase velocity propagation vector direction, defined as

$$\mathbf{n} = (n_1, n_2, n_3) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{2}$$

where θ is the polar angle measured from the positive x_3 (vertical) axis $(0 \le \theta \le \pi)$ and ϕ the azimuthal angle measured in a positive sense from the x_1 axis $(0 \le \phi < 2\pi)$.

To put equation (1) in a form that has been found to be more useful and instructive, add to and subtract from it the quantity (Daley and Krebes, 2004a)

$$n_1^2 n_2^2 (A_{11} + A_{22}) + n_1^2 n_3^2 (A_{11} + A_{33}) + n_2^2 n_3^2 (A_{22} + A_{33}). \tag{3}$$

After some manipulation, the following formula, quadratic in the terms involving n_i, results

$$v_{qP}^2 (n_k) = A_{11} n_1^2 + A_{22} n_2^2 + A_{33} n_3^2 + E_{12} n_1^2 n_2^2 + E_{13} n_1^2 n_3^2 + E_{23} n_2^2 n_3^2 \tag{4}$$

with the quantities E_{ij} being the linearized forms of the anellipsoidal deviation terms in the $x_i x_j - \text{plane}$ or equivalently in the $p_i p_j - (\text{slowness})\,\text{plane}$ and defined as

$$E_{12} = 2(A_{12} + 2A_{66}) - (A_{11} + A_{22}) \tag{5}$$

$$E_{13} = 2(A_{13} + 2A_{55}) - (A_{11} + A_{33}) \tag{6}$$

$$E_{23} = 2(A_{23} + 2A_{44}) - (A_{22} + A_{33}). \tag{7}$$

The components of the slowness vector, p_i, are defined in terms of the *qP* phase (wave front normal) vector components, n_i, and phase velocity as

$$\mathbf{p} = (p_1, p_2, p_3) = [v_{qP} (n_k)]^{-1} (n_1, n_2, n_3), \tag{8}$$

so that the specification of a pseudo eikonal equation may be given as

$$\begin{aligned} G_{qP} (x_k, p_k, n_k) = 1 = {} & A_{11} p_1^2 + A_{22} p_2^2 + A_{33} p_3^2 + \\ & E_{12} [p_1 p_2][n_1 n_2] + E_{13} [p_1 p_3][n_1 n_3] + E_{23} [p_2 p_3][n_2 n_3], \end{aligned} \tag{9}$$

where the equal signs have to be taken within the context that an approximation is being considered. The above equation can be put in a form that is only a function of (x_k, p_k) with the introduction of the identities $n_1^2 + n_2^2 + n_3^2 = 1$ and $v_{qP}^2 (x_k, n_k) / v_{qP}^2 (x_k, n_k) = 1$. The resulting linearized *qP* eikonal equation, obtained after minor rearrangement, is

$$G_{qP}(x_k, p_k) = 1 = A_{11}p_1^2 + A_{22}p_2^2 + A_{33}p_3^2 + \left[\frac{E_{13}p_1^2p_3^2 + E_{12}p_1^2p_2^2 + E_{23}p_2^2p_3^2}{p_1^2 + p_2^2 + p_3^2}\right]. \tag{10}$$

The method of characteristics (Courant and Hilbert, 1962) is used to determine the rays, along which the energy traverses between one point in the medium and another. In the degenerate ellipsoidal case, where the 3 symmetry plane anellipsoidal coefficients are identically zero, the *qP* eikonal becomes

$$G_{qP}(x_k, p_k) = 1 = A_{11}p_1^2 + A_{22}p_2^2 + A_{33}p_3^2. \tag{11}$$

The group (ray)velocity vector and corresponding slowness vector components are given generally in terms of some eikonal equation, $G(x_k, p_k)$, by

$$\frac{dx_i}{dt} = \frac{1}{2}\frac{\partial G(x_k, p_k)}{\partial p_i} \tag{12}$$

$$\frac{dp_i}{dt} = -\frac{1}{2}\frac{\partial G(x_k, p_k)}{\partial x_i}. \tag{13}$$

An initial value problem is fully specified, given some initial conditions

$$\mathbf{x_0} = \mathbf{x}(t_o) \text{ and } \mathbf{p_0} = \mathbf{p}(t_o) \tag{14}$$

at a reference time t_0. The progression of the ray in 3D Cartesian space as well as the magnitude and direction of the slowness vector at these points may be determined. In what follows the elastic anisotropic parameters have been assumed to be spatially independent, therefore $dp_i/dt = 0$. The initial conditions on $\mathbf{p}$ require that $\mathbf{p}_0 = \mathbf{p}(t_0) = \mathbf{p}(t) =$ some constant for all *t*. With this in mind, the group velocity in terms of its components may be given as

$$\frac{d\mathbf{x}}{dt} = \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \frac{dx_3}{dt}\right) = (A_{11}p_1, A_{22}p_2, A_{33}p_3), \tag{15}$$

with the magnitude defined by

$$\left|\frac{d\mathbf{x}}{dt}\right| = \left[\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2 + \left(\frac{dx_3}{dt}\right)^2\right]^{1/2} = \left[A_{11}^2p_1^2 + A_{22}^2p_2^2 + A_{33}^2p_3^2\right]^{1/2}. \tag{16}$$

It is convenient to introduce the group velocity angles, that is, the azimuthal and polar angles at which the ray propagates. The azimuthal angle, Φ, $(0 \le \Phi < 2\pi)$ may be determined from

$$\tan\Phi = \left[\frac{dx_2}{dx_1}\right] = \left[\frac{dx_2/dt}{dx_1/dt}\right] = \frac{A_{22}p_2}{A_{11}p_1} = \frac{A_{22}}{A_{11}}\tan\phi \quad . \tag{17}$$

Defining the projection of the 3D group velocity vector onto the (x_1, x_2) plane as

$$\frac{dr}{dt} = \left[\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_2}{dt}\right)^2\right]^{1/2} = \left[A_{11}^2 p_1^2 + A_{22}^2 p_2^2\right]^{1/2} \tag{18}$$

the group polar angle, Θ, $(0 \le \Theta \le \pi)$ is given by

$$\tan\Theta = \left[\frac{dr}{dx_3}\right] = \left[\frac{dr/dt}{dx_3/dt}\right] = \frac{\left[A_{11}^2 p_1^2 + A_{22}^2 p_2^2\right]^{1/2}}{A_{33}p_3}$$
$$= \frac{A_{11}\tan\theta\cos\phi\left[1 + \left(A_{22}/A_{11}\right)^2\tan^2\phi\right]^{1/2}}{A_{33}} \quad . \tag{19}$$

After a moderate amount of algebra involving basic trigonometric manipulations, first solving equation (17) for Φ, and then substituting the result into equation (19) to obtain Θ, expressions for the components of the slowness vector, p, may be obtained in terms of the group rather than the phase angles and velocity. Defining a unit vector in the direction of ray propagation as

$$\mathbf{N} = (N_1, N_2, N_2) = (\sin\Theta\cos\Phi, \sin\Theta\sin\Phi, \cos\Theta), \tag{20}$$

the magnitude of the *qP* group velocity for the degenerate (ellipsoidal) orthorhombic medium is

$$\frac{1}{V_{qP}^2(\Theta,\Phi)} = \frac{N_1^2}{A_{11}} + \frac{N_2^2}{A_{22}} + \frac{N_3^2}{A_{33}} \quad . \tag{21}$$

From the above relations the phase slowness vector in this special case may be written completely in terms of group velocity and angles as

$$\mathbf{p} = \left(p_1^e, p_2^e, p_3^e\right) = \left(\frac{V_{qP}^e(\Theta,\Phi)N_1}{A_{11}}, \frac{V_{qP}^e(\Theta,\Phi)N_2}{A_{22}}, \frac{V_{qP}^e(\Theta,\Phi)N_3}{A_{33}}\right). \tag{22}$$

This solution will be used as an initial approximation, or trial solution, in an approximate eikonal for the more general case of the quasi-compressional (qP) group velocity in a general, weakly anellipsoidal orthorhombic medium.

