A minimum phase, band-limited delta spike

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ABSTRACT

We look at the simple example of computing a minimum phase signal corresponding to a band-limited delta spike, using an IIR filter. This example suggests a way to compute more generally the minimum phase version of any band-limited signal, using a stable criteria.

INTRODUCTION

In spiking deconvolution, one-way wave propagators, and physical modeling of impulsive sources, a minimum phase condition is often specified to uniquely characterize a signal with a prescribed amplitude spectrum. The minimum phase condition specifies that of all causal signals with a given amplitude spectrum, the minimum phase signal has most of its energy concentrated at the front. There are many equivalent formulations of this condition arising from filter theory, but this physical notion of energy concentration up front is the most useful for physical signals.

The equivalent formulations are useful for calculating the minimum phase version of a given signal. For instance, it is known that the log amplitude spectrum and the phase are Hilbert transform pairs. Equivalently, the minimum phase signal can be computed as an outer function in a Hardy space of analytic functions, using this same log amplitude spectrum.

However, there is a problem: while any causal signal can be converted into a minimum phase signal, there are constraints on the spectrum of the causal signal. Namely, the log amplitude spectrum must be an integrable function of frequency. So, for instance, the amplitude spectrum of a causal signal cannot equal zero (except possibly on a small set). This is a problem since, for real signals, we might only know the spectrum on a certain frequency band. Outside that band, if the spectrum is assumed to be zero, we get an infinity in the logarithm of the spectrum, which does not compute.

It is important to recognize that some signals do not have a minimum phase equivalent. In particular, any band-limited signal does not have a minimum phase equivalent. However, in practice we often have signals for which only a band-limited version is known. The question arise on how to compute the minimum phase version.

We suggest an alternative to the standard computation. Take the band-limited spectrum, wrap it into a full spectrum and compute the minimum phase version. Then resample to get the minimum phase approximation.

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THE DELTA SPIKE EXAMPLE

As a illustrative example, consider the situation of finding the minimum phase version of a band-limited delta spike. The amplitude spectrum of this band-limited signal would be a boxcar, equal to one inside some frequency interval $[-f_0, f_0]$ and zero outside. The zero phase version of the band-limited spike is a sinc function $\sin(x)/x$; however, the minimum phase version does not exist, since no causal signal can have an interval of zeros in its spectrum.

The standard fix to this problem is to add a "stabilization" factor to the minimum phase calculation, to remove the zeros. For instance, in the delta spike case, one replaces the box car with an amplitude spectrum which is one in the interval $[-f_0, f_0]$ and is equal to a small, fixed constant $\epsilon > 0$ outside the interval. The minimum phase calculation can be done, producing a causal signal with amplitude spectrum close to that of the band-limited spike, and with the energy concentrated near the front.

To demonstrate, we compute the minimum phase version of a band-limited delta spike, and show the results in Figure 1. Here, we are using sampled signals, and band-limited the spike to .4 of Nyquist frequency. Three different stablizing factors have been used, $\epsilon = 0.1, 0.001, 0.00001$, corresponding to a cutoff band at -20dB, -60dB and -80dB. The corresponding minimum phase signals are shown on the right of the figure. Notice that as the stabilizing factor decreasing to zero, the first peak of the signal moves more and more to the right, away from the starting time $t = 0$. This is what we expect, since in the limit, the spectrum will have zeros in the spectrum, and no casual version the signal can exist with those zeros. In effect, as ϵ goes to zero, the corresponding signal gets its energy pushed all the way out to infinity.

As an alternative, our method is to pass the delta spike through a minimum phase, lowpass filter, to obtain a better, more stable computation of the minimum phase signal[†].

There are many choices for a minimum phase, lowpass filter. The design criteria for Butterworth, Chebyshev, and Jacobi elliptic filter all specify zeros and poles for a filter that are inside, or on, the unit disk, and thus are mimimum phase. Thus in principle we could use any one of these. Since we are using a delta spike, we are really just examining the impulse response of these IIR filter.

Here are some results:

In Figure 2 we see the results using a Chebyshev Type 2 filter, which is flat in the passband and has a specified cutoff down at 20dB. We chose three different orders, to see the effect on the resulting minimum phase signal. There are not large differences, and notice the maximum peak in the signal occurs at the third sample.

In Figure 3 we see similar results using an elliptic filter, with a passband ripple of 3 dB,

[†] A more general minimum phase calculation is discussed in another article in this report, by the same authors.

