Circular Wavefront Assumptions for Gridded Traveltime Computations

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ABSTRACT

Traveltime computations are an integral part of modelling and imaging seismic data by providing efficient kinematic information on the location of propagated energy. The traveltimes may be computed analytically using simplifying assumptions, or may be estimated on a complex geological structure using raytracing or gridded traveltimes. A basic requirement for the propagation of gridded traveltimes is the estimation of one point on a corner of a square, given the traveltimes on the other three corners. A number of solutions are available to solve for the unknown time and are based on either a plane-wave assumption, a finite difference solution to the Eikonal equation, or an assumption that the wavefront at the square is curved. A solution for a curved wavefront assumption requires estimating the center of curvature, and requires solving a quartic equation. An alternate method is presented to estimate the center of curvature for a curved wavefront that uses an iterative procedure and does not require solving the quartic equation.

INTRODUCTION

Accurate modelling and Kirchhoff depth migrations compute traveltimes by estimating the time along a ray path or on a grid. These times may represent the propagation of energy from a scatterpoint to the surface, defining a diffraction shape for a poststack migration. They may also define the traveltimes between a source and scatterpoint, or scatterpoint to a receiver for a prestack migration.

The gridded method places a grid on the velocity field, then, assuming a given starting or source point, computes the traveltimes on the grid surrounding the point. The traveltimes on the adjacent grid points are computed and the process repeated to expand the area of known traveltimes. This region expands away from the source point until the desired objective (such as traveltimes on the surface, or the arrival at a desired point) is met, as illustrated in Figure 1a. This figure shows a partial grid and the corresponding traveltime contour for a constant velocity. The traveltimes at the surface are then mapped to define a diffraction on a time section in (b).

The velocity in each square formed by the grid is assumed to be locally constant, and in structured areas will vary from square to square. The inclusion of anisotropy parameters with the velocity information enables state-of-the-art anisotropic prestack depth migration (Perez, 2004).

The spreading of times on the grid may be accomplished by a number of techniques that involve estimating the time on a corner of a square when the times on the other three corners are known. The geometry for estimating this time is illustrated in Figure 2, which shows one square taken from the grid with time t_1 at the origin, and times t_2 and t_3 on the adjacent corners. The unknown time t_4 that we are estimating is on the corner opposite t_1. The square has a local velocity v, and each side has a distance h.

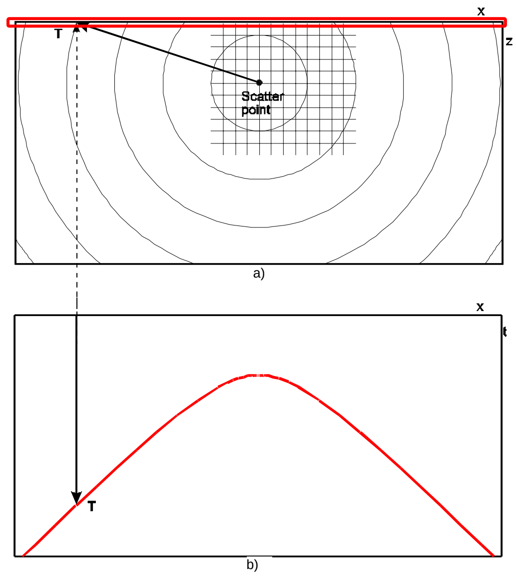

FIG. 1. Part of a grid is shown on a) that also contains constant velocity traveltime contours from a scatterpoint. Times at the surface are mapped to the time section in b) to illustrate a zero-offset diffraction.

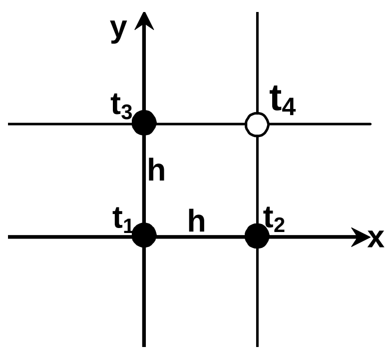

FIG. 2. Geometry for one square for gridded traveltime estimations. Times t_1, t_2, and t_3 are known, t_4 is unknown.

A solution to the problem is illustrated in Figure 3 that assumes the energy is propagating through the square as a plane-wave. The traveltime t_{3x} is interpolated on the x axis using the two known times t_1 and t_2. Connecting the corner time t_3 with the interpolated time t_{3x} defines the angle of a plane-wave. The construction of a parallel plane-wave that passes through the corner point of t_4 will intersect the x axis at time t_{4x}.