GROUP VELOCITY APPROXIMATIONS

In this section two different routes will be investigated to find *qP* group velocity estimates using linearized approximations to the exact eikonal equation. Both approaches involve the use of the method of characteristics. In the first case it has already been employed to derive analytical expressions for the group velocity and slowness vector in the degenerate (ellipsoidal) orthorhombic *qP* problem, where the anellipsoidal terms E_{13}, E_{12}, and E_{23}, were assumed to be equal to zero.

The ellipsoidal phase velocity is

$$\left[v_{qP}^2\left(x_k, p_k\right)\right]_e = A_{11}n_1^2 + A_{22}n_2^2 + A_{33}n_3^2 \tag{23}$$

and the general linearized phase velocity may be recovered as

$$\begin{aligned} G_{qP}\left(x_k,n_k\right) = v_{qP}^2\left(x_k,n_k\right) = \left[v_{qP}^2\left(x_k,n_k\right)\right]_e \Big[1+E_{12}\left[p_1p_2\right]\left[n_1n_2\right]+ \\ E_{13}\left[p_1p_3\right]\left[n_1n_3\right]+E_{23}\left[p_2p_3\right]\left[n_2n_3\right]\Big]_e \\ =\left[A_{11}n_1^2 + A_{22}n_2^2 + A_{33}n_3^2 + +E_{12}n_1^2n_2^2 + E_{13}n_1^2n_3^2 + E_{23}n_2^2n_3^2\right] \end{aligned} \tag{24}$$

where the subscript "e" denotes ellipsoidal and the constraints that the n_k have the ellipsoidal angular values has been removed. In an equivalent manner the general linearized group velocity may be written as

$$\begin{aligned} G\left(x_k,N_k\right) = \frac{1}{V_{qP}^2\left(x_k,N_k\right)} = \left[\frac{1}{V_{qP}^2\left(x_k,N_k\right)}\right]_e \times \\ \left[\frac{1}{1+E_{12}\left[p_1p_2\right]\left[n_1n_2\right]+E_{13}\left[p_1p_3\right]\left[n_1n_3\right]+E_{23}\left[p_2p_3\right]\left[n_2n_3\right]}\right]_e . \end{aligned} \tag{25}$$

Rewriting equation (25) using the approximation $\left(1+a_{ij}\right)^{-1} \approx \left(1-a_{ij}\right)$, introducing the definitions of p_k presented in equation (22), and as in the phase velocity case relaxing the constraints on N_k, results in

$$\begin{aligned} \frac{1}{V_{qP}^2\left(\Theta,\Phi\right)} \approx \Bigg[\frac{N_1^2}{A_{11}} + \frac{N_2^2}{A_{22}} + \frac{N_3^2}{A_{33}} - E_{12}\frac{N_1N_2}{A_{11}A_{22}}\left[n_1n_2\right] - \\ E_{13}\frac{N_1N_3}{A_{11}A_{33}}\left[n_1n_3\right] - E_{23}\frac{N_2N_3}{A_{22}A_{33}}\left[n_2n_3\right]\Bigg] . \end{aligned} \tag{26}$$

Consider now the possibility that $n_in_j \approx N_iN_j$. In the initial linearization process the phase vector components were used to approximate the components of the polarization vector, which in general is not aligned with either the phase or group unit vectors. At that point, the group vector components could have been used as they would serve just as well in approximating the polarization vector components. However, the use of the phase vector components was more convenient in the initial stages of the problem, being the only known quantities. Introducing the above replacement into equation (26) yields the following approximation for the *qP* group velocity in an orthorhombic medium as a function of the related group vector angles

$$\frac{1}{V^2(\Theta,\Phi)} = \frac{N_1^2}{A_{11}} + \frac{N_2^2}{A_{22}} + \frac{N_3^2}{A_{33}} - \frac{E_{12}N_1^2N_2^2}{A_{11}A_{22}} - \frac{E_{13}N_1^2N_3^2}{A_{11}A_{33}} - \frac{E_{23}N_1^2N_2^2}{A_{22}A_{33}} \,. \qquad (27)$$

The perturbed velocity derivation above results from the fact that in ray propagation space, for some given ray, the vector beginning at the origin of the ray surface and normal to the tangent plane associated with the point at which the ray touches the ray surface is the phase velocity vector, $\mathbf{v}_{qP}(n_k) = [\mathbf{p}]^{-1}$. Equivalently, in slowness space, $\left(\mathbf{p} = \left[\mathbf{v}_{qP}(n_k)\right]^{-1}\right)$, for an arbitrary slowness vector, the vector originating at the slowness surface origin and normal to the tangent plane at the point at which the slowness vector contacts the slowness surface is the group velocity vector inverse, $\left[\mathbf{V}_{qP}(n_k)\right]^{-1}$.

Formula (27) is in agreement with that presented by Song and Every (2000) where these results were "… not established … by rigorous derivation but we were lead to [them] by plausibility arguments that are backed up by the numerical results …". As indicated in the quote, the formulae derived by Song and Every (2000) are quite comparable to those obtained by exact methods if they are used in the situations for which they were derived - weakly anellipsoidal orthorhombic media. The advantage of the expression derived above for the *qP* group velocity is that it is in terms of group angles rather than wave front normal vector components or equivalently phase velocity angles. An alternate group velocity approximation may be constructed from the eikonal equation (10), in a similar manner as was equation (27), by introducing the group angle dependent slowness vector components from equation (23). This possibility is not considered here as equation (27) is the simplest of a number of variants that may be derived in the preceding manner.

The second method that will be presented for obtaining an approximation for the *qP* group velocity in an orthorhombic medium is, as would be expected, a bit more complex and results in a more complicated expression than equation (27). As it has already been established that equation (27) produces reasonable results, the motivation for pursuing this further is that it is presumed that for azimuths not aligned with symmetry axes a more accurate approximation might be required to more adequately approximate the exact *qP* group velocity.