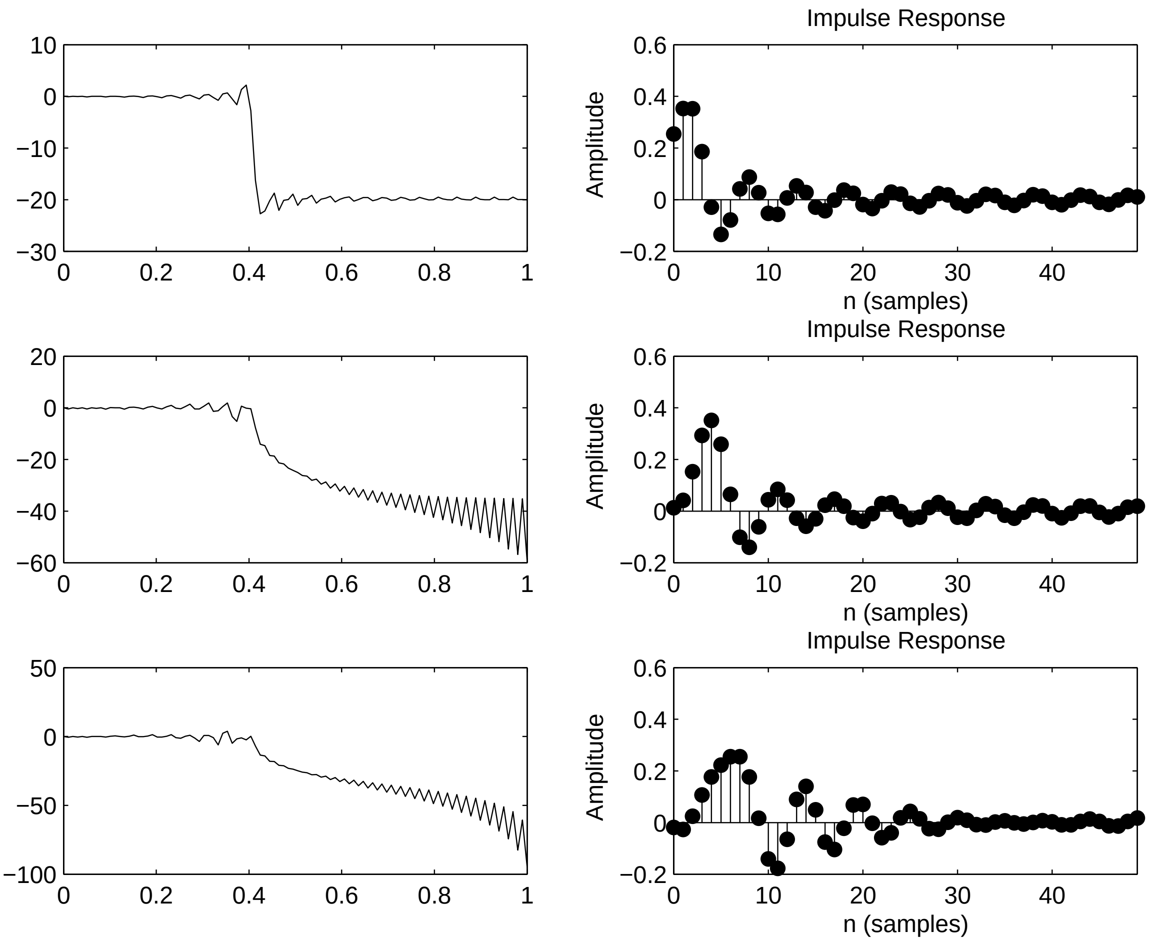

FIG. 1. Amplitude spectra (in dB) and minimum phase signals, from the stabilized rceps calculation, for stability factors of $\epsilon = 0.1, 0.001, 0.00001$.