Note the distance on the x axis between t_{3x} and t_1 is equal to the distance between t_{4x} and t_2, giving a simple estimate for the time on the wavefront t_4, i.e.,

$$t_4 = t_2 + t_3 - t_1 . \tag{1}$$

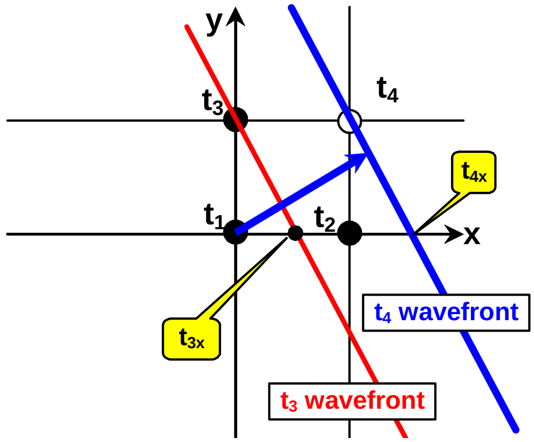

FIG. 3. Plane wave assumption.

Vidale (1988) introduced a method for computing t_4 that was based on a finite difference solution to the Eikonal equation, i.e.,

$$t_4 = t_1 + \sqrt{\frac{2h^2}{V^2} - (t_2 - t_3)^2} \tag{2}$$

where h is the dimension on a side of the square and v is the local velocity of the square. Equation (2) can be rearranged as

$$(t_4 - t_1)^2 + (t_2 - t_3)^2 = \frac{2h^2}{V^2}. \tag{3}$$

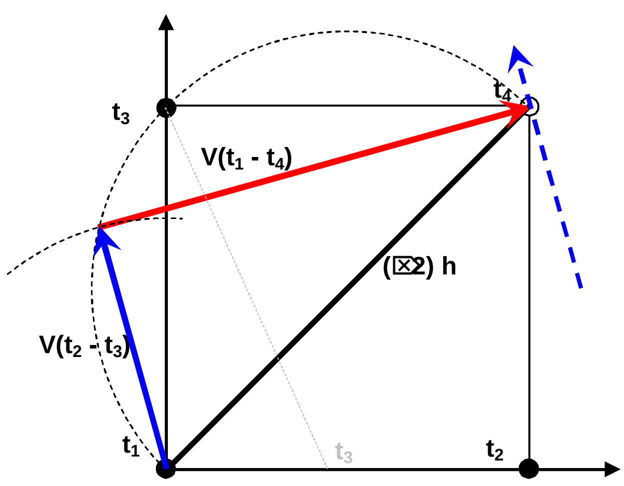

FIG. 4. Triangle construction for Vidale's computation for unknown time t_4.

Equation (3) represents three sides of a triangle that match the construction in Figure 4. Note the solid blue vector represents a new wavefront while the red vector is a normal to the wavefront at t_4. This method still assumes a plane-wave, but chooses a more appropriate angle for the wavefront.

Vidale also mentions in his paper that a curved wavefront may be assumed that is locally circular at the square. This geometry is illustrated in Figure 4 and assumes the velocity *V* is now constant over the region that contains the virtual center of curvature at a point ($-x_0$, $-z_0$). The time at the center of curvature is designated t_0, which is not assumed to be zero, as the wavefronts typically have varying curvature. Once the time and location of the center of curvature is known, then t_4 can be computed from

$$t_4 = \frac{1}{V}\sqrt{\left(x_0 + h\right)^2 + \left(z_0 + h\right)^2} \, . \tag{4}$$

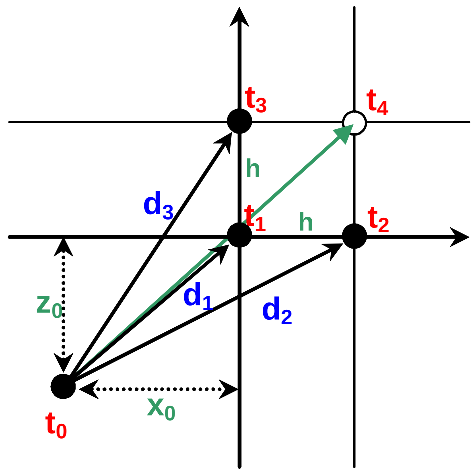

FIG. 5. Geometry for curved wavefront assumption where the center of curvature is ($-x_0$, $-z_0$).

The initial objective is to estimate the location of the center of curvature ($-x_0$, $-z_0$), and to define the time t_0 at this virtual source. Vidale points out that three equations that define the times of t_1, t_2, and t_3, from the center of curvature, may be combined into a quartic equation that can be solved for ($-x_0$, $-z_0$). The solutions to the quartic equation are analytic, and produce four possible solutions. This task is straightforward, but not trivial, and is the reason for the iterative approach.

This problem is identical to the Loran C navigation method in which the <u>differences</u> in traveltimes from radiating antennae are identified as hyperbolic contours on maps. In our application the traveltime differences (t_2 - t_1) and (t_3 - t_1) define the hyperbolae as illustrated in Figure 5. This figure shows four hyperbolae that intersect at four locations, corresponding to the four solutions of the quartic equation. (More often than not, there are two real solutions and two complex solutions indicating only two intersection points). The relative amplitudes of t_2 to t_1 and t_3 to t_1 provide additional information that eliminate one side of the hyperbolic pair, reducing the number of possible solutions to two. Logic must then be used to decide which solution is chosen.