Again employing equations (12) and (13), where the eikonal equation is given by equation (10), the following initial value problem consisting of a system of ordinary differential equations are obtained with the initial conditions specified in a manner similar to equation (14). The assumption that the elastic anisotropic parameters are spatially independent (p_i = constant; i = 1,2,3) will again be made. The following expressions are obtained for the components of the qP group velocity.

$$\frac{dx_1}{dt} = p_1 A_{11} (1 + F_1 / A_{11}) \tag{28}$$

$$F_1 = \frac{1}{(p_k p_k)^2} [(E_{13} p_3^2 + E_{12} p_2^2)(p_2^2 + p_3^2) - E_{23} p_2^2 p_3^2] \tag{29}$$

$$\frac{dx_2}{dt} = p_2 A_{22} (1 + F_2 / A_{22}) \tag{30}$$

$$F_2 = \frac{1}{(p_k p_k)^2} [(E_{12} p_1^2 + E_{23} p_3^2)(p_1^2 + p_3^2) - E_{13} p_1^2 p_3^2] \tag{31}$$

$$\frac{dx_3}{dt} = p_3 A_{33} (1 + F_3 / A_{33}) \tag{32}$$

$$F_3 = \frac{1}{(p_k p_k)^2} [(E_{13} p_1^2 + E_{23} p_2^2)(p_1^2 + p_2^2) - E_{12} p_1^2 p_2^2] \tag{33}$$

with $p_k p_k = p_1^2 + p_2^2 + p_3^2$. The scalar expression for the group velocity is

$$\left| \frac{d\mathbf{x}}{dt} \right| = \left[\left(\frac{dx_1}{dt} \right)^2 + \left(\frac{dx_2}{dt} \right)^2 + \left(\frac{dx_3}{dt} \right)^2 \right]^{1/2} . \tag{34}$$

At this point it might seem appropriate to introduce the approximation

$$(1 + F_i / A_{ii})^2 \approx 1 + 2F_i / A_{ii} \; (i = 1,2,3) \tag{35}$$

under the assumption that the dimensionally equivalent parameters F_i and A_{ii} are such that $F_i << A_{ii}$. Little would be gained by this additional level of approximation, in either computational efficiency or analytic simplification.

The azimuthal and polar group velocity (ray) direction angles, Φ and Θ, may be obtained using equations (17) – (19). However, only the numerical values of dx_i / dt are known here.

For completeness, the problem of exact ray tracing for qP rays in an orthorhombic medium should be addressed and may be found in Appendix A. The motivation for including this was that nothing comparable, which only computed group velocities, could be found in the literature or public domain software.

NUMERICAL RESULTS

The first model that will be considered is the weakly anellipsoidal orthorhombic material, whose anisotropic properties are similar in degree of anisotropy to transversely isotropic clay-shale associated with hydrocarbon deposits that could have been made orthorhombic (azimuthally anisotropic) through the introduction of vertical fracturing. The model is defined by the density normalized anisotropic parameters, A_{ij}, which have the dimensions of velocity squared $(km/s)^2$ and are given in Table 1. The second model is another weakly anellipsoidal orthorhombic medium that is a modification of a dry sandstone model used in the paper by Pšenčík and Gajewski (1998). Only those anisotropic parameters of the 21 in all that define an orthorhombic medium are retained. These values may also be found in Table 1. As an indication of the degree of anisotropy (deviation from the ellipsoidal case), the dimensionless quantities $E_{ij}/2A_{33}$ are given in Table 2. These values are equivalent to $\delta_{ij}-\varepsilon_{ij}$ in a slight modification of the notation introduced by Thomsen (1986).

The group velocities for each of these two models are computed at an azimuth angle of $\phi=\Phi=0$ degrees, in the x_1x_3 symmetry plane, and at phase azimuthal angles of $\phi=30$, 45 and 60 degrees, which are not in symmetry planes and, although nearly the same in value do not, in general correspond to group angles of $\Phi=30$, 45 and 60 degrees. The variation of Φ with ϕ is related to the degree of anisotropy. These angles are measured from the positive x_1 axis. The inclusion of the symmetry plane results for $\phi=\Phi=0$ is to provide a reference comparison from which to determine the quality of fit in the non-symmetry plane examples. The two approximations A_1 (equation (27) and A_2 (equations (28) – (34)) and the exact, *E*, group velocity are compared in Figures 1 through 4 for a polar angle range of 0 to 360 degrees for the two models described above at the azimuthal phase angles specified. The curves in the plots are annotated on the panels; red indicating the exact solution and blue the approximate solution, which in Figures 1 and 2 refers to A_2 and in Figures 3 and 4 to A_1. The group angle inputs for the approximation A_1 are obtained from those angles, computed numerically, and resulting from the phase angle input of the exact solution.

The plotting of the group velocity curves is not done in polar plots but rather in a manner that enhances the differences between exact and approximate group velocity computations. The polar angle Θ is measured from the vertical, or x_3, axis. It is quite evident upon viewing the four figures that for weakly anisotropic media, the match between the approximations and the exact solution are quite reasonable, which is a subjective observation as the fit required is most often problem specific.

The numerical measure of deviation, D_P, given in Tables 3 and 4, is the average deviation of a given approximate group velocity expression $(V_{app.})$ from the exact value (V_{exact}) over a 360 degree polar angle range at *N* equally spaced points obtained using the formula

$$D_p = \left[\frac{1}{N} \sum_{j=1}^{N} \frac{\left| V_{exact}^{(j)} - V_{app.}^{(j)} \right|}{V_{exact}^{(j)}} \right] \times 100\% . \qquad (36)$$

CONCLUSIONS

Two quasi-compressional *(qP)* group velocity approximations for elastic wave propagation in an orthorhombic medium have been presented. The solution methods were facilitated by modifying the standard form of the linearized eikonal equation that is found in the literature for this medium and wave type. The eikonal equation is first put in a form such that the background slowness surface, and hence the group velocity surface, is an ellipsoid, with anellipsoidal correction terms added in each of the three symmetry planes. This rewriting of the eikonal equation has the effect of allowing the group velocity and slowness vector components for the degenerate (ellipsoidal) case to be determined analytically in the first approximate method, using the method of characteristics, and as functions of group rather than phase angles. In this approximation, the exact solution for this degenerate case was then used as a trial solution to obtain the group velocity approximation for the anellipsoidal case. As both approximations have analytic solutions when the anellipsoidal terms, E_{ij}, are zero, they have been referred to as "weakly anellipsoidal" rather than "weakly anisotropic". The second approximation was obtained using the same eikonal equation as in the first case but introducing some algebraic manipulations so that the full *qP* eikonal equation was a homogeneous function of order two in slowness vector components. The group velocity vector components were then obtained using characteristic theory.

Comparisons of both approximations, in a symmetry plane and at phase azimuthal angles of 30, 45 and 60 degrees, with the exact group velocity expression for two realistic geological models were carried out with good matches in all instances. As with any approximate method, care must be taken not to violate the original assumptions used in its development. The models used in the previous section were selected such that they lay within the set which could be designated as "weakly anellipsoidal". In geophysical applications, this assumption is infrequently contravened to a large degree in actual geological models. The second model example presented here approaches this limit as indicated by the values of $E_{ij}/2A_{33}$ in Table 2. It was mentioned early on that the methods employed here to obtain group velocity approximations could be extended to more complex anisotropic structures. This was not pursued here but this contention should receive some validity from the methods examined here.

