Seismic waves in poroviscoelastic media: A tutorial

Edited and partially retranslated by Pat F. Daley[1]

ABSTRACT[2]

Initially, it should be mentioned that this report is a highly edited redraft, modification and partial retranslation of a paper by Frenkel (1944) which was written in Russian and translated by Cheng (1972). As the title of the original paper dealt with *seismoelectric theory*, some of the original text has been deleted, some moved to Appendix A, and additional text and formulae have been added. The writer of this apologizes to the original author and translator for this (possible) misuse of their academic research, but it, compared with other similar works, explains in a fairly uncomplicated manner the implementation of Darcy's law into the elastodynamic equations, producing, at least in principle, the equations of motion for compressional (P) and shear (S) waves propagating in a poroviscoelastic medium. Other papers and texts on this topic will have to be consulted for more numerically specific aspects of this problem. However, a reasonable understanding of the contents of this report should make such undertakings much easier. All equations have been re-derived and typographic errors corrected, and points that may require further investigation are annotated in footnotes. Appendix B has been added, in which all of the parameters used in the text are defined, as the notation of the original translation has been modified to conform to recent works dealing with this topic. Finally, it should be noted that as the original paper was severely modified, this is a preliminary report and is subject to changes to improve its readability or to introduce additional relevant content.

INTRODUCTION[3]

The processing of seismic data by geoscientists is based almost exclusively on elastodynamic theory. For the general area of seismological study, this is a reasonable choice. However, for those dealing with the hydrocarbon exploration and data processing related to this endeavor, the question of whether a more relevant and accurate theory should be incorporated in their work, specifically that of porosviscoelasticity. As the basis of hydrocarbon recovery from reservoirs at depth involves the flow of a fluid through a porous medium, which is recovered and brought to the earth's surface, would indicate that elastostodynamic theory alone might be found lacking when used in explaining the dynamics of this process. Porosviscoelastic theory would be the more

[1] Department of Geoscience, University of Calgary: pdaley@ucalgary.ca

[2] *The theory of motion of a fluid in a porous medium based on Darcy's law does not take into account the fact that the particles comprising the dry porous medium can be elastically compressed and extended and assumes that the external forces and hydrostatic pressure act only on the fluid which occupy the pores.*

[3] *Linearized elastodynamic theory describing seismic wave propagation in an isotropic homogeneous medium is specific to the aforementioned medium type with no theoretical provisions, even in the inhomogeneous extension, for the existence of a static fluid phase, much less a fluid phase that has a flow rate within in the medium measured relative to the isotropic solid.*

correct choice. As the core curricula at post secondary institutions in the geosciences are designed to provide a general treatment of seismic wave propagation, the inclusion of this type of specialized topic most often has to be delegated to a post graduate program. The development of poroviscoelastic theory presented here is directed at the geoscientist. Starting with basic elastodynamic theory, Darcy's law is introduced to construct equations governing the propagation of seismic waves in a porous medium with a fluid occupying the pores. It is by no means an all inclusive exposition of the poroviscoelastic problem, but rather has been designed to provide an introduction of the topic which should allow for more specific problems in this area to be considered.

DRY POROUS MEDIUM

A porous medium, assuming a two – phase system, is characterized, from the point of view of elastic properties, as the propagation of waves in each of its two constituents; the solid and fluid phases. The propagation of each phase is partially dependent on the propagation of the other. Let it initially be assumed that the fluid phase is totally absent so that the volume occupied by the pores is empty. The elastic properties of the medium may then be described, from a macroscopic point of view by equations related to wave propagation in an elastodynamic solid, which will taken here to be isotropic. The quantities

$$\tau_{ik} = \delta_{ik}\lambda\theta + 2\mu e_{ik} \tag{1}$$

are the components of the elastic stress tensor, $\tau_{ik}\ (i,k = 1,2,3)$ where $\delta_{ik} = 1$ for $i = k$ and $\delta_{ik} = 0$, $i \neq k$ The related quantities

$$e_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) \tag{2}$$

(Aki and Richards, 1980) are the components of the strain tensor, where $u_i\ (i = 1,2,3)$ are the components of particle displacement vector, dependent on the Cartesian coordinates $x_i\ (i = 1,2,3)$. Letting $\theta = \nabla \cdot \mathbf{u} = \sum \partial u_i / \partial x_i$ defines the relative change of the effective volume of the dry porous medium. Also, λ and μ are Lamé's coefficients, which specify the elastic properties of the dry medium, which is a porous elastic medium with empty pores. From equation (1) the expressions for the components of the elastic force acting upon a unit volume of the dry porous medium are

$$\Phi_i^{(s)} = \sum_k \frac{\partial \tau_{ik}}{\partial x_k} = \lambda \frac{\partial \theta}{\partial x_i} + \mu \sum_k \frac{\partial^2 u_i}{\partial x_k^2} + \mu \frac{\partial}{\partial x_i} \sum_k \frac{\partial u_k}{\partial x_k} \ . \tag{3}$$

In vector notation the above becomes

$$\mathbf{\Phi}^{(s)} = (\lambda + \mu)\nabla\theta + \mu\nabla^2\mathbf{u} \tag{4}$$

In the macroscopic theory of a porous elastic medium, only those distances that are large when compared to the dimensions of the pores[4], and such volumes that contain a large number of pores within a "reasonable" volume of the solid medium are considered. The presence and degree of porosity is accounted for by the coefficient ϕ[5] which is equal to the ratio of the volume of pores, V_f to the total (macroscopic) volume occupied by the porous medium, that is, $V = V_s + V_f$, where V_s is the volume actually filled by the solid constituent. The actual density of the solid constituent will be denoted by ρ_s and the mean (macroscopic or effective) density of the porous medium by γ_s. Referring these quantities to the volumes V_s and $V_s + V_f$ for a unit mass results in

$$\rho_s = \frac{1}{V_s} \ , \quad \gamma_s = \frac{1}{V_s + V_f} \ . \tag{5}$$

As a consequence

$$\gamma_s = \rho_s\left(\frac{V_s}{V_s + V_f}\right) = \rho_s\left(1 - \frac{V_f}{V_s + V_f}\right). \tag{6}$$

Employing the definition of ϕ

$$\gamma_s = \rho_s(1 - \phi). \tag{7}$$

The change in the volume of the porous medium consists of two parts:

1. the change in volume of a unit mass of the solid phase, ΔV_s, and

2. the change in the volume of the pores associated with this unit mass, ΔV_f.

For the case of small strains[6], which are routinely considered in elastodynamic theory, these quantities can be taken to be proportional to one another, so that

$$\Delta V_s = \alpha \Delta V_f \tag{8}$$

where α is some proportionality coefficient[7], which together with the porosity ϕ specifies the mechanical properties of the dry porous medium. With the assistance of this parameter, it is possible to express the variation of the degree of porosity of the porous

[4] Probably more accurately, a distribution of pore sizes that does not violate the macroscopic theory.

[5] This is an overly cautious manner of introducing porosity. In what follows, ϕ **is** porosity.

[6] Small strains imply small deformations, which is the foundation of linearization, as well as other related approximations to the state equations derived from physical principles.