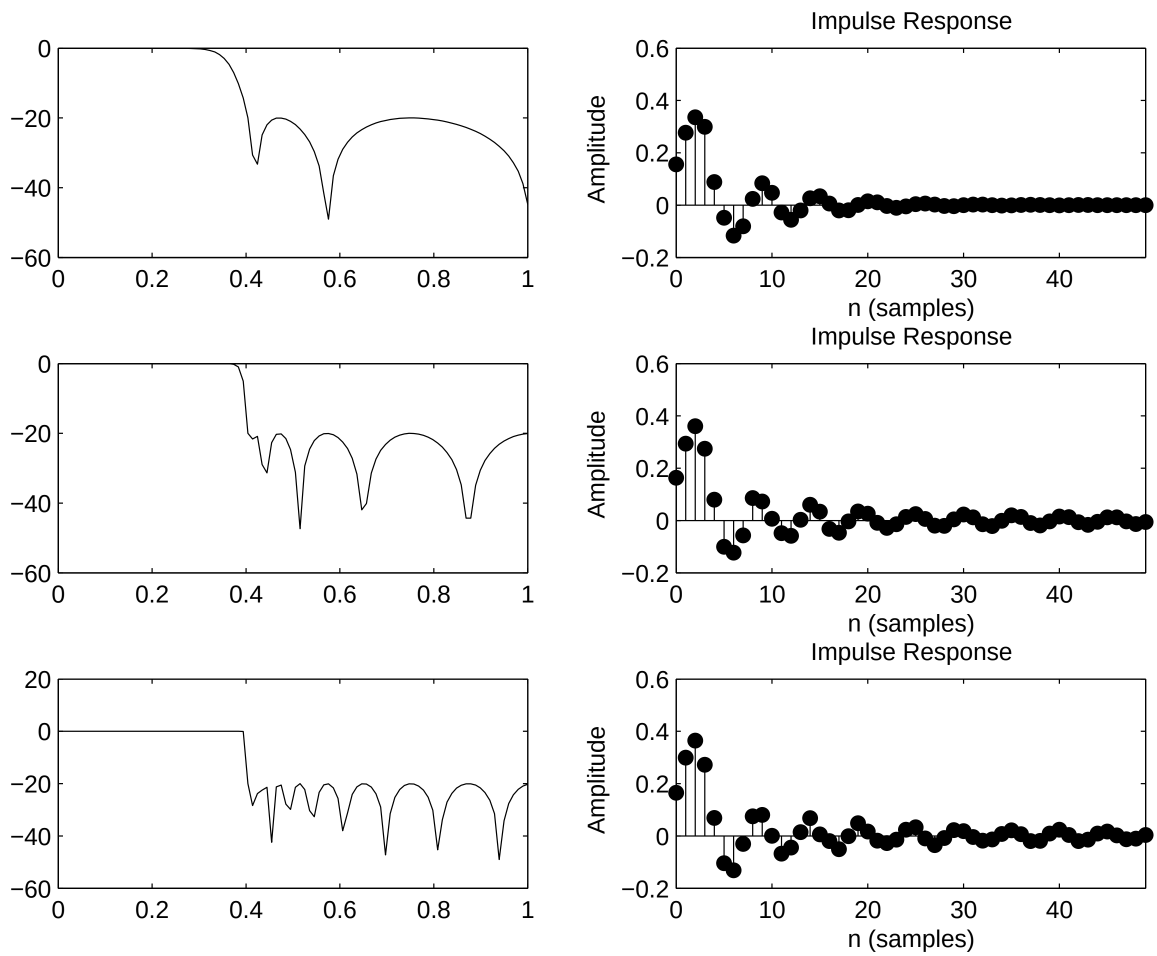

FIG. 2. Amplitude spectra and corresponding minimum phase signals, formed through a Chebyshev filter at 20dB cutoff, with orders 5, 10, and 20.

and down 20dB in the stop band. Again we use orders 5, 10, and 20, and note the maximum sample in the signal is at sample 3.