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Anisotropic Parameters $(\mathrm{km/s})^2$ for the Two Models Considered

Material	A_{11}	A_{22}	A_{33}	A_{44}	A_{55}	A_{66}	A_{12}	A_{13}	A_{23}
Clay-shale	19.56	11.94	14.20	3.90	4.72	4.78	4.01	4.33	4.57
Sandstone	19.30	17.40	14.10	5.10	5.50	4.60	0.90	1.30	0.20

Table 1. Density normalized anisotropic parameter specification of the models used in the text. The A_{ij} have the units of $(km/s)^2$.

Anellipsoidal Deviation Parameters

Material	E_{12}	E_{13}	E_{23}	$E_{12}/2A_{33}$	$E_{13}/2A_{33}$	$E_{23}/2A_{33}$
Clay-shale	-4.37	-6.25	-1.40	-0.154	-0.220	-0.050
Sandstone	-16.50	-8.80	-10.70	-0.585	-0.312	-.380

Table 2. Density normalized anisotropic deviation parameters for the two models. The E_{ij} have the units of $(km/s)^2$. These parameters, when normalized with respect to $2A_{33}$, are dimensionless and are equivalent to $(\delta_{ij} - \varepsilon_{ij})$ in a modification of the notation introduced by Thomsen (1986).

Percentage Deviation from the Exact Solution for Clayshale Model

Clay-shale	**0 Degrees**	**30 Degrees**	**45 Degrees**	**60 Degrees**
Approximation 1	0.0816	0.1049	0.2913	0.3541
Approximation 2	0.2603	0.5105	0.6947	0.4833

Table 3. Average percentage deviation for the clay-shale model over a 360° range, equally sampled, in the group polar angle, Θ, for azimuths of phase angles $\phi = 0$, 30, 45 and 60 degrees.

Percentage Deviation from the Exact Solution for Sandstone Model

Sandstone	**0 Degrees**	**30 Degrees**	**45 Degrees**	**60 Degrees**
Approximation 1	0.2697	1.0894	0.7075	0.7509
Approximation 2	0. 3148	1.150	0.4485	0.5011

Table 4. Average percentage deviation for the sandstone model over a 360° range, equally sampled, in the group polar angle, Θ, for azimuths of phase angles $\phi = 0$, 30, 45 and 60 degrees.

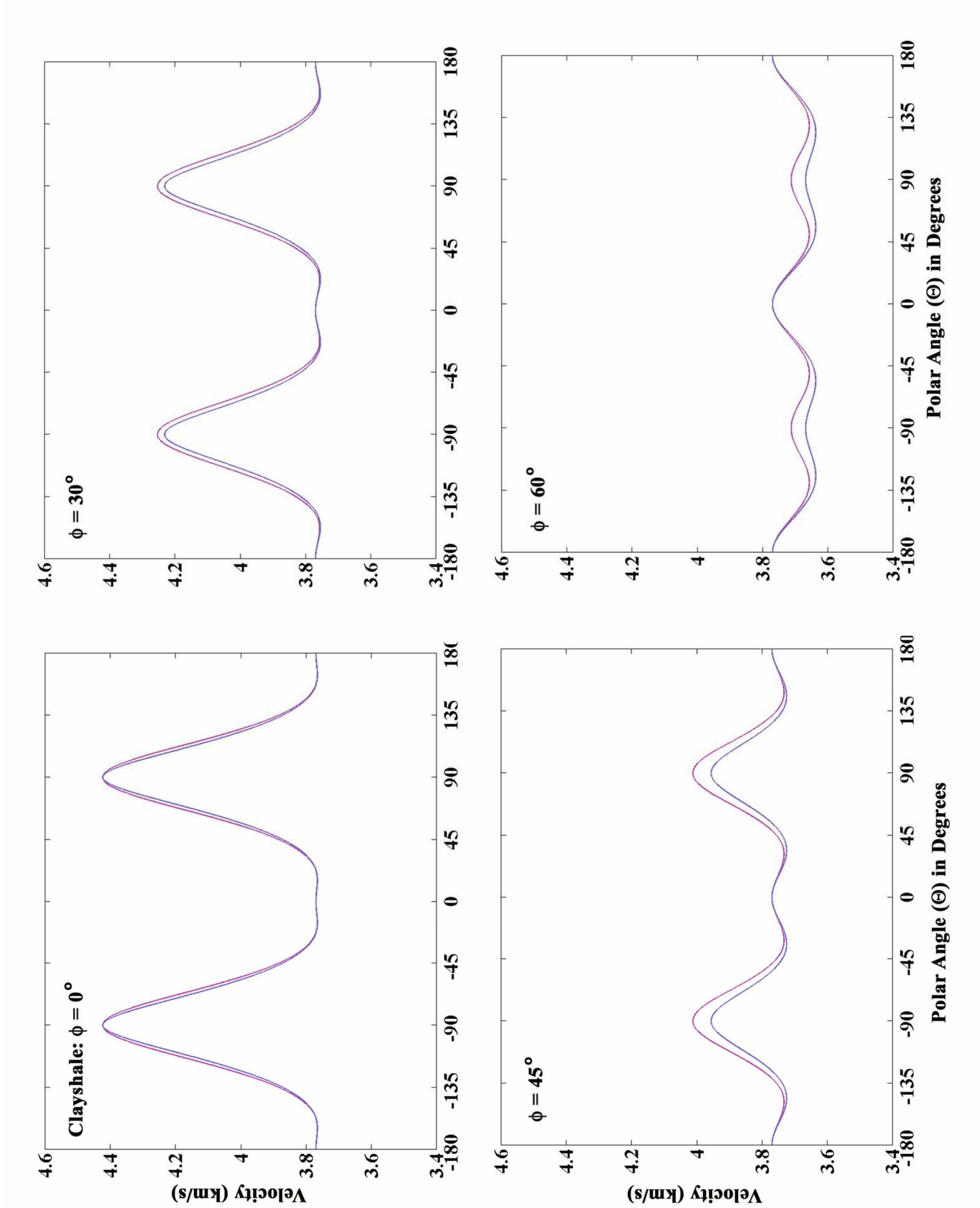

FIG. 1. Clay-shale orthorhombic model. The exact group velocity (red curve) is compared to the second approximation for the for the polar (Θ) angle range 0 to 360 degrees and shown at four different (phase) azimuth angles. The azimuth of 0 degrees coincides with the x_1x_3 symmetry plane. The other panels are at phase azimuthal angles of 30, 45 and 60 degrees, respectively, and do not correspond to symmetry planes.