[7] Although this may seem fairly straightforward, α is a very important quantity, which if not estimated correctly, can cause large inaccuracies in any related computations.

medium due to its deformation. Using the definition of ϕ for a unit mass, the following results

$$\Delta\phi = \frac{\Delta V_f}{\left(V_s + V_f\right)} - \frac{V_f\left(\Delta V_s + \Delta V_f\right)}{\left(V_s + V_f\right)^2} = \frac{V_s \Delta V_f - V_f \Delta V_s}{\left(V_s + V_f\right)^2} \tag{9}$$

which leads to

$$\frac{\Delta\phi}{\phi} = \frac{\left(V_s/V_f\right)\Delta V_f - \Delta V_s}{\left(V_s + V_f\right)} = \left(\frac{\left(V_s/V_f\right) - \alpha}{\left(V_s + V_f\right)}\right)\Delta V_f \ . \tag{10}$$

Since

$$\frac{V_s}{V_f} = \frac{V_s + V_f}{V_f} - 1 = \frac{1}{\phi} - 1 = \frac{1-\phi}{\phi} \tag{11}$$

[8]

then

$$\Delta\phi = \frac{1-\phi\left(1+\alpha\right)}{\left(V_s + V_f\right)}\Delta V_f \ . \tag{12}$$

From another perspective, according to the definition of the quantity $\theta = \nabla \cdot \mathbf{u}$ it follows that

$$\theta = \frac{\Delta V}{V} = \frac{\Delta V_s + \Delta V_f}{\left(V_s + V_f\right)} = \frac{\left(1+\alpha\right)\Delta V_f}{\left(V_s + V_f\right)} \ . \tag{13}$$

Thus the subsequent relationship between $\Delta\phi$ and θ is obtained as

$$\Delta\phi = \frac{1-\phi\left(1+\alpha\right)}{\left(1+\alpha\right)}\,\theta \ . \tag{14}$$

In the case of high porosity, the coefficient α must be small compared with unity, so that the compression or expansion of the porous medium is realized primarily as the result of the compression or expansion of its pores. With a decrease in porosity, α must increase. It is natural to assume, however, that for values of ϕ not equal to zero, the product $\phi\left(1+\alpha\right)$ must be smaller than unity, i.e., $1+\alpha < 1/\phi$.

The forces acting on a solid body and characterized by the stress tensor τ_{ik} can be divided into a pressure, $p_s = \left(1/3\right)\left(\tau_{11} + \tau_{22} + \tau_{33}\right)$ and shear stresses, specified by the tensor

[8] Compare this to what is termed *tortuosity*.

$$\tau'_{ik} = \tau_{ik} - \delta_{ik}\frac{1}{3}\sum_i \tau_{ii} = \tau_{ik} + \delta_{ik} p_s \ . \tag{15}$$

In the absence of shear stresses, the deformation of the body is reduced to either a simple expansion or compression, with the pressure being defined by the formula

$$-p_s = \lambda\theta + \frac{2}{3}\mu\left(e_{11}+e_{22}+e_{33}\right)\tau'_{ik} = \left(\lambda+\frac{2}{3}\mu\right)\theta \tag{16}$$

which follows from equation (1) together with the definition of θ. Thus

$$\theta = -\frac{1}{K_b} p_1 \tag{17}$$

where $K_b = \lambda + \frac{2}{3}\mu$ is the bulk modulus[9] of the dry porous medium, and as a result, taking equation (14) into consideration,

$$\Delta\phi = -\frac{1-\phi(1+\alpha)}{(1+\alpha)K_b}\ p_s \ . \tag{18}$$

Hence, if the condition $1+\alpha < 1/\phi$ holds, the compression of the dry porous medium must be accompanied by a decrease in porosity, whereas for the case $1+\alpha > 1/\phi$ the opposite would be true. The following relations will also be needed in what follows

$$e_{ik} = \delta_{ik}\sigma p_1 + \vartheta\tau_{ik} \tag{19}$$

which are obtained by employing equations (1) for the quantities e_{ik}. The coefficients σ and ϑ are defined by the formulae

$$\sigma = \frac{\lambda+(2/3)\mu}{2\mu(\lambda+2\mu)} p_s \quad , \quad \vartheta = \frac{1}{2\mu} \quad . \tag{20}$$

The absence of shear stresses is characterized by the relationships $\tau_{11} = \tau_{22} = \tau_{33} = -p_s$ which together with $\tau_{12} = \tau_{23} = \tau_{31} = 0$[10], requires that equation (19) reduces to equation (17) in this case.

SATURATED POROUS MEDIUM

To this point a dry porous medium has been considered. Now assume that all of the pores are totally filled with a fluid, which can flow freely within the pore spaces. The question that arises is: "What will be the influence of the fluid phase under such a situation on the macroscopic properties of the porous medium?"

---

[9] See the definition of compressibility in Appendix B.

[10] $\boldsymbol{\tau}$ is symmetric, so that $\tau_{21} = \tau_{32} = \tau_{13} = 0$ also holds in this case.

In order to remain in equilibrium the fluid phase must, in the absence of external forces, be subjected to the same hydrostatic pressure, p_f, at all points in the connected[11] space formed by the pores. This pressure is also exerted on the solid skeleton of the porous medium. The resulting deformation of this skeleton must be proportional to the variation of the volume of the solid phase in the same ratio as that of the fluid phase in the following manner

$$\frac{\Delta V_s}{V_s} = \frac{\Delta V_f}{V_f} = -\frac{1}{K_s} p_f \tag{21}$$

where K_s is the actual (effective) bulk modulus[12] of the solid phase. It follows that the entire (macroscopic) volume of the porous medium must vary at the same ratio

$$\frac{\Delta V}{V} = \theta = -\frac{1}{K_s} p_f \tag{22}$$

and that the porosity percentage must remain unaltered. Consequently

$$\Delta\phi = 0 \ . \tag{23}$$

Comparing equations (17) and (22) indicates that the hydrostatic pressure, p_f, is equivalent to the compression of the porous medium, produced by it, so that the pressure in the solid given by

$$p_s = \frac{K_b}{K_s} p_f \tag{24}$$

which is such that it is smaller than p_f because $K_s > K_b$.[13]

Equation (22) requires an additional condition to complete its specification. This is given by

$$\frac{\Delta\rho_f}{\rho_f} = \frac{1}{K_f} p_f \tag{25}$$

relating the true density of the fluid (as well as the bulk modulus of the fluid) with the hydrostatic pressure and represents the approximate form of the equation of state of the fluid at a constant temperature.[14]

[11] What is meant here is that all of the pore space is effective. That is, there are no pore spaces that are not connected to another pore space. Taken to the limit, all pore spaces are connected in some fashion. If all the pore space is not effective, the following derivations are questionable without some correction factor that compensates for the unconnected pore space. Again see *tortuosity*.

[12] Again see the definition of compressibility in Appendix B.

[13] $K_s > K_b = \lambda + (2/3)\mu$.

[14] K_f is the bulk modulus of the fluid. The definition of compressibility is given in Appendix B.