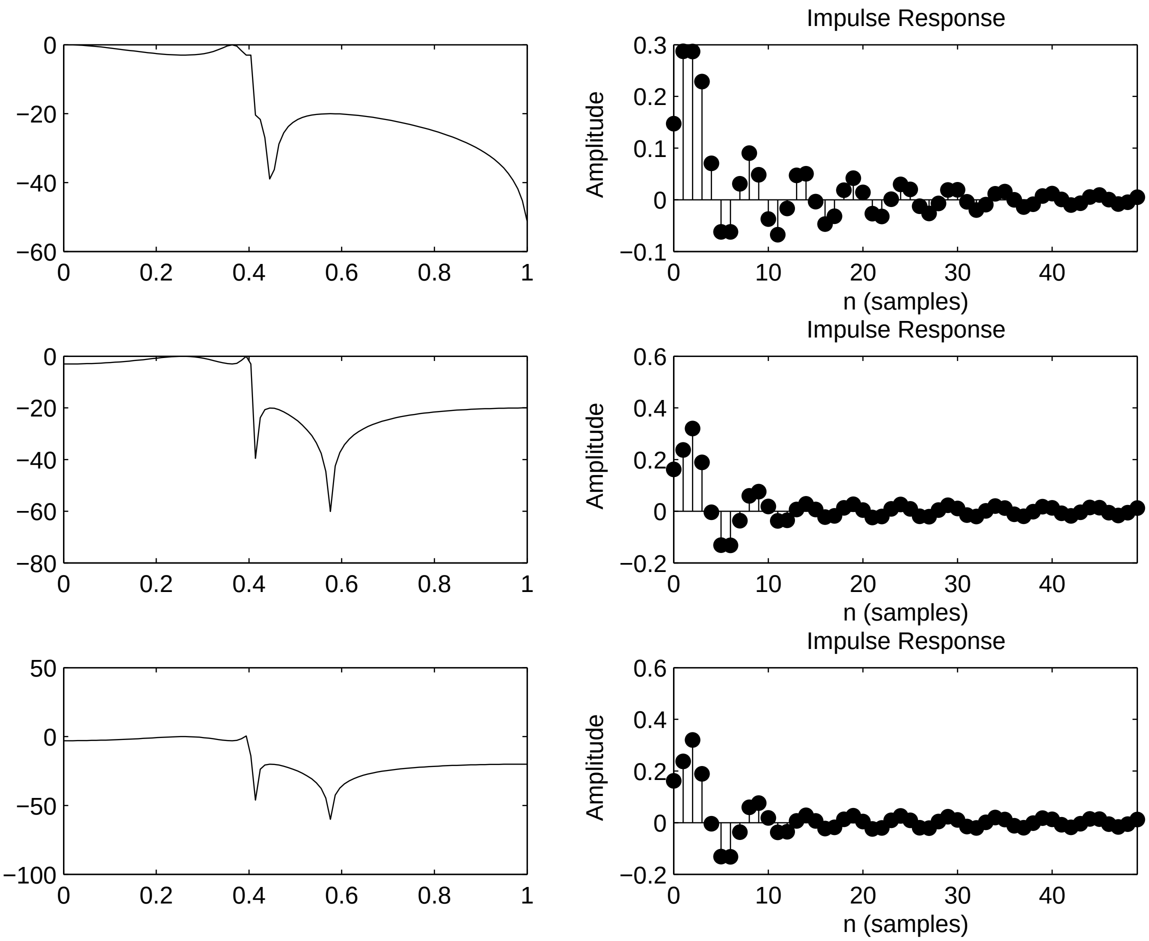

FIG. 3. Amplitude spectra and corresponding minimum phase signals, formed through an elliptic filter, at 20dB cutoffwith orders 5, 10, and 20.

In Figure 4 we repeat the Chebyshev calculation with a specified cutoff of 60dB. For the order 5 filter, the frequency response is not that close to the boxcar, as the order is not high enough to achieve such a sharp cutoff. The corresponding minimum phase signal is overly smoth. We note in all three that the maximum sample occurs at sample 6 or 7.

In Figure 5 we see similar results with the elliptic filter, with a passband ripple of 3 dB, and down 60dB in the stop band. Again we use orders 5, 10, and 20, and note the maximum sample in the signal is at sample 5.

Just to verify the trend, we repeat the calculation for an elliptic filter with a 100dB cutoff, with results shown in Figure 6. Here again, the 5th order filter is not big enough to achieve the desired spectral response. However, the 10th and 20th order do. The maximum spike in the minimum phase signal occurs at sample 7 or 8.