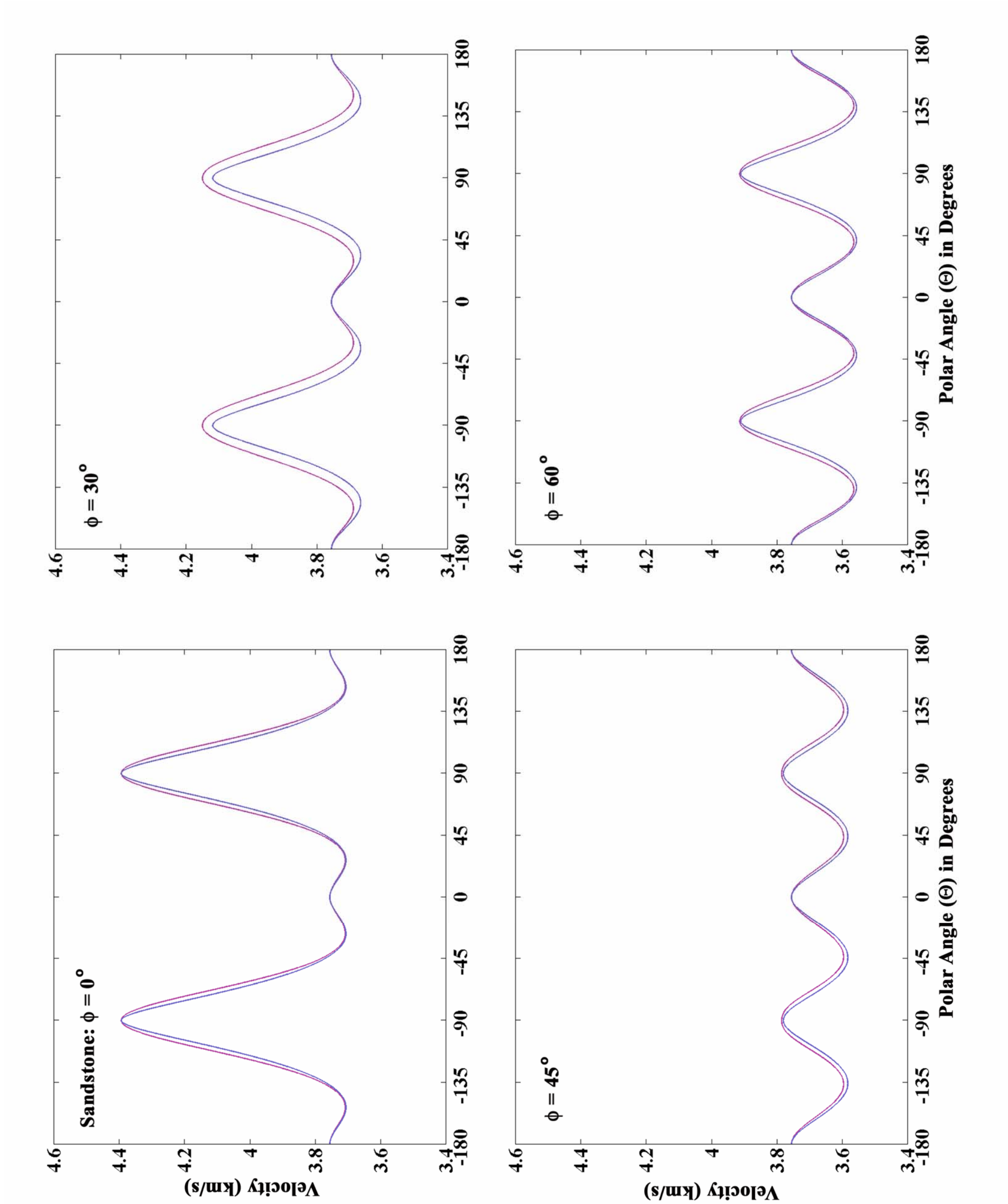

FIG. 2. Orthorhombic sandstone model. The exact group velocity (red curve) is compared to the second approximation for the for the polar (Θ) angle range 0 to 360 degrees and shown at four different (phase) azimuth angles. The azimuth of 0 degrees which coincides with the x_1x_3 symmetry plane The other panels are at phase azimuthal angles of 30, 45 and 60 degrees, respectively, and do not correspond to symmetry planes.

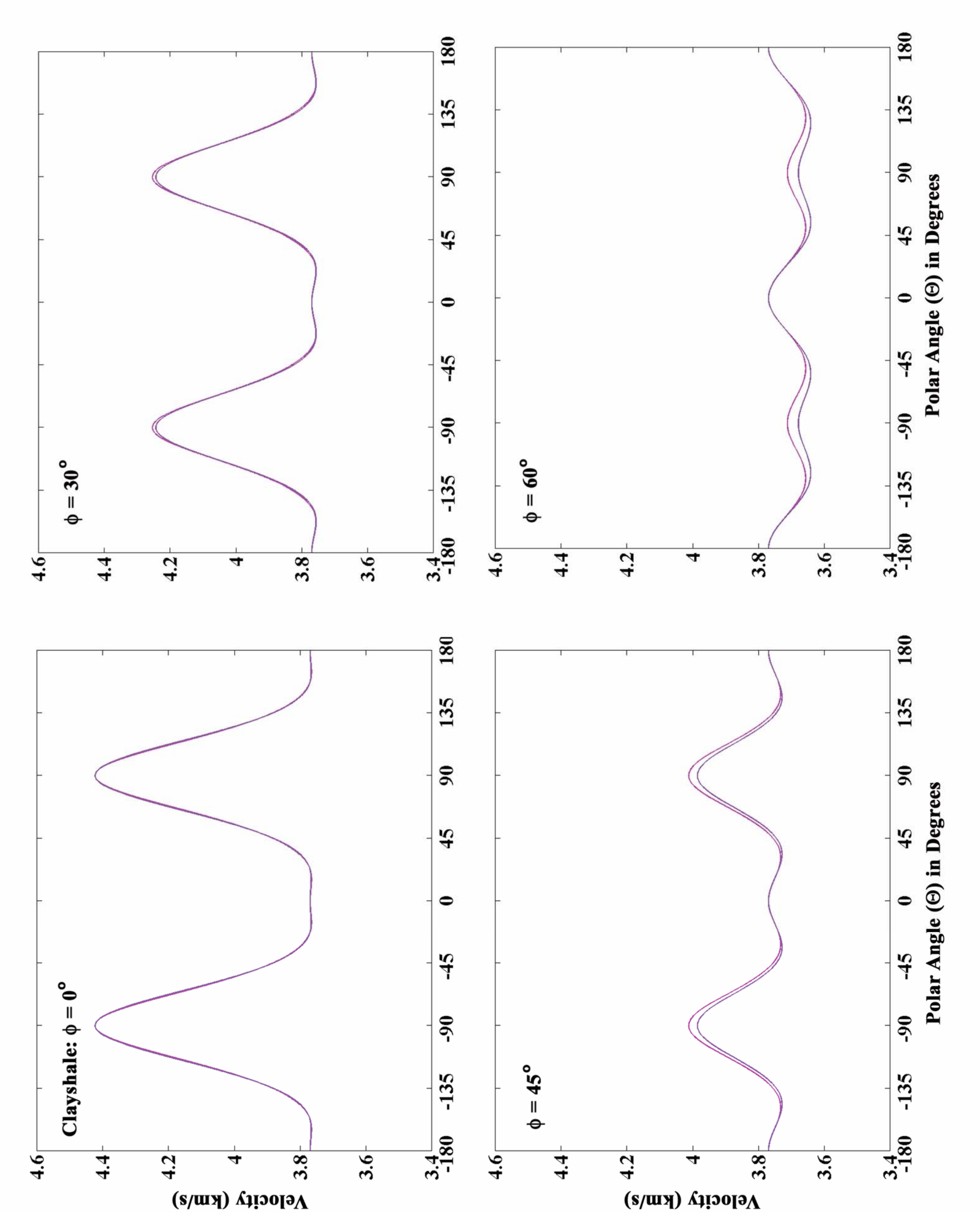

FIG. 3. Clay-shale orthorhombic model. The exact group velocity (red curve) compared with the first approximation for the for the polar (Θ) angle range 0 to 360 degrees are shown at four different (phase) azimuth angles. The azimuth of 0 degrees coincides with the x_1x_3 symmetry plane. The other panels are for phase azimuthal angles of 30, 45 and 60 degrees, respectively, and do not correspond to symmetry planes.