Forming the quartic equation

Assume that the origin is at the location of the point identified as t_1, that the grid spacing is <u>two</u> units to the point that defines time t_2 and that the corresponding normalised velocity of the square *V* is locally constant. The times t_1 and t_2 define the focus points of the hyperbolae as illustrated in Figure 6.

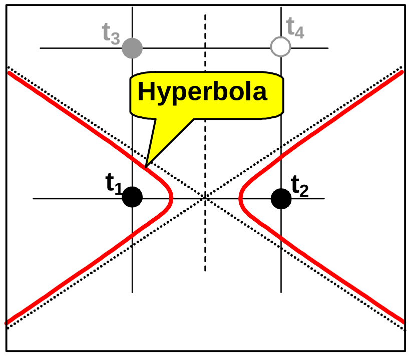

FIG. 6. Hyperbolic pair from traveltimes t_1 ands t_2.

A conventional representation of the hyperbola is defined by

$$\frac{(x-c_1)^2}{a_1^2}-\frac{z^2}{b_1^2}=1, \tag{5}$$

where $c_1 = 1$. The distance between the two hyperbolae on the x axis is $2a_1$ with a corresponding time difference (t_2 -t_1) or

$$a_1=\frac{V(t_1-t_2)}{2}. \tag{6}$$

The value of b_1 is defined using $c_1^2=a_1^2+b_1^2$ giving

$$b_1=\sqrt{1-a_1^2}\ . \tag{7}$$

The hyperbolae for the time pair t_1 and t_3 can also be defined using similar constants as

$$\frac{(x-c_2)^2}{a_2^2}-\frac{z^2}{b_2^2}=1. \tag{8}$$

We could at this point define the asymptotes and use them to approximate the source location, however our objective is to obtain an accurate source location. Solving equation (5) for z we get

$$z=b_1\left[\frac{(x-1)^2}{a_1^2}-1\right]^{1/2}. \tag{9}$$

This value of z may be substituted into equation (8) giving one equation that only contains the variable x, i.e.,

$$\frac{\left\{ b_1 \left[\frac{(x-1)^2}{a_1^2} - 1 \right]^{\frac{1}{2}} - 1 \right\}^2}{b_2^2} - \frac{x^2}{a_2^2} = 1 . \tag{10}$$

Solving for *x*, we use considerable algebraic manipulation to get

$$\begin{aligned}
&-2b_1^2\left[\frac{x^2-2x+1}{a_1^2}-1\right]+1-2b_2^2\left(1+\frac{x^2}{a_2^2}\right)\\
&+b_1^4\left[\frac{x^4-4x^3+6x^2-4x+1}{a_1^4}-2\frac{x^2-2x+1}{a_1^2}+1\right]\\
&-2b_1^2b_2^2\left[\frac{x^2-2x+1}{a_1^2}+\frac{x^4-2x^3+x^2}{a_1^2a_2^2}-1-\frac{x^2}{a_2^2}\right] \quad ,\\
&+b_2^4\left(\frac{x^4}{a_2^4}+\frac{2x^2}{a_2^2}+1\right)\\
&=0
\end{aligned} \tag{11}$$

which may then be reorganized to define the coefficients of *x*, giving our desired quartic solution, i.e.,

$$\begin{aligned}
&x^4\left(\frac{b_1^4}{a_1^2}-\frac{2b_1^2b_2^2}{a_1^2a_2^2}+\frac{b_2^4}{a_2^4}\right)\\
&+x^3\frac{4b_1^2}{a_1^2}\left(\frac{b_2^2}{a_2^2}-b_1^2\right)\\
&+x^2\left(+\frac{6b_1^4}{a_1^4}+\frac{2b_2^4}{a_2^2}-\frac{2b_1^4}{a_1^2}-\frac{2b_1^2b_2^2}{a_1^2}-\frac{2b_1^2b_2^2}{a_1^2a_2^2}+\frac{2b_1^2b_2^2}{a_2^2}-\frac{2b_1^2}{a_1^2}-\frac{2b_2^2}{a_2^2}\right).\\
&+x\frac{4b_1^2}{a_1^2}\left(1-\frac{b_1^2}{a_1^2}+b_1^2+b_2^2\right)\\
&+\left(b_1^4+b_2^4+2b_1^2-2b_2^2+2b_1^2b_2^2+\frac{b_1^4}{a_1^4}-\frac{2b_1^4}{a_1^2}-\frac{2b_1^2}{a_1^2}-\frac{2b_1^2b_2^2}{a_1^2}+1\right)\\
&=0
\end{aligned} \tag{12}$$

At this point we could substitute the values of t_1, t_2, t_3, and *V*, but that is not efficient, and we will assume that the above coefficients are computed to give a quartic equation with new variables defined as

$$a_0x^4+a_1x^3+a_2x^2+a_3x+a_4=0 . \tag{13}$$

Note that some of these new coefficients are similar to the variables used to define the hyperbola, i.e. a_1 and a_2, but they are now different, and the following will assume these new values.