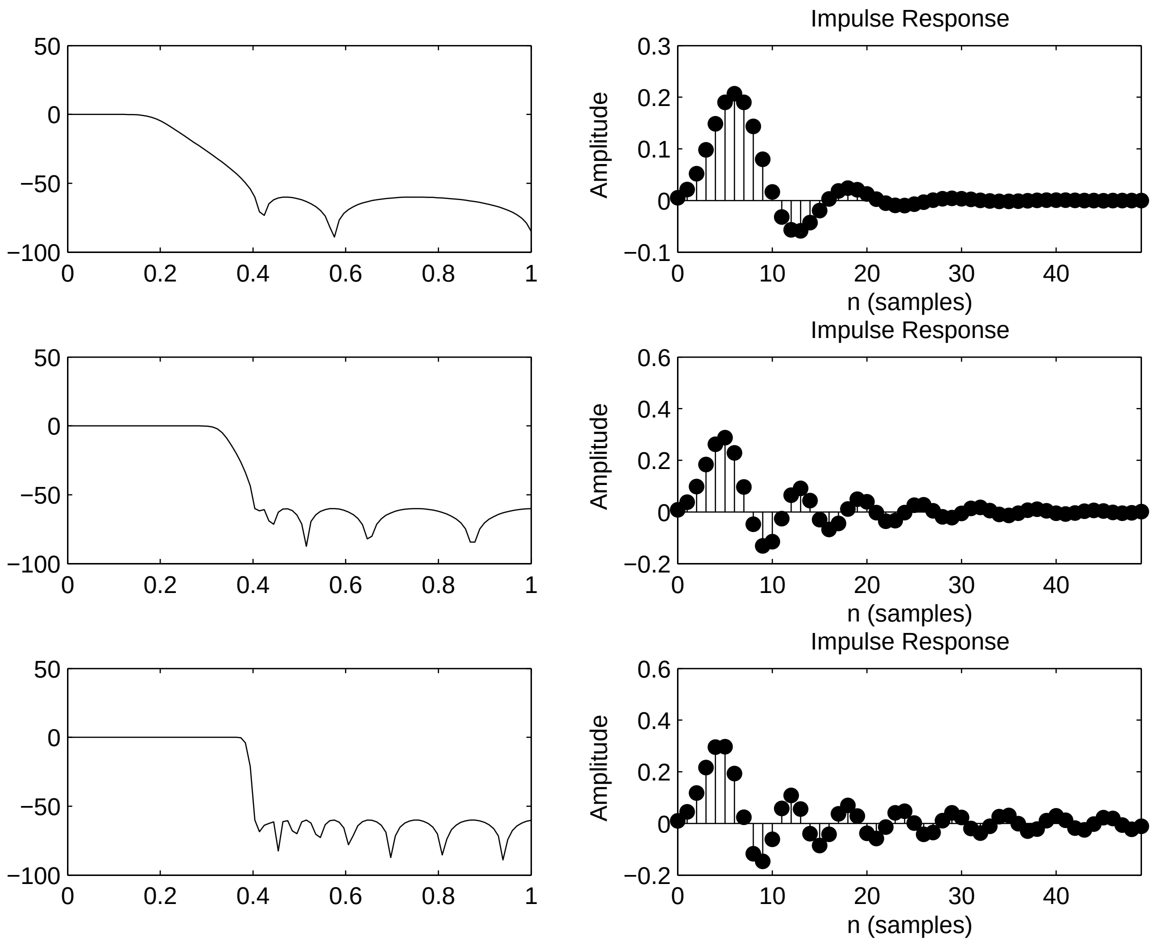

FIG. 4. Amplitude spectra and corresponding minimum phase signals, formed through a Chebyshev filter at 60dB cutoff, with orders 5, 10, and 20.