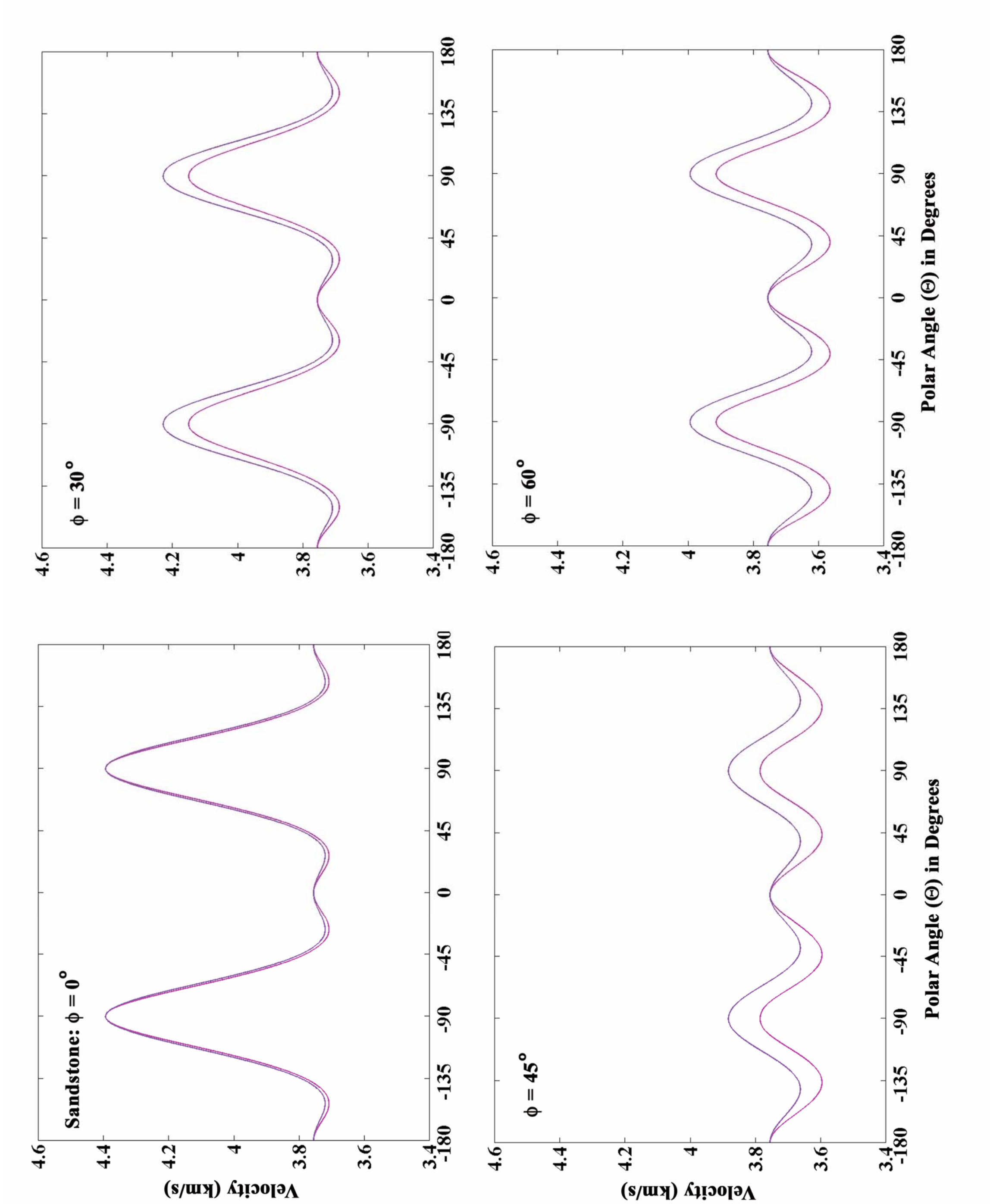

FIG. 4. Orthorhombic sandstone model. The exact group velocity (red curve) compared with the first approximation for the polar (Θ) angle range 0 to 360 degrees are shown at four different (phase) azimuthal angles. The azimuth of 0 degrees coincides with the x_1x_3 symmetry plane. The other panels are for phase azimuthal angles of 30, 45 and 60 degrees, respectively, and do not correspond to symmetry planes.

APPENDIX A: EXACT QUASI- COMPRESSIONAL RAY TRACING EQUATIONS IN AN ORTHORHOMBIC ANISOTROPIC MEDIUM

Discussions of the problem of wave propagation in a general anisotropic medium may be found in numerous texts and papers (for example, Cerveny, 2001). After substitution of a standard high frequency, asymptotic ray (geometrical optics) solution into the equation of particle motion, a condition for the existence of a solution is obtained in terms of the eigenvalues (eikonal equations) and the elements, $\Gamma_{mn}\left(m,n=1,2,3\right)$, of the Christoffel matrix $\mathbf{\Gamma}$. The elements of $\mathbf{\Gamma}$ are functions of the anisotropic parameters of the medium and are homogeneous functions of order two in powers of slowness, p_i, so that with *G* denoting the eigenvalues $\left(G\left(x_k, p_k\right)=1\right)$ the following equation is obtained

$$\Gamma_{jk} - G\,\delta_{jk} = 0 \tag{A.1}$$

As $\mathbf{\Gamma}$ is known to be a symmetric matrix (Cerveny, 2001), it is positive definite and as a consequence its three eigenvalues are real, positive and generally distinct quantities. These eigenvalues, $G_i\left(x_k, p_k\right)\left(i=1,2,3\right)$ are the solution of the above characteristic equation which results in the following cubic equation

$$G^3 - Tr\left(\mathbf{\Gamma}\right)G^2 + Tr\left[Cof\left(\mathbf{\Gamma}\right)\right]G - Det\left(\mathbf{\Gamma}\right) = 0\,. \tag{A.2}$$

where

$$Tr\left(\mathbf{\Gamma}\right) = \Gamma_{11} + \Gamma_{22} + \Gamma_{33} \tag{A.3}$$