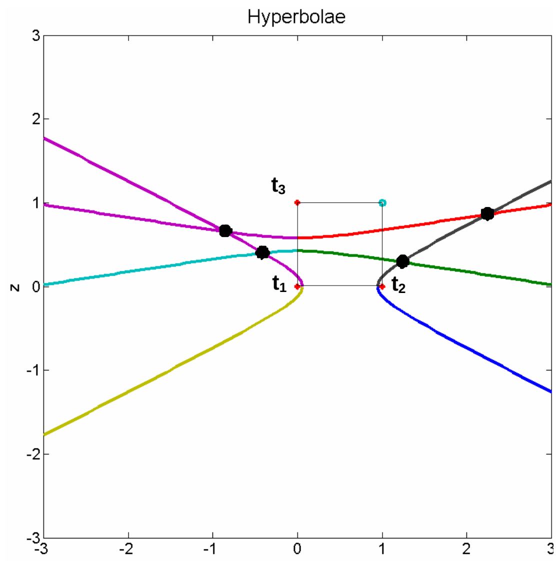

FIG. 7. Two sets of hyperbolae that result from two time differences, t_2 - t_1 and t_3 - t_1.

The significant effort to solve equation (13) is illustrated by the solution produced by MATHEMATICA which occupies nearly a page of text as illustrated in the appendix. The quartic equation produces four possible solutions as illustrated in Figure 7, but usually there are two real solution and two complex solutions. Logic is now required to choose the desired solution. The desired solution x defines x_0, then equation (5) can be used to solve for z_0 and then t_0 is computed from

$$t_0 = t_1 - \frac{d_1}{V}. \tag{14}$$

THE ITERATIVE SOLUTION

The method that follows has simpler equations to solve for the center of curvature, but does involve iterating to an acceptable accuracy.

From the geometry of Figure 4 we can write three equations for the distances as

$$\begin{aligned} d_1 &= v(t_1 - t_0) = \sqrt{x_0^2 + z_0^2} \\ d_2 &= v(t_2 - t_0) = \sqrt{(h - x_0)^2 + z_0^2} \\ d_3 &= v(t_3 - t_0) = \sqrt{x_0^2 + (h - z_0)^2} \end{aligned}. \tag{15}$$

We can rewrite these equations as

$$\begin{aligned} x_0^2 + z_0^2 &= v^2\left(t_1^2 - 2t_1t_0 + t_0^2\right) \\ x_0^2 - 2x_0h + h^2 + z_0^2 &= v^2\left(t_2^2 - 2t_2t_0 + t_0^2\right) \\ x_0^2 + z_0^2 - 2z_0h + h^2 &= v^2\left(t_3^2 - 2t_3t_0 + t_0^2\right) \end{aligned}. \tag{16}$$

Subtracting the first equation from the second and third we obtain

$$x_0 = \frac{v^2}{2h}\left[2t_0\left(t_2 - t_1\right) - t_2^2 + t_1^2\right] + \frac{h}{2}$$
$$z_0 = \frac{v^2}{2h}\left[2t_0\left(t_3 - t_1\right) - t_3^2 + t_1^2\right] + \frac{h}{2} . \tag{17}$$

These new equations are only dependent on t_0; all the other terms are constant. They are then substituted into equation (15)a, i.e.,

$$t_0 = t_1 - \frac{1}{v}\sqrt{x_0^2 + z_0^2} , \tag{18}$$

giving one equation where the only variable is t_0, i.e.,

$$t_0 = t_1 - \frac{1}{v}\sqrt{\left\{\frac{v^2}{2h}\left[2t_0\left(t_2 - t_1\right) - t_2^2 + t_1^2\right] + \frac{h}{2}\right\}^2 + \left\{\frac{v^2}{2h}\left[2t_0\left(t_3 - t_1\right) - t_3^2 + t_1^2\right] + \frac{h}{2}\right\}^2} \tag{19}$$

This equation could be simplified by squaring, but that introduces additional solutions that must be identified and eliminated. In its present form equation (19) can be written as a function $f(t_0) = 0$ that is suitable for a Newton-Raphson (NR) iterative solution. The function is differentiable, i.e.,

$$\frac{d\,f(t_0)}{d\,t_0} = 1 + \frac{1}{v}\left[\left\{\frac{v^2\left(t_2 - t_1\right)}{2h}\left[2t_0 - t_2 - t_1\right] + \frac{h}{2}\right\}^2 + \left\{\frac{v^2\left(t_3 - t_1\right)}{2h}\left[2t_0 - t_3 - t_1\right] + \frac{h}{2}\right\}^2\right]^{-1/2} \times$$
$$\left[2\left\{\frac{v^2\left(t_2 - t_1\right)}{2h}\left[2t_0 - t_2 - t_1\right] + \frac{h}{2}\right\}\frac{v^2\left(t_2 - t_1\right)}{h} + 2\left\{\frac{v^2\left(t_3 - t_1\right)}{2h}\left[2t_0 - t_3 - t_1\right] + \frac{h}{2}\right\}\frac{v^2\left(t_3 - t_1\right)}{h}\right] . \tag{20}$$

Assuming a starting point at $t_{0,1}$, the NR method uses the derivative $f^{1}(t_{0,1})$ to predict an improved solution $t_{0,2}$:

$$t_{0,2} = t_{0,1} + \frac{f\left(t_{0,1}\right)}{f^1\left(t_{0,1}\right)} . \tag{21}$$

Two solutions for $f(t_0) = 0$ are possible that depend on the starting point of the iterative solution. Starting with $t_0 = 0$ is a good choice and is suitable for all points in the third quadrant where both x_0 and z_0 are less than zero. Solutions in the third quadrant represent an expanding wavefield that is moving away from the three known points as designed in the mapping of the traveltimes. Visualization of the NR solution for typical values is shown in Figure 8.