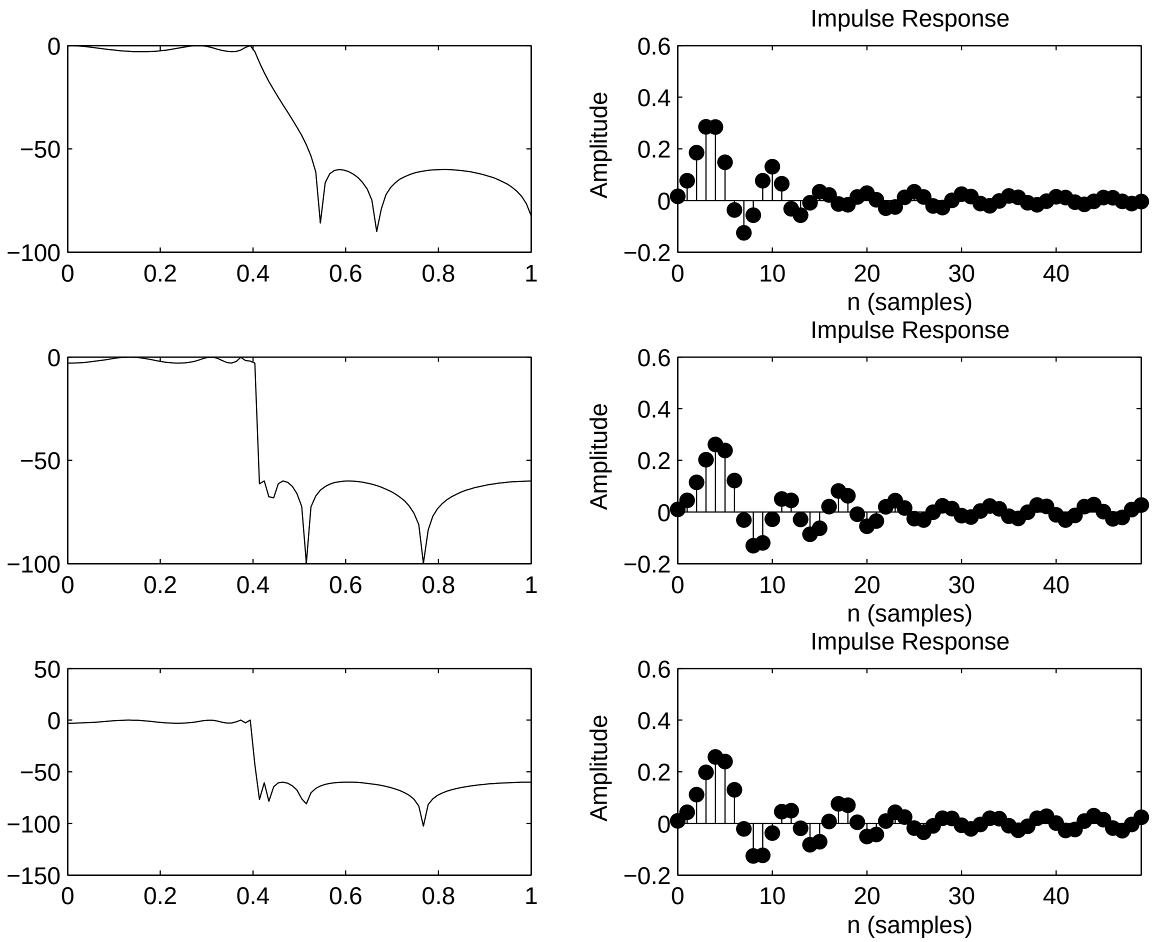

FIG. 5. Amplitude spectra and corresponding minimum phase signals, formed through an elliptic filter, at 60dB cutoffwith orders 5, 10, and 20.

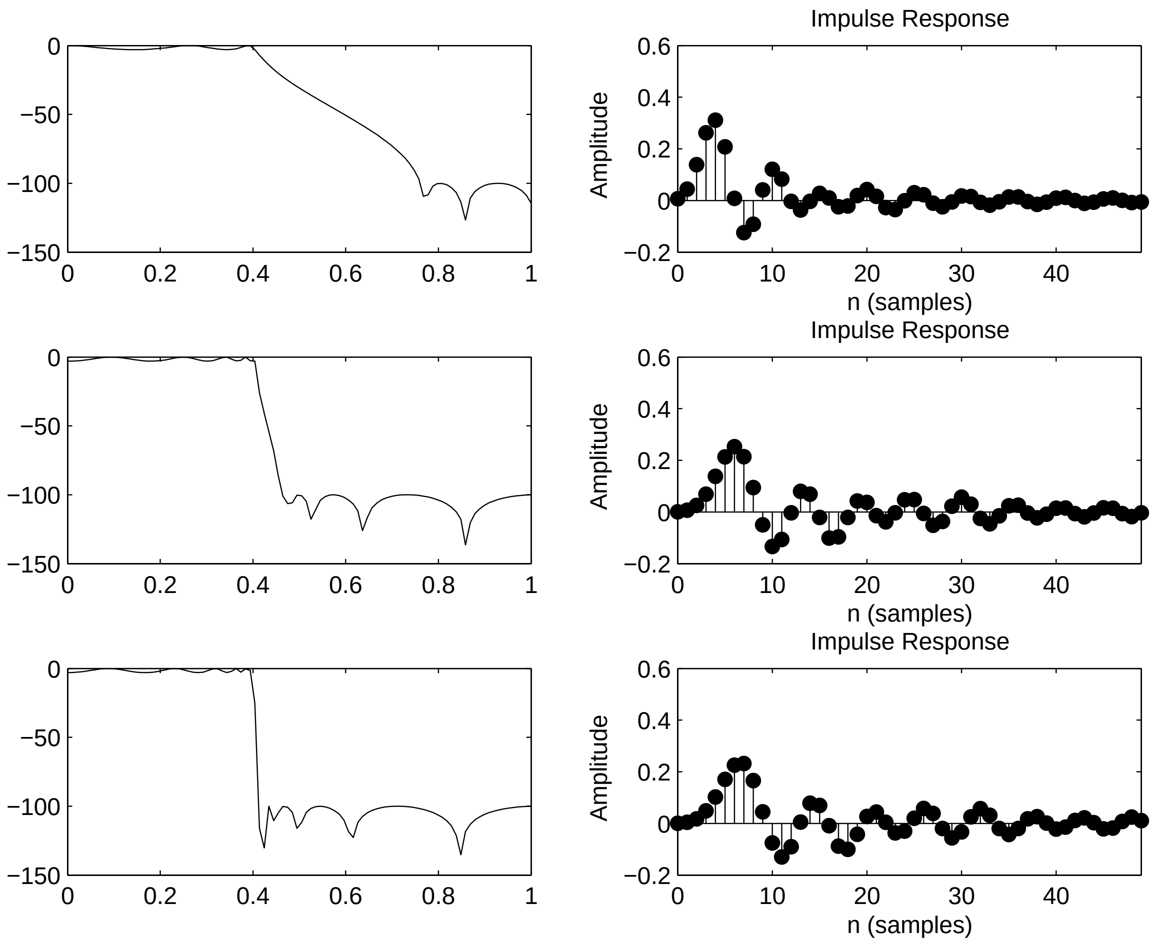

FIG. 6. Amplitude spectra and corresponding minimum phase signals, formed through an elliptic filter, at 100dB cutoffwith orders 5, 10, and 20.