the two invariants in equation (A.2) for *Y* are

$$Tr\left[Cof\left(\mathbf{\Gamma}\right)\right] = \Pi_{\mathbf{\Gamma}} = \Gamma_{11}\Gamma_{33} + \Gamma_{11}\Gamma_{22} + \Gamma_{22}\Gamma_{33} - \Gamma_{13}^2 - \Gamma_{12}^2 - \Gamma_{23}^2 \tag{A.4}$$

$$Det\left(\mathbf{\Gamma}\right) = \left|\mathbf{\Gamma}\right| = \Gamma_{11}\Gamma_{22}\Gamma_{33} + 2\Gamma_{12}\Gamma_{13}\Gamma_{23} - \Gamma_{11}\Gamma_{23}^2 - \Gamma_{12}^2\Gamma_{33} - \Gamma_{13}^2\Gamma_{22} \tag{A.5}$$

and *Tr* denotes trace, *Cof*, cofactor and *Det*, determinant. A variable change (Every, 1980 and Schoenberg and Helbig, 1997), which allows for the determination of analytic expressions for the three eigenvalues, is given by

$$G = Y + Tr\left(\mathbf{\Gamma}\right)/3 = 1\,. \tag{A.6}$$

Introducing equation (A.6) into (A.2) results in

$$Y^2 - 3K_1Y - 2K_2 = 0\,. \tag{A.7}$$

There is no quadratic term in equation (A.7) and as a consequence a solution method exists or this equation, provided certain constraints related to the A_{ij} are satisfied. The quantities K_1 and K_2 in equation (A.7) are defined as

$$K_1 = \left[Tr(\mathbf{\Gamma})\right]^2 \big/ 2 - Tr\left[Cof(\mathbf{\Gamma})\right] \big/ 3 \tag{A.8}$$

$$K_2 = \left[Tr(\mathbf{\Gamma})\right]^3 \big/ 27 - \left\{\left[Tr\left[Cof(\mathbf{\Gamma})\right]\right]\left[Tr(\mathbf{\Gamma})\right]\right\} \big/ 6 + Det(\mathbf{\Gamma})/2 \; . \tag{A.9}$$

where the three roots of the polynomial correspond to the eikonal (characteristic) equations associated with each of the three possible modes of elastic wave propagation in the medium, one quasi-compressional, *qP*, and two quasi-shear, qS_1 and qS_2.

What is of interest here is *qP* wave propagation, where the eikonal equation is given by equation (A.6), and *Y* and related quantities are defined through the series of equations

$$Y = \sqrt{L}\,\cos(\, X/3\,) \tag{A.10}$$

$$X = \cos^{-1}\left(\, H/L^{3/2}\,\right) \tag{A.11}$$

$$L = \frac{[Tr\,(\mathbf{\Gamma}\,)]^3}{9} - \frac{\Pi_{\mathbf{\Gamma}}}{3} \quad > \quad 0 \tag{A.12}$$

$$H = \frac{[Tr\,(\mathbf{\Gamma}\,)]^3}{27} - \frac{\Pi_{\mathbf{\Gamma}}\,[Tr\,(\mathbf{\Gamma}\,)]}{6} + \frac{|\mathbf{\Gamma}|}{2} \; . \tag{A.13}$$

The expression for *Y* and related quantities were approximated by Tsvankin (1997) to obtain a linearized expression for the *qP* phase velocity in an orthorhombic medium. The resultant formula is equivalent to that presented by Backus (1965) (his equation (1)) expressed in terms of a generalization of the anisotropic parameters introduced by Thomsen (1986).

The *qP* eigenvalue of the matrix $\mathbf{\Gamma}$, is by definition, homogeneous of order two in powers of the slowness vector components p_i, which is masked in the above equations. The components of the group velocity vector may be obtained using equations (12) and (13) in the text. For *qP* wave propagation in an orthorhombic medium obtaining the expressions $\partial G/\partial p_i \;\; (i = 1,2,3)$ is an exercise in differentiation, the results of which are given below. From these results, together with some initial spatial, $\mathbf{x}_0 = \mathbf{x}(t_0)$, and slowness vector $\mathbf{p}_0 = \mathbf{p}(t_0)$, conditions at a reference time, t_0, a computer code may be written to compute the exact *qP* group velocity vector components.

The following sequence of steps produce a system of first order differential equations which may be used to trace rays in an arbitrarily inhomogeneous orthorhombic medium. The expressions for derivatives with respect to spatial variables, $\partial G/\partial x_i \;\; (i = 1,2,3)$ are not given here as they are cumbersome and are not required for the homogeneous medium type being discussed here. The quasi-compressional eikonal equation is given by equation (A.6) and repeated here

$$G\left(p_i, x_i\right) = Y + Tr\left(\mathbf{\Gamma}\right)/3 \tag{A.6}$$

The derivative of $G\left(p_i\right)$ with respect to a slowness vector component, p_i, is obtained from the following sequence of equations

$$\frac{\partial G}{\partial p_i} = \frac{\partial Y}{\partial p_i} + \frac{1}{3}\frac{\partial[Tr(\mathbf{\Gamma})]}{\partial p_i} \tag{A.14}$$

$$\frac{\partial Y}{\partial p_i} = \frac{1}{\sqrt{L}}\cos(X/3)\frac{\partial L}{\partial p_i} - \frac{2\sqrt{L}}{3}\sin(X/3)\frac{\partial X}{\partial p_i} \tag{A.16}$$

$$\frac{\partial X}{\partial p_i} = -\frac{1}{\sqrt{1 - H^2/L^3}}\frac{\partial\left(H/L^{3/2}\right)}{\partial p_i} = \frac{1}{\sqrt{1 - H^2/L^3}}\left\{\frac{\partial H/\partial p_i\ L - \frac{3}{2}H\ \partial L/\partial p_i}{L^{5/2}}\right\} \tag{A.16}$$

where Y and X are defined by equations (A.10) and (A.11). The derivatives of the intermediate variables L and H (equations A.12) and A.13) are given by

$$\frac{\partial L}{\partial p_i} = \frac{2}{9}Tr(\mathbf{\Gamma})\frac{\partial[Tr(\mathbf{\Gamma})]}{\partial p_i} - \frac{1}{3}\frac{\partial\Pi_{\mathbf{\Gamma}}}{\partial p_i} \tag{A.17}$$

$$\frac{\partial H}{\partial p_i} = \frac{1}{9}[Tr(\mathbf{\Gamma})]^2\frac{\partial[Tr(\mathbf{\Gamma})]}{\partial p_i} - \frac{1}{6}\frac{\partial\Pi_{\mathbf{\Gamma}}}{\partial p_i}Tr(\mathbf{\Gamma}) - \frac{\Pi_{\mathbf{\Gamma}}}{6}\frac{\partial[Tr(\mathbf{\Gamma})]}{\partial p_i} + \frac{1}{2}\frac{\partial|\mathbf{\Gamma}|}{\partial p_i} \tag{A.18}$$

where $Tr\left[\mathbf{\Gamma}\right]$, $\Pi_{\mathbf{\Gamma}}$ and $\left|\mathbf{\Gamma}\right|$ have been previously defined in equations (A.3) – (A.5).