It must be kept in mind that the mass of the fluid, $\rho_f V_f$ **,** filling the pores in a certain part of the solid skeleton, V_s, is, generally speaking, a variable quantity[15], when compared to the solid constituent, $\rho_s V_s$, which remains constant. The pressures p_s and p_f are totally different in their origin and nature, and as such are completely independent of one another. The total variation of the macroscopic volume of the porous medium, due to their combined action, is equal to the sum of the variations (elastodynamic), due to each of them being considered separately. Adding the expressions (17) and (22) results in

$$\theta = -\frac{1}{K_b} p_s - \frac{1}{K_s} p_f \ . \tag{26}$$

This formula is valid in the special case of the absence of shear stresses in the solid skeleton of the rock. Given that these stresses are absent when the fluid has no relative velocity (motion) to the rock (fluid at rest), the stress tensor in the rock reduces to, in the general case, to the sum of equation (19) and the tensor $\left[-\left(1/3K_s\right) p_f \delta_{ik}\right]$. This leads to the relationship

$$e_{ik} = \delta_{ik}\left(\sigma p_s - \frac{1}{3K_s} p_f\right) + \vartheta \tau_{ik} \ . \tag{27}$$

The above equations, together with equations (25) specify the deformations of the fluid and solid phases as functions of the stresses. To derive the equations of motion in the solid, the stresses must be expressed as functions of deformation (strain). Define $\varphi = \Delta\rho_f / \rho_f$ such that φ (*not to be mistaken for* ϕ) becomes a characterization of the liquid phase in a similar manner as θ does for the solid phase. Replacing p_f in compliance with equation (25) $\left(p_f = -K_f \varphi\right)$ equations (27) may be rewritten in the form

$$e_{ik} - \delta_{ik}\frac{K_f}{3K_s}\varphi = \delta_{ik}\sigma p_s + \vartheta\tau_{ik} \ . \tag{28}$$

$$\tau_{ik} = \delta_{ik}\left(\lambda\theta - \frac{K_f}{K_s}\varphi\right) + 2\mu\left(e_{ik} - \delta_{ik}\frac{K_f}{3K_s}\varphi\right) \tag{29}$$

or since $\lambda + \frac{2}{3}\mu = K_b$

$$\tau_{ik} = \delta_{ik}\left(\lambda\theta - \frac{K_b K_f}{K_s}\varphi\right) + 2\mu e_{ik} \tag{30}$$

These equations together with the equation of state of the fluid phase

[15] That is, the fluid is compressible. If it were incompressible then the mass of the fluid would not be variable.

$$p_f = -K_f \varphi \tag{31}$$

allows for the determination of the elastodynamic volume forces on the solid in the case when the quantities τ_{ik} and p_f vary arbitrarily within the medium. In classic elastodynamic theory, the vector components of the elastic force, with reference to some unit volume of the medium, are given by

$$\Phi_i^{(s)} = \sum_k \frac{\partial \tau_{ik}}{\partial x_k}. \tag{32}$$

In the problem being considered here, the above expression refers only to the solid phase, which is contained in a unit volume of the medium. In the presence of a hydrostatic pressure, a unit volume of the solid is also acted upon by the force $-\nabla p_f$, which is distributed between the fluid and solid phases according to the ratio of their respective volumes, which is, $\phi/(1-\phi)$. Equations for the components of $\Phi_i^{(s)}$ may now be written as

$$\Phi_i^{(s)} = \sum_k \frac{\partial \tau_{ik}}{\partial x_k} - (1-\phi)\frac{\partial p_f}{\partial x_i}. \tag{33}$$

The force acting on the fluid phase per unit volume of the solid is

$$\mathbf{\Phi}^{(f)} = -\phi \, \nabla p_f \,. \tag{34}$$

Substituting equations (33) and (34) into the expressions (30) and (31) results in (for the solid)

$$\begin{aligned} \Phi_i^{(s)} &= \frac{\partial}{\partial x_k}\left(\lambda\theta - \frac{K_b K_f}{K_s}\varphi \right) + \mu\left(\sum_k \frac{\partial^2 u_i}{\partial x_k^2} + \frac{\partial}{\partial x_i}\sum_k \frac{\partial u_k}{\partial x_k} \right) + (1-\phi) K_f \frac{\partial \varphi}{\partial x_i} \\ &= \frac{\partial}{\partial x_k}\left((\lambda+\mu)\theta - \frac{K_b K_f}{K_s}\varphi \right) + \mu \sum_k \frac{\partial^2 u_i}{\partial x_k^2} + (1-\phi) K_f \frac{\partial \varphi}{\partial x_i} \end{aligned} \tag{35}$$

or in vector notation

$$\mathbf{\Phi}^{(s)} = (\lambda+\mu)\nabla\theta + \mu\nabla^2\mathbf{u} + K_f\left(1-\phi-\left(K_f/K_s\right)\right)\nabla\varphi. \tag{36}$$

For the liquid phase

$$\mathbf{\Phi}^{(s)} = \phi K_s \nabla \varphi. \tag{37}$$

In the next section, $\mathbf{\Phi}^{(s)}$ and $\mathbf{\Phi}^{(f)}$ will be related through the introduction of Darcy's law.

EQUATIONS OF MOTION IN A POROUS MEDIUM: THE INTERACTION BETWEEN SOLID AND FLUID PHASES

From the theory of hydrodynamics, the mean velocity of the flow of a fluid, which completely occupies the pores in a porous solid, under the additional assumption of absolute rigidity of the solid skeleton, is determined by Darcy's equation (Craft and Hawkins, 1991)

$$\mathbf{v}_f = \frac{k}{\eta}\left(-\nabla p_f + \mathbf{F}_f\right) = \frac{k}{\eta}\left(-\nabla p_f + \mathbf{F}_f\right) \qquad (38)$$

where η is the viscosity, k is the permeability and $\mathbf{F}_f$ denotes the external forces acting on the fluid contained within a unit volume of the porous medium. The permeability is proportional to the porosity, ϕ, and the average values of a cross section of the pores in the porous medium. This would indicate that the relationship

$$k = \text{constant} \times \phi \ell^2 \qquad (39)$$

is valid, where ℓ is a linear dimension in the porous medium. To complete the specification of the problem, as in a standard hydrodynamics problem, a continuity equation is required. This is given by

$$\frac{\partial \gamma_f}{\partial t} + \nabla \cdot \left(\gamma_f \mathbf{v}_f\right) = 0 \qquad (40)$$

where $\gamma_f = \phi \rho_f$ is the mean (effective) density of the fluid in some macroscopically small region, that contains a sufficient pore density to provide an adequately accurate definition of γ_f.[16] Equation (38) refers to the case of steady flow. For variable flow, it is replaced by

[16] In some of the literature on this topic, γ_f in the second term on the left side of equation (40), is replaced by ρ_f .so that the continuity equation has the form

$$\phi \frac{\partial \gamma_f}{\partial t} + \nabla \cdot \left(\rho_f \mathbf{v}_f\right) = 0 .$$

It can be shown that in this form it contradicts the law of conservation of mass of the fluid, when compared to the definition of mean macroscopic velocity of flow, $\mathbf{v}_f$. This circumstance is of no consequence as long as the liquid is assumed to be incompressible or if the solid skeleton of the porous medium is taken to be absolutely rigid. (In the latter case, the definition of $\mathbf{v}_f$ must be somewhat altered.) In the case of a deformable skeleton, the use of the equation contained in this footnote will lead to erroneous results.

$$\gamma_f \frac{\partial \mathbf{v}_f}{\partial t} = -\nabla p_f + \mathbf{F}_f - \frac{\eta}{k}\mathbf{v}_f \,. \tag{41}$$

The higher order term $(\nabla \cdot \mathbf{v}_f)\mathbf{v}_f$ has been neglected in the above equation. Equation (41) is inexact[17] because of the absence of the factor ϕ preceding the gradient of the pressure, p_f. Introducing this factor, the corrected equation for the motion in the fluid is obtained as

$$\gamma_f \frac{\partial \mathbf{v}_f}{\partial t} = -\phi\nabla p_f + \mathbf{F}_f - \frac{\eta}{\kappa}\mathbf{v}_f \tag{42}$$

where the permeability has been replaced by the porosity normalized permeability defined as

$$\kappa = \frac{k}{\phi} = \text{constant} \times \ell^2 \tag{43}$$

whose introduction ensures Darcy's law will be satisfied for the case of steady flow of the fluid.