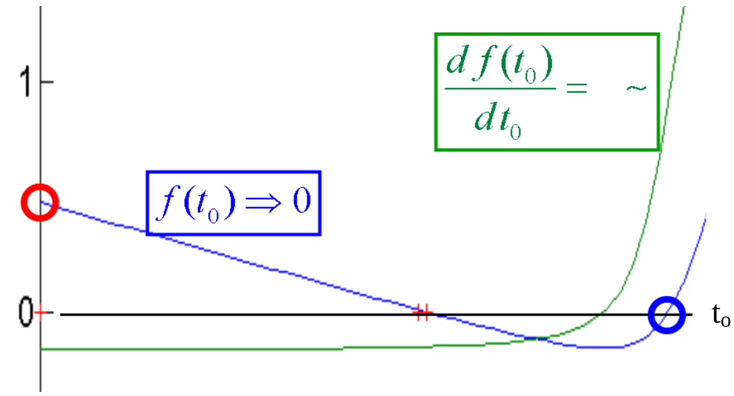

FIG. 8. Plot of f(t_o) and its derivative with red "+" to indicate the iteration results.

This function $f(t_o)$ is ideally suited for a NR solution as the derivative at the starting point, indicated by the red circle at $t_o = 0$, points directly to the solution. In this case, only two iterations are required to achieve an error in the estimate of t_o to be less than 10^{-10}. The blue circle represents the alternate solution point if approached from the right.

Testing and evaluating the methods

The accuracy of the different methods were evaluated using an array of points that represented many different centers of curvature. They were defined in the third quadrant with 100 × 100 points and with a range of $5h \times 5h$ from the axis. At each center point, the traveltimes t_1, t_2, and t_3 were computed, and then, using only those times, the traveltime t_4 was estimated.

The number of iterations required to reach a minimum threshold was recorded and displayed in Figure 9. A threshold of 10^{-8} for the normalized error in t_0 was used, and an average iteration number appears to be less than 5. The number of iterations outside this quadrant may be higher, especially on the x and y axis. However, the solution on the axis can be predetermined and eliminated for the iterative solution by noting that a ray on an axis will have a time difference that is equal to the normalized distance 2 divided by the normalized velocity V.

The relative errors in estimating t_4 in the third quadrant are compared in Figure 10 using (a) the plane-wave assumption, (b) the Vidale finite difference method, and (c) the iterative method. The plots in (a) and (b) have a maximum error of one percent, while (c) has a maximum error of 1.5E-12. In this region, the plane wave solution has the largest error, while the eikonal method also showed some significant error. However it should be noted that their computational requirements are significantly less than the iterative method.

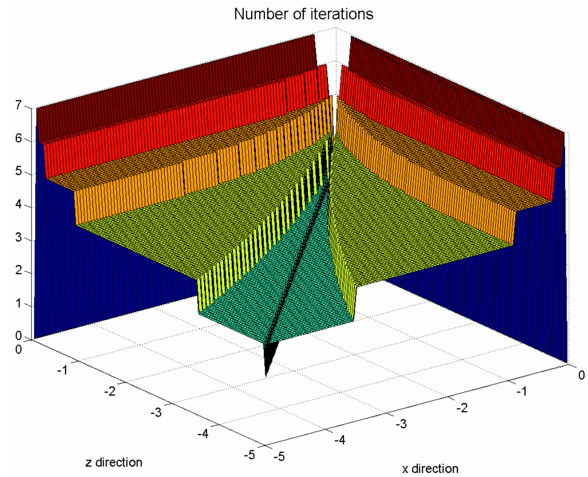

FIG. 9. Iterations required to for an error in the third quadrant that is less than 10^{-8}.

When the center of curvature is relatively close to the dimensions of the grid, i.e. less than 5*h*, then it may be more appropriate to use the curved wavefront assumption than Vidale's method.

CONCLUSIONS AND COMMENTS

An iterative method was presented to compute a gridded traveltime estimate that assumes curved wavefronts. The method is simpler than the quartic solution. The number of iterations required for very accurate estimates is typically five or less. Its accuracy was compared with that of a plane-wave assumption and with Vidale's finite difference method.