THE BUTTERWORTH CASE

For completeness, we show the case in using a Butterworth IIF filter. In Figure 7, we show the resulting signal given by passing the delta spike through Butterworth filters of order 5, 10, and 20. Unfortunately, these signals do not look very minimum phase – the energy is pushed too far over. A quick check with MATLAB shows that some of the zeros of this filter lie outside the unit circle, hence this is not a minimum phase filter. Which is odd, since the design criteria for a Butterworth filter implies that the resulting filter should be minimum phase. This suggests there is a problem in the MATLAB code that produces Butterworth filters. We have not investigated this further.

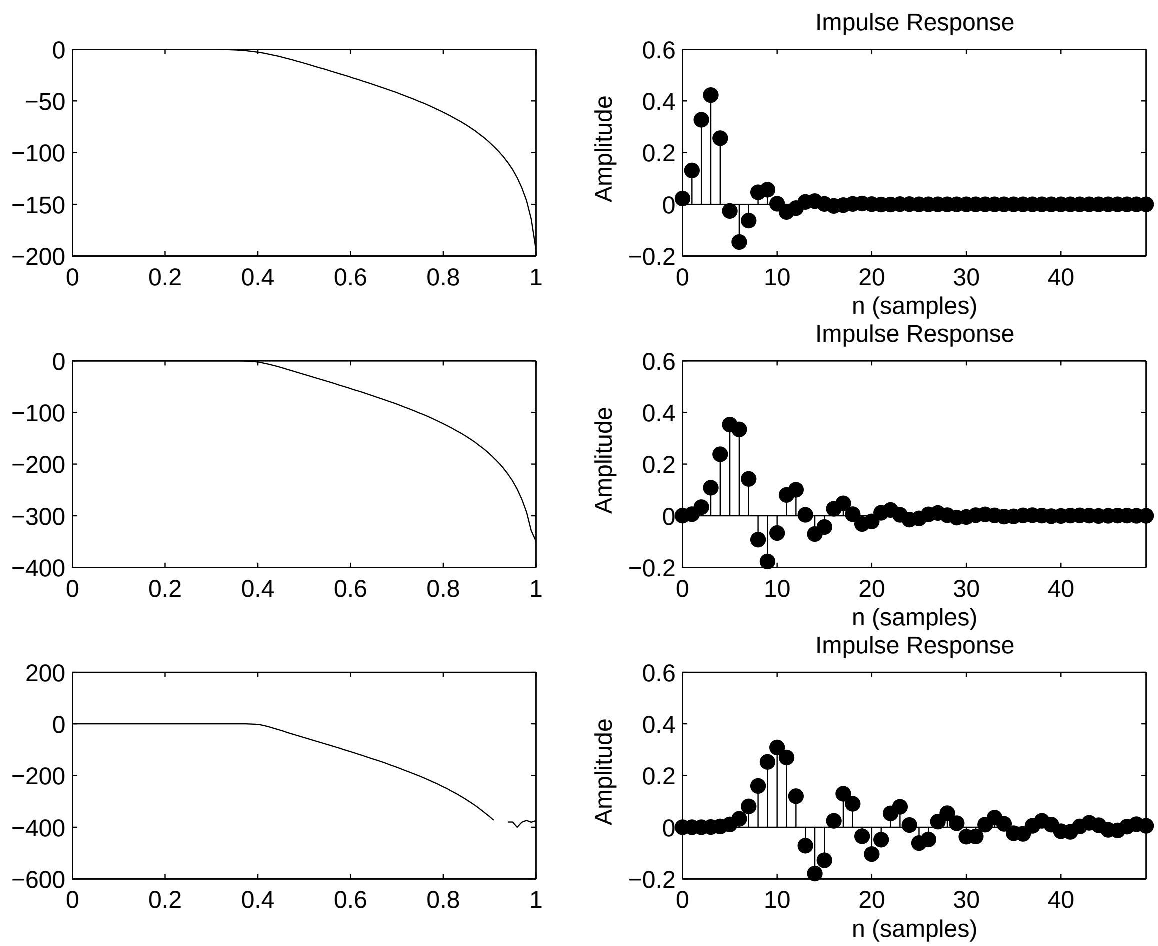

FIG. 7. Amplitude spectra and corresponding minimum phase signals, formed through a Butterworth filter, of orders 5, 10, and 20.