Equation (42) may be generalized to the case when the deformability and the mobility of the solid skeleton become important, in the propagation of elastic vibrations.. Here, the absolute velocity of the liquid, $\mathbf{v}_f$, must be replaced by its velocity relative to the solid phase, $\mathbf{v}_f - \mathbf{v}_s$. The quantity $\mathbf{v}_s$, where, $\mathbf{v}_s = \partial\mathbf{u}/\partial t$ ($\mathbf{u}$ being particle displacement) denotes the mean macroscopic velocity of the particles in the solid phase at some arbitrary point within the porous medium. The relative velocity is connected to the friction force acting on the fluid in a unit volume of the porous medium through the formula

$$\mathbf{F}_{fs} = -\frac{\eta}{\kappa}\left(\mathbf{v}_f - \mathbf{v}_s\right) \tag{44}$$

the solid phase being acted on by the fluid phase in a unit volume by an equal but opposite force, $\mathbf{F}_{sf} = -\mathbf{F}_{fs}$.

Replacing $\mathbf{v}_2$ in equation (42) by $\mathbf{v}_f - \mathbf{v}_s$ and p_f by $-K_f\varphi$ results finally in

$$\gamma_f \frac{\partial \mathbf{v}_f}{\partial t} = \phi K_f \nabla\varphi + \mathbf{F}_f - \frac{\eta}{\kappa}\left(\mathbf{v}_f - \mathbf{v}_s\right). \tag{45}$$

In the absence of external forces equation (45) may be written as

[17] Both of the normalizations $\kappa = k/\phi$ and $\gamma_f = \phi\rho_f$ cause this inconsistency.

$$\rho_f \frac{\partial \mathbf{v}_f}{\partial t} = K_f \nabla \varphi - \frac{\eta}{\kappa\phi}\left(\mathbf{v}_f - \mathbf{v}_s\right). \tag{46}$$

The equation of motion of the solid phase in the general case of a relative motion of the fluid can be written as

$$\gamma_s \frac{\partial \mathbf{v}_s}{\partial t} = \mathbf{\Phi}^{(s)} + \mathbf{F}_s + \frac{\eta}{\kappa}\left(\mathbf{v}_f - \mathbf{v}_s\right) \tag{47}$$

As in elastodynamic theory, velocity is considered to be a function of time and the spatial coordinates of a solid particle, so that $\partial \mathbf{v}_s / \partial t$ is the specification of acceleration. In the fluid phase, $\partial \mathbf{v}_f / \partial t$ is not the exact expression for the corresponding quantity. Rather, the exact expression in this case is $\partial \mathbf{v}_f / \partial t + \left(\mathbf{v}_f \cdot \nabla\right)\mathbf{v}_f$, which includes a term that was previously dropped. In practice, however, the motion of the fluid is so slow that the addition of the extra term does not play significant role, as it is negligibly small. It should also be mentioned that the effective density of the solid phase, $\gamma_s = \left(1-\phi\right)\rho_s$, in equation (47) refers to the unstressed state and must as a consequence must be assumed to be a constant quantity.

Substituting expression (36) for $\mathbf{\Phi}^{(s)}$ in equation (47) produces, in the absence of body forces,

$$\gamma_s \frac{\partial \mathbf{v}_s}{\partial t} = \left(\lambda+\mu\right)\nabla\theta + \mu\nabla^2\mathbf{u} + K_f\left(1+\phi-K_b/K_s\right)\nabla\varphi + \frac{\eta}{\kappa}\left(\mathbf{v}_f - \mathbf{v}_s\right). \tag{48}$$

Equations (45) or (46) and (48) are functions of the quantities: $\mathbf{u}$, $\mathbf{v}_f$, ρ_f, φ and ϕ ($\mathbf{v}_s = \partial\mathbf{u}/\partial t$, $\theta = \nabla\cdot\mathbf{u}$ and $\gamma_s = \text{constant}$). The quantities ρ_f and φ are related by the expression

$$\rho_f = \rho_f^{(0)}\left(1-\varphi\right) \tag{49}$$

where $\rho_f^{(0)}$ is the average density of the fluid. Its effective density is related to the velocity $\mathbf{v}_f$ through the continuity equation (40). The variation of the porosity may be expressed in terms of θ as

$$\Delta\phi = \frac{1-\phi\left(1-\alpha\right)}{\left(1+\alpha\right)}\left(\theta - \frac{K_f}{K_s}\varphi\right) \tag{50}$$

This relation is obtained from equation (27) if θ is replaced by $\theta = -p_s/K_b$, which is in terms of the solid phase pressure and as has been shown in the derivation of equation (30), is equal to $\theta - \left(K_f/K_s\right)\varphi$..

Thus there are five equations in five unknowns such that the equations of motion in the porous solid are fully determined.

COMPRESSIONAL WAVE PROPAGATION IN A POROUS MEDIUM

In this section compressional wave propagation is investigated, under the assumption of small vibrations in the porous medium, which allows for the linearization of the equations of motion. With this supposition, the small coefficients of quantities of interest may be replaced by their values in an unstressed medium, as is done in classical elastodynamic and hydrodynamic problems. The quantities in question include: $\mathbf{u}$, $\mathbf{v}_s$, $\mathbf{v}_f$, θ, φ and $\Delta\phi$. For the analysis of compressional waves the operator "$\nabla\cdot$" is applied to the equations of motion. The following formula will be used

$$\nabla\cdot\mathbf{v}_s=\frac{\partial}{\partial t}\nabla\cdot\mathbf{u}=\frac{\partial\theta}{\partial t} \tag{51}$$

together with the equation

$$\nabla\cdot\mathbf{v}_f=-\frac{1}{\gamma_f}\frac{\partial\gamma_f}{\partial t}=-\frac{1}{\phi}\frac{\partial\phi}{\partial t}-\frac{1}{\rho_f}\frac{\partial\rho_f}{\partial t}=-\frac{1}{\phi}\frac{\partial\Delta\phi}{\partial t}+\frac{\partial\varphi}{\partial t} \tag{52}$$

which follows from equation (20) and in its linearized form, relating equation (23) with equation (50), can be written as

$$\nabla\cdot\mathbf{v}_f=-(\beta-1)\frac{\partial\theta}{\partial t}+\beta'\frac{\partial\varphi}{\partial t} \tag{53}$$

where the following intermediate variables have been introduce to reduce the complexity of notation within the problem

$$\beta=\frac{1}{\phi(1+\alpha)},\quad \beta'=1+(\beta-1)\frac{K_f}{K_s}. \tag{54}$$

Applying the operator "$\nabla\cdot$" to both sides of the linearized equation (48) and utilizing the previous equations in this section, the following results

$$\frac{\partial^2\theta}{\partial t^2}=\frac{E}{\gamma_s}\nabla^2\theta+\frac{K_f}{\gamma_s}\left(1-\phi-\frac{K_b}{K_s}\right)\nabla^2\varphi+\frac{\eta}{\kappa\gamma_s}\left(\beta'\frac{\partial\varphi}{\partial t}-\beta\frac{\partial\theta}{\partial t}\right). \tag{55}$$ [18]

In a similar manner equation (43) becomes

$$\frac{\partial^2\varphi}{\partial t^2}-\frac{\beta-1}{\beta'}\nabla^2\theta=\frac{K_f}{\beta'\rho_f}\nabla^2\varphi-\frac{\eta}{\kappa\gamma_s}\left(\frac{\partial\varphi}{\partial t}-\frac{\beta}{\beta'}\frac{\partial\theta}{\partial t}\right). \tag{56}$$

Equations (55) and (56) contain, at least in principle, the specification of the propagation of compressional vibrations in a saturated porous medium. Before proceeding to possible solution methods, a plane wave solution will be considered in the next section.