Before proceeding further it should be noted that the expressions for $\partial G/\partial x_i$ are the same as the above sequence of equations with the exception that p_i is replaced by x_i. For this reason they will not be presented here. Additionally, for the case of an infinite medium with constant anisotropic coefficients, A_{jk}, the result of computing $\partial G/\partial x_i$ will be zero, so that $dp_i/dt = 0$, $\left(p_i \text{ is a constant}\right)$ for $i = 1,2,3$.

For the formulae for ray tracing to be complete, the individual Γ_{mn} terms for an orthorhombic medium, defined below, must be differentiated with respect to the p_i

$$\Gamma_{11} = p_1^2 A_{11} + p_2^2 A_{66} + p_3^2 A_{55} \tag{A.19}$$

$$\Gamma_{22} = p_1^2 A_{66} + p_2^2 A_{22} + p_3^2 A_{44} \tag{A.20}$$

$$\Gamma_{33} = p_1^2 A_{55} + p_2^2 A_{44} + p_3^2 A_{33} \tag{A.21}$$

$$\Gamma_{13} = p_1 p_3 \left(A_{13} + A_{55} \right) \tag{A.22}$$

$$\Gamma_{12} = p_1 p_2 \left(A_{12} + A_{66} \right) \tag{A.23}$$

$$\Gamma_{23} = p_2 p_3 \left(A_{23} + A_{44} \right) \tag{A.24}$$

As the partial derivatives of $Tr\left[\mathbf{\Gamma}\right]$ with respect to the p_i are easily determined they are given first as

$$\frac{\partial[Tr(\mathbf{\Gamma})]}{\partial p_1} = 2p_1 (A_{11} + A_{66} + A_{55}) \tag{A.25}$$

$$\frac{\partial[Tr(\mathbf{\Gamma})]}{\partial p_2} = 2p_2 (A_{66} + A_{22} + A_{44}) \tag{A.26}$$

$$\frac{\partial[Tr(\mathbf{\Gamma})]}{\partial p_3} = 2p_3 (A_{55} + A_{44} + A_{33}) \tag{A.27}$$

The following relations are required for the partial derivatives of $\Pi_{\mathbf{\Gamma}}$ and $\left|\mathbf{\Gamma}\right|$ with respect to p_i

$$\frac{\partial\Gamma_{11}}{\partial p_1} = 2p_1 A_{11}, \quad \frac{\partial\Gamma_{11}}{\partial p_2} = 2p_2 A_{66}, \quad \frac{\partial\Gamma_{11}}{\partial p_3} = 2p_3 A_{55} \tag{A.28}$$

$$\frac{\partial\Gamma_{22}}{\partial p_1} = 2p_1 A_{66}, \quad \frac{\partial\Gamma_{22}}{\partial p_2} = 2p_2 A_{22}, \quad \frac{\partial\Gamma_{22}}{\partial p_3} = 2p_3 A_{44} \tag{A.29}$$

$$\frac{\partial\Gamma_{33}}{\partial p_1} = 2p_1 A_{55}, \quad \frac{\partial\Gamma_{33}}{\partial p_2} = 2p_2 A_{44}, \quad \frac{\partial\Gamma_{33}}{\partial p_3} = 2p_3 A_{33} \tag{A.30}$$

$$\frac{\partial\Gamma_{13}}{\partial p_1} = p_3 \left(A_{13} + A_{55} \right), \quad \frac{\partial\Gamma_{13}}{\partial p_2} = 0, \quad \frac{\partial\Gamma_{13}}{\partial p_3} = p_1 \left(A_{13} + A_{55} \right) \tag{A.31}$$

$$\frac{\partial\Gamma_{12}}{\partial p_1} = p_2 \left(A_{12} + A_{66} \right), \quad \frac{\partial\Gamma_{12}}{\partial p_2} = p_1 \left(A_{12} + A_{66} \right), \quad \frac{\partial\Gamma_{12}}{\partial p_3} = 0 \tag{A.32}$$

$$\frac{\partial\Gamma_{23}}{\partial p_1} = 0, \quad \frac{\partial\Gamma_{23}}{\partial p_2} = p_3 \left(A_{23} + A_{44} \right), \quad \frac{\partial\Gamma_{23}}{\partial p_3} = p_2 \left(A_{23} + A_{44} \right) \tag{A.33}$$

so that the partial derivatives of $\Pi_{\mathbf{\Gamma}}$ and $\left|\mathbf{\Gamma}\right|$ have the form

$$\frac{\partial \Pi_{\mathbf{\Gamma}}}{\partial p_i} = \frac{\partial \Gamma_{11}}{\partial p_i}\Gamma_{33} + \Gamma_{11}\frac{\partial \Gamma_{33}}{\partial p_i} + \frac{\partial \Gamma_{11}}{\partial p_i}\Gamma_{22} + \Gamma_{11}\frac{\partial \Gamma_{22}}{\partial p_i} + \frac{\partial \Gamma_{22}}{\partial p_i}\Gamma_{33} + \Gamma_{22}\frac{\partial \Gamma_{33}}{\partial p_i} - 2\left[\Gamma_{13}\frac{\partial \Gamma_{13}}{\partial p_i} + \Gamma_{12}\frac{\partial \Gamma_{12}}{\partial p_i} + \Gamma_{23}\frac{\partial \Gamma_{23}}{\partial p_i}\right] \qquad \text{(A.34)}$$

and

$$\frac{\partial |\mathbf{\Gamma}|}{\partial p_i} = \frac{\partial \Gamma_{11}}{\partial p_i}\Gamma_{22}\Gamma_{33} + \Gamma_{11}\frac{\partial \Gamma_{22}}{\partial p_i}\Gamma_{33} + \Gamma_{11}\Gamma_{22}\frac{\partial \Gamma_{33}}{\partial p_i} + 2\left[\frac{\partial \Gamma_{12}}{\partial p_i}\Gamma_{13}\Gamma_{23} + \Gamma_{12}\frac{\partial \Gamma_{13}}{\partial p_i}\Gamma_{23} + \Gamma_{12}\Gamma_{13}\frac{\partial \Gamma_{23}}{\partial p_i}\right] - \frac{\partial \Gamma_{11}}{\partial p_i}\Gamma_{23}^2 - \frac{\partial \Gamma_{22}}{\partial p_i}\Gamma_{13}^2 - \frac{\partial \Gamma_{33}}{\partial p_i}\Gamma_{12}^2 - 2\left[\Gamma_{11}\Gamma_{23}\frac{\partial \Gamma_{23}}{\partial p_i} + \Gamma_{22}\Gamma_{13}\frac{\partial \Gamma_{13}}{\partial p_i} + \Gamma_{33}\Gamma_{12}\frac{\partial \Gamma_{12}}{\partial p_i}\right] \qquad \text{(A.35)}$$

With the above equations it is possible to construct a computer algorithm that can be used to compute the exact vector components of the group velocity, given appropriate initial conditions. From these expressions for these 3 vector components other related quantities such as the magnitude of the group velocity and the azimuthal and polar angles describing the direction of the group velocity vector may be obtained.