REFERENCES

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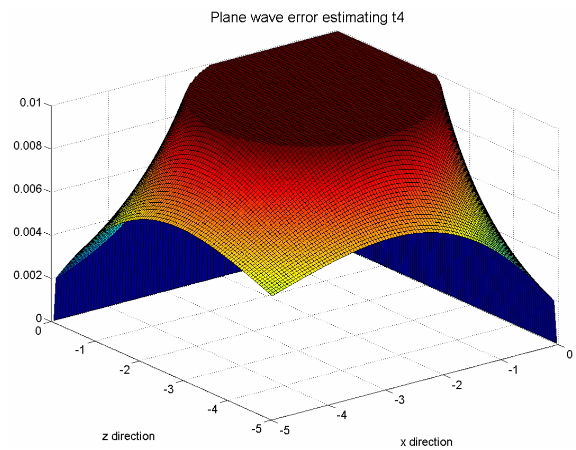
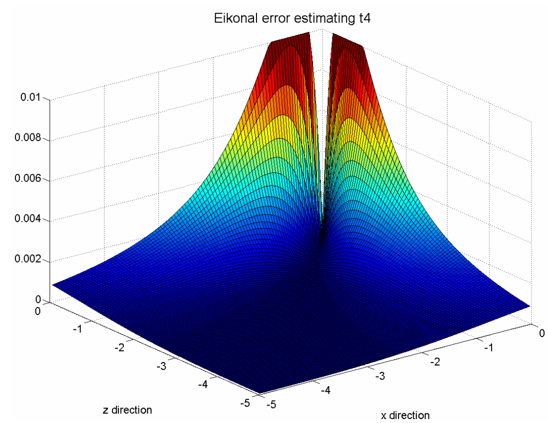
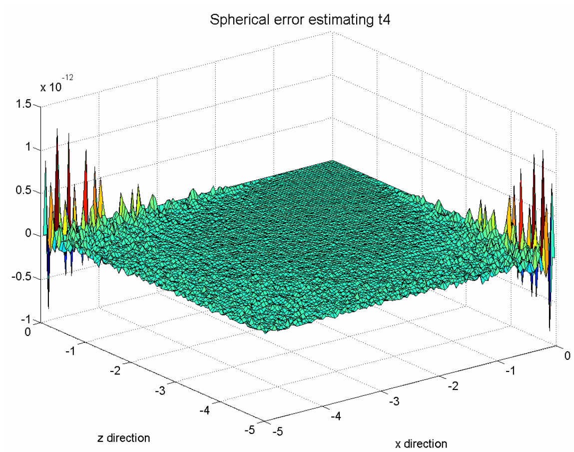
Vidale, J., 1988, Finite-difference calculation of traveltimes, Bulletin of the Seismological Society of America, **78**, 2062-2076.



(a)



(b)



(c)

FIG. 10. Comparison of the errors in estimating t_4, with a) the plane wave method, b) the Vidale method, and c) the iterative method.

APPENDIX 1

The following is a solution to the quartic equation obtained using MATHEMATICA:

Solve[a + b*x + c*x^2 + d*x^3 + f*x^4==0,x]

$$\Big\{\Big\{x \to -\frac{d}{4 f} - \frac{1}{2}\sqrt{\frac{d^2}{4 f^2} - \frac{2 c}{3 f} + \frac{2^{1/3} (c^2 - 3 b d + 12 a f)}{3 f \left(2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f + \sqrt{-4 (c^2 - 3 b d + 12 a f)^3 + (2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f)^2}\right)^{1/3}} + \frac{1}{3\, 2^{1/3} f}\left(2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f + \sqrt{-4 (c^2 - 3 b d + 12 a f)^3 + (2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f)^2}\right)^{1/3}} - \frac{1}{2}\sqrt{\frac{d^2}{2 f^2} - \frac{4 c}{3 f} - \frac{2^{1/3} (c^2 - 3 b d + 12 a f)}{3 f \left(2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f + \sqrt{-4 (c^2 - 3 b d + 12 a f)^3 + (2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f)^2}\right)^{1/3}} - \frac{1}{3\, 2^{1/3} f}\left(2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f + \sqrt{-4 (c^2 - 3 b d + 12 a f)^3 + (2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f)^2}\right)^{1/3} - \frac{-\frac{d^3}{f^3} + \frac{4 c d}{f^2} - \frac{8 b}{f}}{4\sqrt{\frac{d^2}{4 f^2} - \frac{2 c}{3 f} + \frac{2^{1/3} (c^2 - 3 b d + 12 a f)}{3 f \left(2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f + \sqrt{-4 (c^2 - 3 b d + 12 a f)^3 + (2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f)^2}\right)^{1/3}} + \frac{1}{3\, 2^{1/3} f}\left(2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f + \sqrt{-4 (c^2 - 3 b d + 12 a f)^3 + (2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f)^2}\right)^{1/3}}}}\Big\},$$