[18] Recall that $\lambda+2\mu=E$ is Young's modulus for an elastic medium (dry porous medium).

PLANE COMPRESSIONAL WAVES

Rather than attempt to solve the most general case of propagation of compressional waves in a porous medium, a plane wave solution will be considered (with a damping coefficient or viscosity term).

Assume a plane wave type of solution for the quantities θ and φ of the form

$$e^{i(wt-qx)} \tag{57}$$

where $t-\text{time}$ and the direction of propagation may be arbitrarily chosen. Let this direction be the x direction so the plane wave solution is $e^{i(wt-qx)}$. Here, $\omega/2\pi = f$ is the frequency of the vibrations and $q/2\pi$ the complex wave number in the direction of wave propagation[19]. It is equal, in the absence of damping, to the reciprocal of the wave length, λ. The ratio ω/q is the generally complex valued velocity of wave propagation.

Under these conditions, the differential equations (55) and (56) reduce to a system of two linear algebraic equations for the amplitudes θ and φ, as[20]

$$\left(\frac{E}{\gamma_s}\xi - 1 + \frac{i\eta\beta'}{\kappa\gamma_s\omega}\right)\theta + \left(\varepsilon\xi - \frac{i\eta\beta'}{\kappa\gamma_s\omega}\right)\varphi = 0 \tag{58}$$

and

$$\left(\frac{\beta-1}{\beta'} - \frac{i\eta\beta}{\kappa\gamma_f\beta'\omega}\right)\theta + \left(\frac{K_f}{\beta'\rho_f}\xi - 1 + \frac{i\eta}{\kappa\gamma_f\beta'\omega}\right)\varphi = 0 \tag{59}$$

where $\xi = q^2/\omega^2 = 1/W_P^2$ and $\varepsilon = K_f/\gamma_s\left(1-\phi-K_b/K_s\right)$. This quantity, the propagation velocity of the wave, is determined from the solution of the quadratic equation[21]

$$\left(\frac{E}{\gamma_s}\xi - 1 + \frac{i\eta\beta'}{\kappa\gamma_s\omega}\right)\left(\frac{K_f}{\beta'\rho_f}\xi - 1 + \frac{i\eta}{\kappa\gamma_f\beta'\omega}\right) - \left(\varepsilon\xi - \frac{i\mu\beta'}{\kappa\gamma_s\omega}\right)\left(\frac{\beta-1}{\beta'} - \frac{i\eta\beta}{\kappa\gamma_f\beta'\omega}\right) = 0 \tag{60}$$

which represents the compatibility condition of equation (59)[22]. After some manipulation the above quadratic equation reduces to

---

[19] q/ω is the complex slowness in the direction of propagation.

[20] $\varepsilon = K_f/\gamma_s\left(1-\phi-K_b/K_s\right)$ See equation (A.3).

[21] A quadratic equation implies two velocities, these are the *fast* and *slow* compressional wave velocities.

[22] For the system of equations $\mathbf{A\theta} = 0$, $\mathbf{A}$ a matrix, $\boldsymbol{\theta}$ a vector, to have a solution, $\det[\mathbf{A}]$ must be equal to zero.

$$\frac{EK_f}{\beta'\gamma_s\rho_f}\xi^2-\left\{\frac{E}{\gamma_s}+\frac{K_f}{\beta'\rho_f}+\frac{\beta-1}{\beta'}\varepsilon-\frac{i\eta}{\kappa\omega}\frac{1}{\gamma_s\gamma_f}\left[E+\frac{\beta}{\beta'}K_f\left(1-\frac{K_b}{K_s}\right)\right]\right\}\xi$$
$$+1-\frac{i\eta}{\kappa\omega}\left(\frac{1}{\gamma_s}+\frac{1}{\gamma_f}\right)=0 \tag{61}$$

The two roots of equation (61) will not be given here as they are easily obtained numerically.[23] It should be noted that for large values of the parameter, $\zeta=\eta/\kappa$ one of the roots corresponds to a wave with a very small damping factor (fast compressional) while the other, to a very large damping factor (slow compressional). Waves of the second type may be difficult to detect, however, recent papers have indicated their existence (See for example, Coussy and Bourbie,1984). An approximate determination of the value of ξ, corresponds to a wave of the first kind. Approximating ξ in a series in terms of the powers of the small parameter, $i\omega\kappa/\eta=i\chi$, results in

$$\xi=\xi_0+i\chi\xi_1+\ldots \tag{62}$$

Substituting this truncated series into equation (61) and equating the coefficients of the various powers of χ, starting with χ^{-1} produces

$$\frac{1}{\gamma_s\gamma_f}\left[E+\frac{\beta}{\beta'}K_f\left(1-\frac{K_b}{K_s}\right)\right]\xi_0-\left(\frac{1}{\gamma_s}+\frac{1}{\gamma_f}\right)=0 \tag{63}$$

$$\frac{EK_f}{\beta'\gamma_s\rho_f}\xi_0^2-\left(\frac{E}{\gamma_s}+\frac{K_f}{\beta'\rho_f}+\frac{\beta-1}{\beta'}\varepsilon\right)\xi_0-\frac{1}{\gamma_s\gamma_f}\left[E+\frac{\beta}{\beta'}K_f\left(1-\frac{K_b}{K_s}\right)\right]\xi_1+1=0 \tag{64}$$

$$\frac{2EK_2}{\beta'\gamma_s\rho_f}\xi_0\xi_1-\left(\frac{E}{\gamma_s}+\frac{K_f}{\beta'\rho_f}+\frac{\beta-1}{\beta'}\varepsilon\right)\xi_1-\frac{1}{\gamma_s\gamma_f}\left[E+\frac{\beta}{\beta'}K_f\left(1-\frac{K_b}{K_s}\right)\right]\xi_2=0 \tag{65}$$

etc.

The first of these equations leads to

$$W_{P_0}=\sqrt{\frac{E+(\beta/\beta')K_f\left(1-K_b/K_s\right)}{\gamma_s+\gamma_f}} \tag{66}$$

as $\xi_0=1/W_{P_0}^2$ where W indicates velocity and the subscripts the type of wave and order of approximation.

[23] Equation (61) has two roots corresponding to the *fast* and *slow* compressional waves. However, the expressions for these two compressional wave velocities, obtained by solving the quadratic, are complex and consequently of limited usefulness. For this reason, the analysis that follows, which provides an approximation to the *fast* compressional wave, will be pursued as the resulting expression provides a usable point of reference from which numerical experiments may be undertaken to ascertain the affects of varying specific quantities on the velocity and damping factor.