COMPARISON

We compare the standard method, with the three filter methods, in Figure 8. All at 60dB cutoff, so we can see how different they are. Notice that Butterworth is unacceptable.

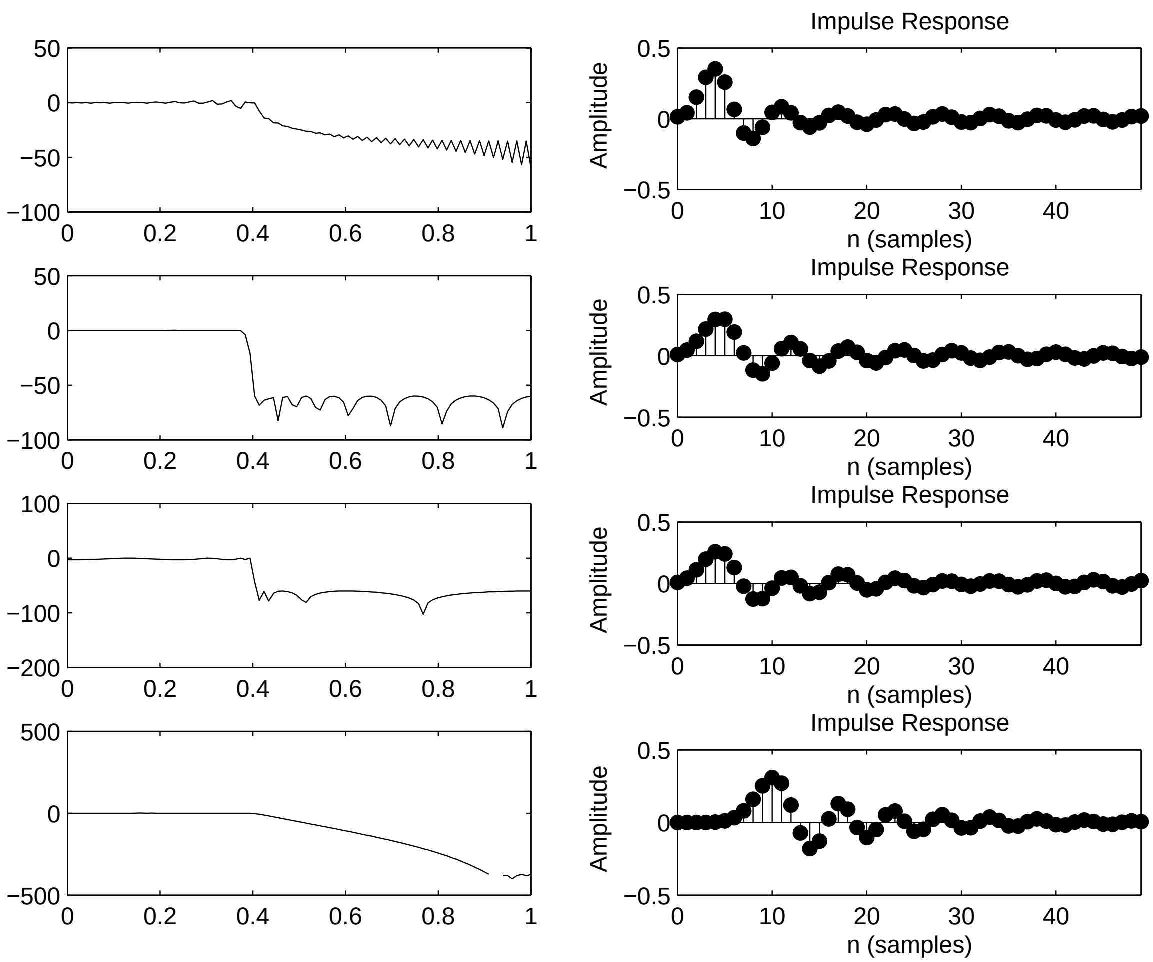

FIG. 8. Amplitude spectra and corresponding minimum phase signals, standard method, Chebyshev, Elliptic, and Butterworth.

SUMMARY

No minimum phase version of a band limited signal exists. However, it is possible to make certain approximations, making use of IIR filters as an alternative to the Hilbert transform.

We have demonstrated how to use an IIR filter to build an approximate minimum phase version of a band-limited delta spike, avoiding the Hilbert transform and stabilization factors necessary in the standard approach, such as the MATLAB command rceps. We observe that the delay introduced into the causal signal is a function of the sharpness of the IIR filter cutoff, not something intrinsic to the delta spike. We have compared the rceps method with different IIR filters, including Chebyshev, elliptic, and Butterworth. The Chebyshev and elliptic filters give good performance. We have not compared these methods with other Hilbert transform methods, such as in the CREWES toolbox.

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