$$\Big\{x \to -\frac{d}{4 f} - \frac{1}{2}\sqrt{\frac{d^2}{4 f^2} - \frac{2 c}{3 f} + \frac{2^{1/3} (c^2 - 3 b d + 12 a f)}{3 f \left(2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f + \sqrt{-4 (c^2 - 3 b d + 12 a f)^3 + (2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f)^2}\right)^{1/3}} + \frac{1}{3\, 2^{1/3} f}\left(2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f + \sqrt{-4 (c^2 - 3 b d + 12 a f)^3 + (2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f)^2}\right)^{1/3}} + \frac{1}{2}\sqrt{\frac{d^2}{2 f^2} - \frac{4 c}{3 f} - \frac{2^{1/3} (c^2 - 3 b d + 12 a f)}{3 f \left(2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f + \sqrt{-4 (c^2 - 3 b d + 12 a f)^3 + (2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f)^2}\right)^{1/3}} - \frac{1}{3\, 2^{1/3} f}\left(2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f + \sqrt{-4 (c^2 - 3 b d + 12 a f)^3 + (2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f)^2}\right)^{1/3} - \frac{-\frac{d^3}{f^3} + \frac{4 c d}{f^2} - \frac{8 b}{f}}{4\sqrt{\frac{d^2}{4 f^2} - \frac{2 c}{3 f} + \frac{2^{1/3} (c^2 - 3 b d + 12 a f)}{3 f \left(2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f + \sqrt{-4 (c^2 - 3 b d + 12 a f)^3 + (2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f)^2}\right)^{1/3}} + \frac{1}{3\, 2^{1/3} f}\left(2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f + \sqrt{-4 (c^2 - 3 b d + 12 a f)^3 + (2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f)^2}\right)^{1/3}}}}\Big\},$$

$$\Big\{x \to -\frac{d}{4 f} + \frac{1}{2}\sqrt{\frac{d^2}{4 f^2} - \frac{2 c}{3 f} + \frac{2^{1/3} (c^2 - 3 b d + 12 a f)}{3 f \left(2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f + \sqrt{-4 (c^2 - 3 b d + 12 a f)^3 + (2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f)^2}\right)^{1/3}} + \frac{1}{3\, 2^{1/3} f}\left(2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f + \sqrt{-4 (c^2 - 3 b d + 12 a f)^3 + (2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f)^2}\right)^{1/3}} - \frac{1}{2}\sqrt{\frac{d^2}{2 f^2} - \frac{4 c}{3 f} - \frac{2^{1/3} (c^2 - 3 b d + 12 a f)}{3 f \left(2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f + \sqrt{-4 (c^2 - 3 b d + 12 a f)^3 + (2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f)^2}\right)^{1/3}} - \frac{1}{3\, 2^{1/3} f}\left(2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f + \sqrt{-4 (c^2 - 3 b d + 12 a f)^3 + (2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f)^2}\right)^{1/3} + \frac{-\frac{d^3}{f^3} + \frac{4 c d}{f^2} - \frac{8 b}{f}}{4\sqrt{\frac{d^2}{4 f^2} - \frac{2 c}{3 f} + \frac{2^{1/3} (c^2 - 3 b d + 12 a f)}{3 f \left(2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f + \sqrt{-4 (c^2 - 3 b d + 12 a f)^3 + (2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f)^2}\right)^{1/3}} + \frac{1}{3\, 2^{1/3} f}\left(2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f + \sqrt{-4 (c^2 - 3 b d + 12 a f)^3 + (2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f)^2}\right)^{1/3}}}}\Big\},$$

$$\Big\{x \to -\frac{d}{4 f} + \frac{1}{2}\sqrt{\frac{d^2}{4 f^2} - \frac{2 c}{3 f} + \frac{2^{1/3} (c^2 - 3 b d + 12 a f)}{3 f \left(2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f + \sqrt{-4 (c^2 - 3 b d + 12 a f)^3 + (2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f)^2}\right)^{1/3}} + \frac{1}{3\, 2^{1/3} f}\left(2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f + \sqrt{-4 (c^2 - 3 b d + 12 a f)^3 + (2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f)^2}\right)^{1/3}} + \frac{1}{2}\sqrt{\frac{d^2}{2 f^2} - \frac{4 c}{3 f} - \frac{2^{1/3} (c^2 - 3 b d + 12 a f)}{3 f \left(2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f + \sqrt{-4 (c^2 - 3 b d + 12 a f)^3 + (2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f)^2}\right)^{1/3}} - \frac{1}{3\, 2^{1/3} f}\left(2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f + \sqrt{-4 (c^2 - 3 b d + 12 a f)^3 + (2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f)^2}\right)^{1/3} + \frac{-\frac{d^3}{f^3} + \frac{4 c d}{f^2} - \frac{8 b}{f}}{4\sqrt{\frac{d^2}{4 f^2} - \frac{2 c}{3 f} + \frac{2^{1/3} (c^2 - 3 b d + 12 a f)}{3 f \left(2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f + \sqrt{-4 (c^2 - 3 b d + 12 a f)^3 + (2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f)^2}\right)^{1/3}} + \frac{1}{3\, 2^{1/3} f}\left(2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f + \sqrt{-4 (c^2 - 3 b d + 12 a f)^3 + (2 c^3 - 9 b c d + 27 a d^2 + 27 b^2 f - 72 a c f)^2}\right)^{1/3}}}}\Big\}\Big\}$$

APPENDIX II

Vidale (1988) also used three inline points to propagate a point away from one side of a square shaped traveltime map as illustrated in Figure 11. (More recent methods expand minimum times that approximate a wave front.) His solution is for three inline points was

$$t_4 = t_1 + \sqrt{\frac{h^2}{v^2} - \frac{(t_3 - t_2)^2}{4}}, \tag{22}$$

which can be shown to represent a plane wave that is defined from times t_2 and t_3 as also illustrated in Figure 11.