Inserting the expression for $\xi_0 = 1/W_{P_0}^2$ into equation (64) results in the following first order correction

$$i\chi\xi_1 = i\chi \frac{\dfrac{EK_f}{\beta'\gamma_s\rho_f}\xi_0^3 - \left(\dfrac{E}{\gamma_s} + \dfrac{K_f}{\beta'\rho_f} + \dfrac{\beta-1}{\beta'}\varepsilon\right)\xi_0^2 + \xi_0}{\left(\dfrac{1}{\gamma_s} + \dfrac{1}{\gamma_f}\right)}. \tag{67}$$

With a first order accuracy with respect to χ for the complex propagation velocity of the (fast) compressional wave, W_{P_1}, is determined by the formula

$$\frac{1}{W_{P_1}} = \sqrt{\xi_1} = \xi_0^{1/2}\left(1 + i\chi\frac{\xi_1}{\xi_0}\right)^{1/2} \approx \xi_0^{1/2} + i\chi\frac{\xi_1}{2\xi_0^{1/2}} \text{ [24]} \tag{68}$$

that is

$$\frac{1}{W_{P_1}} = \frac{1}{W_{P_0}} + i\frac{\kappa}{\eta}\frac{\omega W_{P_0}\xi_1}{2} \tag{69}$$

where ξ_1 is given by equation (67). Inserting this expression in the exponent of the factor $\exp\left[i\left(\omega t - qx\right)\right] = \exp\left[i\left(\omega t - \omega x/W_{P_1}\right)\right]$ and writing the later in the form $\exp\left[i\left(\omega t - \omega x/W_{P_0}\right) - \delta x/2\right]$, where δ is the damping coefficient of the wave per unit length, the following expression for this coefficient is obtained

$$\delta = \frac{\kappa\omega^2}{\eta W_{P_0}^3}\frac{\left(\dfrac{E}{\gamma_s} + \dfrac{K_f}{\beta'\rho_f} + \dfrac{\beta-1}{\beta'}\varepsilon\right) - W_{P_0}^2 - \dfrac{EK_f}{\gamma_s\beta'\rho_f}W_{P_0}^{-2}}{\left(\gamma_s + \gamma_f\right)} \tag{70}$$

The damping coefficient has been shown to be proportional to the square of the vibration frequency, that is to $\omega^2 = (2\pi f)^2$. This is the same as for the case of the propagation of compressional waves in an ordinary viscous liquid.

In conclusion, consider the limiting case of wave propagation in a medium with vanishing porosity. In this case, $\phi \to 0$, so that as a consequence, $\gamma_s = \rho_s$ and $\gamma_f = 0$. Also, $\kappa \to 0$ [25], $\beta = 1/\phi(1+\alpha)$ which follows from equation (50), $\beta' = 1$ and $K_b = K_s$. Under these conditions, δ vanishes and W_{P_0} reduces to $\sqrt{E/\rho_s}$ - the standard expression for the compressional wave propagation in an isotropic elastic solid medium.

[24] A "2" appears in the denominator of the imaginary part of this equation. It does not appear in Frenkel.

[25] $k = \text{constant} \times \phi\,\ell^2$, $\kappa = k/\phi$.

Equation (33) enables the determination of the variation of this velocity with an increase in the number and size of pores, which are assumed to be fluid saturated. An essential role is played here by the decrease of the elastic modulus K_b of the solid skeleton. This topic is not dealt with here as it is complex and should be the topic of an independent study.

PROPAGATION OF SHEAR WAVES IN A POROUS MEDIUM

The equations governing the propagation of shear waves in a porous medium can be obtained from the fundamental equations of motion, (45) and (48), by applying the operator "$\nabla\times$" to them. Introducing the notation, $\mathbf{\Omega}_s = \frac{1}{2}\nabla\times\mathbf{v}_s$ and $\mathbf{\Omega}_f = \frac{1}{2}\nabla\times\mathbf{v}_f$, which are the angular velocities of the solid skeleton and of the fluid phase, respectively, results in

$$\gamma_s \frac{\partial^2 \mathbf{\Omega}_s}{\partial t^2} = \mu\nabla^2\mathbf{\Omega}_s + \frac{\eta}{\kappa}\frac{\partial}{\partial t}\left(\mathbf{\Omega}_f - \mathbf{\Omega}_s\right)$$
$$\gamma_f \frac{\partial^2 \mathbf{\Omega}_f}{\partial t^2} = -\frac{\eta}{\kappa}\frac{\partial}{\partial t}\left(\mathbf{\Omega}_f - \mathbf{\Omega}_s\right). \tag{71}$$

If a plane wave solution, which propagates in the (arbitrary) x direction $\left(e^{i(\omega t - qx)}\right)$, is assumed, the above equations reduce to the linear equations

$$\left(\mu\xi - \gamma_s\right)\mathbf{\Omega}_s = \frac{i\eta}{\kappa\omega}\left(\mathbf{\Omega}_f - \mathbf{\Omega}_s\right) \tag{72}$$

$$\gamma_f \mathbf{\Omega}_f = \frac{i\eta}{\kappa\omega}\left(\mathbf{\Omega}_f - \mathbf{\Omega}_s\right) \tag{73}$$

where $\xi = q^2/\omega^2$ and q and ω have been previously defined. Eliminating $\mathbf{\Omega}_s$ and $\mathbf{\Omega}_f$ from equations (73) results in the following equation for ξ

$$\frac{i}{\chi}\left(\frac{1}{\gamma_f} - \frac{1}{\mu\xi - \gamma_s}\right) = 1. \tag{74}$$

From this it follows that

$$\mu\xi = \gamma_s + \frac{1}{1/\gamma_f + i\chi} \quad \text{or} \quad \xi = \frac{\gamma_s}{\mu} + \frac{\gamma_f}{\mu}\left(1 + i\chi\gamma_f\right)^{-1}. \tag{75}$$

Thus, in the first approximation with respect to χ

$$\xi = \frac{\gamma_s + \gamma_f}{\mu} - i\frac{\gamma_f^2}{\mu}\chi \tag{76}$$

This formula shows that the shear waves propagate in a fluid filled porous medium with the velocity

$$W_{S_0} = \sqrt{\frac{\mu}{\gamma_s + \gamma_f}} \qquad \left[\frac{1}{W_{S_1}^2} = \frac{\gamma_s + \gamma_f}{\mu} - i \frac{\gamma_f^2}{\mu} \frac{\omega\kappa}{\chi} \right] \tag{77}$$

with the damping coefficient

$$\delta = \frac{\gamma_f^2}{\mu} \frac{\omega^2 \kappa}{\eta} = \frac{\gamma_f^2}{\left(\gamma_s + \gamma_f\right)} \frac{\omega^2 \kappa}{W_{S_0} \eta} \tag{78}$$

It can be seen from equation (78), that as in the case of compressional waves, the damping coefficient is proportional to the square of the circular frequency, ω**.**

CONCLUSIONS

Equations of particle motion have been derived for a porous medium whose pores are fluid saturated. Fluid flow within the pores of the solid skeleton results in friction between the two phases. This affects the perceived (measured) values of many of the quantities describing the total medium in such a manner that they may differ when compared to results obtained using elastodynamic theory. Employing poroviscoelastic theory in seismic exploration and data processing should, based on the derivations presented here, produce a more accurate description of the physical processes and parameters involved in hydrocarbon recovery from reservoirs at depth within the earth. Granted, a number of assumptions and approximations have been made in the course of the investigation. However, it is difficult to justify not using at least some aspects of the theory presented when dealing with seismological problems related to fluid filled porous media, which are a reasonable approximation of hydrocarbon reservoirs.

What are conspicuous by their absence are references to other texts and papers on this topic. A survey of the literature produces numerous texts and papers on this subject. As this work has been indicated as being tutorial in nature, with the specific intent of introducing the topic of seismic wave propagation in a poroviscoelastic medium, it was thought that as this report is essentially self contained, the listing of these citations would not contribute, and possibly be a detriment. A subsequent report, which is under preparation, deals with numerical solution possibilities. It contains numerous references applicable to the general topic of wave propagation in a poroviscoelastic medium.

REFERENCES

Aki K. and Richards P.G., 1980. *Quantitative Seismology,* W.H. Freeman and Company, San Francisco.

Coussy, O. and Bourbie, T., 1984, Propagation des ondes acoustiques dans les milieux poreux satures, Rev. Instr. Fr. Petrole, **39**, no. 1, 47-66 (in French).