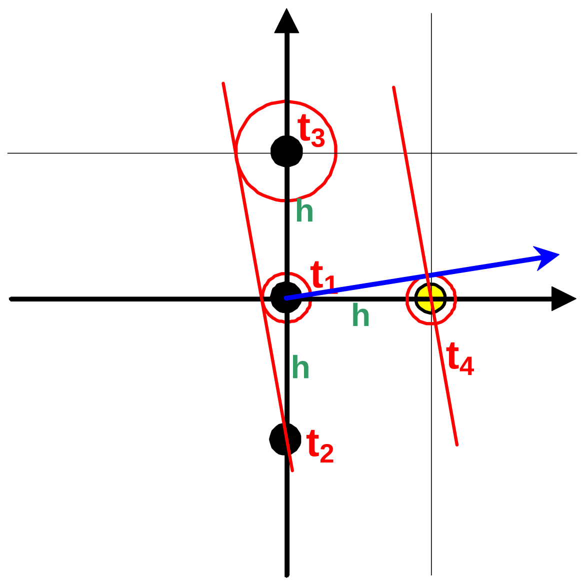


FIG. 11. Geometry of estimating t_4 given the inline times t_1, t_2, and t_3.

A circular wavefront could also be assumed with the center of curvature at ($-x_0$, $-z_0$) and an apparent time t_0 as illustrated in Figure 12. The process for estimating x_0, z_0, and t_0 are much simpler than the three known points on a square and are given by explicit equations, i.e., t_0 is found from

$$t_0 = \frac{t_2^2 + t_3^2 - 2t_1^2 - \dfrac{2h^2}{v^2}}{2\left(t_2 + t_3 - 2t_1\right)}, \tag{23}$$

the depth z_0 from

$$z_0 = \frac{v^2}{4h}\left[\left(t_2 - t_0\right)^2 - \left(t_3 - t_0\right)^2\right], \tag{24}$$

and the spatial location x_0 from

$$x_0 = \pm \operatorname{sqrt}\left[v^2 \left(t_2 - t_0 \right)^2 - z_0^2 \right]. \tag{25}$$

The location of x_0 needs to be questioned for its sign, but that can be determined from the nature of the problem, or might not be required.

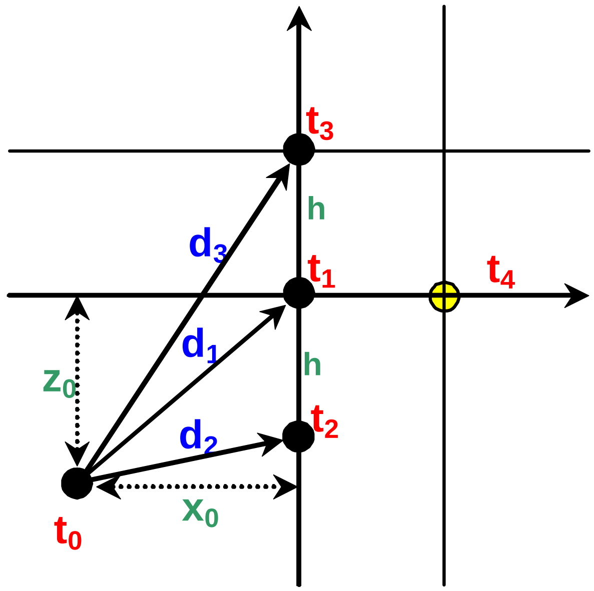


FIG. 12. Geometry of estimating the center ($-x_0$, $-z_0$, t_0) of a curved wavefront from the aligned times t_1, t_2, and t_3.

The above solution leads back to the original problem of three known times on the corners of a square, but now we include the possibility of using additional points in the known traveltime neighbourhood as illustrated in Figure 13.

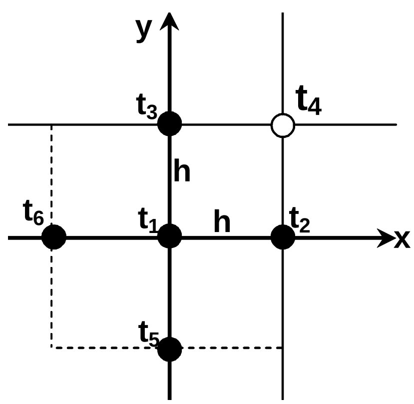


FIG. 13. Geometry of using two sets of three inline points for estimating t_4.

The times t_3, t_1, and t_5 could be used to estimate t_0 and z_0, then t_6, t_1, and t_2 could be used to get an additional estimate of t_0 and a single estimate of x_0. Using these additional points will require the assumption of a locally constant velocity that is estimated from a larger area of four squares. Since these computations are explicit, an error analysis will require a variable velocity field for evaluation.