Craft, B.C. and Hawkins, M.S. 1991, *Applied Petroleum Reservoir Engineering*, Prentice – Hall, Englewood Cliffs, N.J.

Frenkel, J., 1944, On the theory of seismic and seismoelectric phenomena in a moist soil, Journal of Physics, **3**, No. 4, 230-241 (in Russian). [Editor of English translation: Alexander Cheng, 1972.]

APPENDIX A: A SPECIAL CASE AND A SOLUTION OF THE EQUATIONS OF MOTION BASED ON THE METHOD OF SUCESSIVE APPROXIMATIONS

An important special, or rather limiting case, corresponding to an extremely large value of the parameter $\eta/\kappa\gamma_f$ (an extreme smallness of pores) will be considered first. Dividing equations (55) and (56) by this parameter and noting that the quantities θ and φ must have finite values, the following expression is obtained relating them in this instance

$$\varphi = \frac{\beta}{\beta'}\theta \tag{A1}$$

This relationship indicates that the two velocities $\mathbf{v}_s$ and $\mathbf{v}_f$ are identical. That is, there is no relative motion of the fluid with respect to the solid.

Under the conditions (A1), equation (55) reduces to

$$\frac{\partial^2\theta}{\partial t^2} = \left(\frac{E}{\gamma_s} + \varepsilon\frac{\beta}{\beta'}\right)\nabla^2\theta \tag{A2}$$

where

$$\varepsilon = \frac{K_f}{\gamma_s}\left(1-\phi-\frac{K_b}{K_s}\right) \tag{A3}$$

while equation (56) becomes

$$\frac{\partial^2\varphi}{\partial t^2} = \frac{\beta}{\beta'}\frac{K_f}{\rho_f}\nabla^2\varphi \tag{A4}$$

The later equation contradicts equation (A2), since the functions θ and φ must be connected by the relation (A1), unless the velocity of the propagation of waves, given by equation (A2) $\left[\sqrt{(E/\gamma_s)+(\varepsilon\beta/\beta')}\right]$ which coincides with the wave velocity defined by equation (A4) - $\sqrt{(K_f/\rho_f)(\beta/\beta')}$.

Proceeding to the next topic in this Appendix, it can be seen that when the parameter η/κ or $\eta/(\kappa\gamma_f)$ tends to infinity, the difference of the velocities $\mathbf{v}_f - \mathbf{v}_s$ or their divergences tend to zero in a manner such that its product with this parameter remains finite.

Keeping this in mind, this difference will be represented in the form of a series in powers of the small parameter $\kappa/\eta = \zeta$, such that

$$\varphi = \frac{\beta}{\beta'}\theta + \zeta\psi_1 + \zeta^2\psi_2 + \zeta^3\psi_3 + \ldots \qquad \text{(A5)}$$

where $\psi_1, \psi_1, \ldots$ are some unknown functions with finite values.

Before substituting this expression into equations (55) and (56) it must be noted that in solving these equations by the method of successive approximations the function θ must also be expanded in a power series in ζ, so that

$$\theta = \theta_0 + \zeta\theta_1 + \zeta^2\theta_2 + \ldots \qquad \text{(A6)}$$

and as a consequence

$$\varphi = \frac{\beta}{\beta'}\theta_0 + \zeta\left(\frac{\beta}{\beta'}\theta_1 + \psi_1\right) + \zeta^2\left(\frac{\beta}{\beta'}\theta_2 + \psi_2\right) + \ldots. \qquad \text{(A7)}$$

After substituting these expressions into equations (55) and (56) and equating like powers of ζ in both of them, a system of equations is obtained for the determinations of the functions θ_k and ψ_k. In the zero order approximation (terms not containing the parameter ζ) and making use of the notation in equation (A2) results in

$$\frac{\partial^2\theta_0}{\partial t^2} = \left(\frac{E}{\gamma_s} + \varepsilon\frac{\beta}{\beta'}\right)\nabla^2\theta_0 + \frac{\beta'}{\gamma_s}\frac{\partial\psi_1}{\partial t} \qquad \text{(A8)}$$

$$\frac{\partial^2\theta_0}{\partial t^2} = \left(\frac{K_f\beta}{\rho_f\beta'}\right)\nabla^2\theta_0 + \frac{\beta'}{\gamma_f}\frac{\partial\psi_1}{\partial t} \qquad \text{(A9)}$$

Multiplying the first of these equations by γ_s and the second by γ_f and then adding them, results in the following expression for θ_0 being obtained

$$\left(\gamma_s + \gamma_f\right)\frac{\partial^2\theta_0}{\partial t^2} = \left(E + \varepsilon\gamma_s\frac{\beta}{\beta'} + K_f\phi\frac{\beta}{\beta'}\right)\nabla^2\theta_0\,. \qquad \text{(A10)}$$

Upon introducing the definition of ε leads to

$$\left(\gamma_s + \gamma_f\right)\frac{\partial^2\theta_0}{\partial t^2} = \left[E + \frac{\beta}{\beta'}K_f\left(1 - \frac{K_b}{K_s}\right)\right]\nabla^2\theta_0 \qquad \text{(A11)}$$

This equation describes waves which are propagated without damping at velocity

$$W_{P_0} = \sqrt{\frac{E + (\beta/\beta')K_f\left(1 - K_b/K_s\right)}{\gamma_s + \gamma_f}}\,. \tag{A12}$$

To obtain the next term in the approximation expressions (A6) and (A7) must be substituted in equations (55) and (56), and equate the first order terms in ζ to obtain the system of equations

$$\frac{\partial^2\theta_1}{\partial t^2} = \left(\frac{E}{\gamma_s} + \varepsilon\frac{\beta}{\beta'}\right)\nabla^2\theta_1 + \varepsilon\nabla^2\psi_1 + \frac{\beta'}{\gamma_s}\frac{\partial\psi_1}{\partial t} \tag{A13}$$

$$\frac{\partial^2\theta_1}{\partial t^2} + \beta'\frac{\partial^2\psi_1}{\partial t^2} = \frac{K_f}{\rho_f}\nabla^2\left(\frac{\beta}{\beta'}\theta_1 + \psi_1\right) - \frac{\beta'}{\gamma_f}\frac{\partial\psi_1}{\partial t} \tag{A14}$$

Multiplying the first of these by γ_s and the second by γ_f and adding, the following relation between θ_1 and ψ_1 or θ_1 and φ_1 results

$$\begin{aligned}\left(\gamma_s + \gamma_f\right)\frac{\partial^2\theta_1}{\partial t^2} - \left[E + \frac{\beta}{\beta'}\left(\varepsilon\gamma_s + \frac{K_f}{\rho_f}\gamma_f\right)\right]\nabla^2\theta_1 &= \\ \beta'\gamma_f\frac{\partial^2\psi_1}{\partial t^2} + &\left(\varepsilon\gamma_s + \frac{K_f}{\rho_f}\gamma_f\right)\nabla^2\psi_1\,.\end{aligned} \tag{A15}$$

Introducing equation (A12)

$$\frac{\partial^2\theta_1}{\partial t^2} - W_{P_0}\nabla^2\theta_1 = -\frac{\beta'\gamma_f}{\gamma_s + \gamma_f}\frac{\partial^2\psi_1}{\partial t^2} + \frac{\varepsilon\gamma_s + K_f\phi}{\gamma_s + \gamma_f}\nabla^2\psi_1 \tag{A16}$$

or by virtue of the definition of ε (equation (A3))

$$\frac{\partial^2\theta_1}{\partial t^2} - W_{P_0}\nabla^2\theta_1 = -\frac{\beta'\gamma_f}{\gamma_s + \gamma_f}\frac{\partial^2\psi_1}{\partial t^2} + \frac{K_f\left(1 - K_b/K_s\right)}{\gamma_s + \gamma_f}\nabla^2\psi_1\,. \tag{A16}$$

Comparing this equation with equation (A11) there follows, among other things, that the right hand side must be orthogonal to the function θ_0. For the determination of θ_1, the function ψ in (A16) must be replaced by its expression in θ_0 using one of equations (A8) or (A9). Thus

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial^2\theta_1}{\partial t^2} - W_{P_0}\nabla^2\theta_1\right) &= -\frac{\gamma_s\gamma_f}{\gamma_s + \gamma_f}\frac{\partial^2}{\partial t^2}\left[\frac{\partial^2\theta_0}{\partial t^2} - \left(\frac{E}{\gamma_s} + \varepsilon\frac{\beta}{\beta'}\right)\nabla^2\theta_0\right] \\ &\quad + \frac{\gamma_s K_f\left(1 - K_b/K_s\right)}{\beta'\left(\gamma_s + \gamma_2\right)}\nabla^2\left[\frac{\partial^2\theta_0}{\partial t^2} - \left(\frac{E}{\gamma_s} + \varepsilon\frac{\beta}{\beta'}\right)\nabla^2\theta_0\right]\end{aligned} \tag{A17}$$

This process can be continued to obtain equations of higher order.

APPENDIX B: NOTATION

What follows are definitions of the parameters used in this report. Additional support in this may be found on http://www.glossary.oilfield.slb.com and related links.

compressibility – $(\text{bulk modulus})^{-1}$ See K_b, K_s and K_f. (B1)

$$e_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) \quad - \quad \text{elastic strain tensor.} \tag{B2}$$

$E -$ Young's modulus = $\lambda + 2\mu$ [(tensile stress)/(tensile strain)] $\left[\mathrm{N/m^2}\right]$.
– A measure of the stiffness of a given material. (B3)

$\phi,\ \Delta\phi -$ porosity - the percentage of pore volume that can contain fluids. Effective porosity excludes isolated pores and refers only to the connected pore volume in a rock that contributes to fluid flow. Total porosity is the total pore volume of of the rock. $\Delta\phi -$ change in porosity. (B4)

$\mathbf{F}_s -$ external force acting on the solid phase in a unit volume. (B5)

$\mathbf{F}_f -$ external force acting on the fluid phase in a unit volume. (B6)

$\mathbf{F}_{sf} = -\dfrac{\eta}{\kappa}\left(\mathbf{v}_f - \mathbf{v}_s\right) -$ friction force acting on the fluid in a unit volume of the porous medium. (B7)

$\mathbf{F}_{fs} = -\mathbf{F}_{sf} \rightarrow$ an equal but opposite force acting on the solid phase due to fluid friction in a unit volume. (B8)

$\mu -$ Lamé's coefficient in the elastic limit. (B9)

immisible - the inability of two fluids to mix to form form a homogeneous mixture; oil and water are immisible fluids. (B10)

$k -$ permeability, the ability or measurement of a rock's ability to transmit fluids, measured in darcies or milllidarcies. (Relative and absolute.) (See κ.) (B11)

$K_b,\ K_s,\ K_f$ – bulk modulus - the ratio or percent change in volume to the change of pressure applied to a fluid or rock. $\left(K = \lambda + (2/3)\mu\right)$.

K_b – bulk modulus of the dry solid.

K_s – actual bulk modulus of the solid - $\left(K_s > K_b\right)$. (B12)

K_f – bulk modulus of the fluid.

Inverse is the related compressibility coefficient.

λ – Lamé's coefficient in the elastic limit. (B13)

$p_i\ (i = s, f)$ – pressure in the solid phase (s) and fluid phase (f). (B14)

$\tau_{ik} = \delta_{ik}\lambda\theta + 2\mu e_{ik}$ – stress tensor for an elastic medium. (B15)

Tortuosity, often defined as: $T = (1/2)(1 + 1/\phi)$ (B16)

u_i – i^{th} component ($i = 1, 2, 3$) of the particle displacement vector. (B17)

$V,\ V_s,\ V_f$ – volume. $V = V_s + V_f$ where V_s is the volume actually filled by the solid phase and V_f the volume occupied by the liquid phase in a porous medium. (B18)

$\Delta V,\ \Delta V_s,\ \Delta V_f$ – variation of volume - total, porous medium, fluid. (B19)

$\mathbf{u}$ – particle displacement in the porous medium. (B20)

$\mathbf{v}_s = \partial\mathbf{u}/\partial t$ – particle velocity in the solid phase. (B21)

$\mathbf{v}_f$ – mean flow velocity of the fluid phase. (B22)

α – proportionality constant which together with porosity, ϕ, specifies the mechanical properties of the dry porous medium. $\left(\Delta V_s = \alpha\,\Delta V_f\right)$ $\left(\alpha = 1 - K_b/K_s\right)$ (B23)

$$\beta = \frac{1}{\phi(1+\alpha)} \tag{B24}$$

$$\beta' = 1 + (\beta - 1)\frac{K_f}{K_s} \tag{B25}$$

δ – exponential damping function assuming plane wave incidence. (See χ.) (B26)

$$\varepsilon = \frac{K_f}{\gamma_s}\left(1 - \phi - \frac{K_b}{K_s}\right) \tag{B27}$$

$$\varphi = \frac{\Delta\rho_f}{\rho_f} \tag{B28}$$

$\Phi_i^{(s)}\left(\mathbf{\Phi}^{(s)}\right) = \sum_k \frac{\partial \tau_{ik}}{\partial x_k}$ – componentsof elastic force in a unit volume for the solid phase. (B29)

$\Phi_i^{(f)}\left(\mathbf{\Phi}^{(f)}\right) = -\phi \nabla p_f$ – force acting of the fluid phase in a unit volume. (B30)

$\gamma_i \; (i = s, f)$ – effective density - density of the rock matrix with pores (γ_s). Effective fluid density (γ_f). (B31)

χ – a parameter introduced to specify the damping of a wave's amplitude, given plane wave propagation. $\left[\exp\left(-(\delta * x)/2\right)\right]$ (B32)

$\kappa = k/\phi$ – porosity normalized permeability. (See *k*.) (B33)

λ – wave length. (B34)

η – viscosity - a property of fluids that indicates their resistance to flow, defined as the ratio of shear stress to shear rate. Measured in poises (dyne-sec/cm^2) or centipoise - 1/100 of a poise. One centipoise equals one millipascal-sec. Viscosity must have a stated or understood shear rate in order to have meaning. (B35)

$\theta = \nabla \cdot \mathbf{u}$ where $\mathbf{u}$ is particle displacement in the solid phase. (B36)

$\rho_i \; (i = s, f)$ – grain density - the density of a rock with no porosity (ρ_s). Fluid density (ρ_f). (B37)

$$\sigma = \frac{\lambda + (4/3)\mu}{2\mu(\lambda + 2\mu)} \tag{B38}$$

$$\vartheta = \frac{1}{2\mu} \tag{B39}$$

$f = \omega/2\pi$ – ω is the circular frequency related to the vibrational frequency f. (B40)

$\zeta = \eta/\kappa$ (viscosity/permeability [effective]) (B41)

$$\chi = (\omega\eta)/\kappa \tag{B42}$$

$$\theta \text{ and } \varphi \rightarrow \text{dilation of solid and fluid phases, } \theta = \nabla \cdot \mathbf{u} \tag{B43}$$