Finite difference methods for wave propagation in an elastic anisotropic plane layered medium with orthorhombic symmetry

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ABSTRACT

This report presents a simulation system for the solution of the forward problem of elastic wave propagation in a plane layered anisotropic (orthorhombic) elastic media. Forward modeling has become a useful tool for interpretation in exploration seismology. The method discussed here employs finite Fourier transforms to temporarily remove the x and y coordinates resulting in a coupled system of three finite difference equations in the 3 Cartesian coordinate particle displacements in terms of depth (z) and time (t). The return to the (x, y, z, t) domain is done using a double summation over the two horizontal wave numbers (k_x, k_y). At the present time, $3D$ seismic modeling realistically requires a high-performance multiprocessor computer using efficient $3D$ algorithms. for the geological model mentioned above. Non-geometrical wave types are minimal in this model type, but the development of the method presented here may serve as a basis from which to pursue more complicated $3D$ geometries.

INTRODUCTION

In both theory and in numerical methods of solution of the forward problem difficulties arising are not to a large extent connected with a type of the given equations but to a larger extent with media dimension and the complexity of the properties of coefficients defining the media. Different physical effects when simulating elastic wave fields in seismology and seismic prospecting, which are not described by the acoustic wave equation, are obtained in this modeling procedure. Such effects should in reality be taken into account.

The finite difference method is a popular methodology for solving elastic wave equations. However, in seismic problems when we have to deal with very large computation domains, and essentially varying elastic parameters, the use of the conventional numerical methods is strongly limited by high computer costs and insufficient accuracy of these methods. This situation makes us focus on the creation of efficient numerical and analytical algorithms which allow the solution of 3D seismic problems.

This report describes a numerical algorithm for the forward seismic problem, some aspects connected with theoretical numerical and analytical method are not included, as they have been treated in earlier reports. The main concept underlying the algorithm is the splitting of $3D$ problems to a series of coupled $1D$ problems in the (k_x, k_y, z, t) domain and their solution one after another with the help of the finite difference technique.

Depending on the behavior of coefficient variation, the algorithm may take on numerous forms. The version of the algorithm used here is based on a combination of the analytical method of separation of variables (using the Fourier transforms) with the finite difference techniques for solving the resulting $1D$ problems. The algorithm was suggested and developed in Mikhailenko (1985).

In what follows, Lamb's problem for the anisotropic (orthorhombic) inhomogeneous half-space with the axis of symmetry parallel to the horizontal axes (x, y) is used. Elastic wave propagation in such a medium is described by equations given in the next section.

THEORY

In a plane layered orthorhombic medium with no lateral inhomogeneities the particle displacement may be specified in a Cartesian coordinate system, (x, y, z), as $\mathbf{u} = (u, v, w)$, where the vector components of displacement (u, v, w) are the solutions of the coupled equations

$$\rho \frac{\partial^2 u}{\partial t^2} = C_{11} \frac{\partial^2 u}{\partial x^2} + C_{13} \frac{\partial^2 w}{\partial x \partial z} + C_{66} \frac{\partial^2 u}{\partial y^2} + \left(C_{12} + C_{66}\right) \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial}{\partial z}\left(C_{55} \frac{\partial u}{\partial z}\right) + \frac{\partial}{\partial z}\left(C_{55} \frac{\partial w}{\partial x}\right) + F_x(x, y, z, t) \tag{1}$$

$$\rho \frac{\partial^2 v}{\partial t^2} = \left(C_{66} \frac{\partial^2 u}{\partial x \partial y} + C_{66} \frac{\partial^2 v}{\partial^2 x}\right) + C_{12} \frac{\partial^2 u}{\partial x \partial y} + C_{22} \frac{\partial^2 v}{\partial y^2} + C_{23} \frac{\partial^2 w}{\partial y \partial z} + \frac{\partial}{\partial z}\left(C_{44} \frac{\partial v}{\partial z}\right) + \frac{\partial}{\partial z}\left(C_{44} \frac{\partial w}{\partial y}\right) + F_y(x, y, z, t) \tag{2}$$

$$\rho \frac{\partial^2 w}{\partial t^2} = C_{55} \frac{\partial^2 u}{\partial x \partial z} + C_{55} \frac{\partial^2 w}{\partial x^2} + C_{44} \frac{\partial^2 v}{\partial y \partial z} + C_{44} \frac{\partial^2 w}{\partial y^2} + \frac{\partial}{\partial z}\left(C_{13} \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial z}\left(C_{23} \frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial z}\left(C_{33} \frac{\partial w}{\partial z}\right) + F_z(x, y, z, t) \tag{3}$$

In the above equations, the C_{ij} are the 9 stiffnesses which define an orthorhombic medium (Schoenberg and Helbig 1997). The volume density is ρ and $\mathbf{F}(x, y, z, t) = \left(F_x, F_y, F_z\right)$ is some source term. The related quantities $A_{ij} = C_{ij} / \rho$ have the dimensions of velocity squared.

The (stress free) boundary conditions at the free surface are specified by

$$\tau_{zz}\big|_{z=0} = C_{13}\frac{\partial u}{\partial x} + C_{23}\frac{\partial v}{\partial y} + C_{11}\frac{\partial w}{\partial z} = 0 \tag{4}$$

$$\tau_{xz}\big|_{z=0} = C_{55}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = 0 \tag{5}$$

$$\tau_{yz}\big|_{z=0} = C_{44}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0 \tag{6}$$

and the problem is solved with zero initial data:

$$u\big|_{t=0} = \frac{\partial u}{\partial t}\bigg|_{t=0} = v\big|_{t=0} = \frac{\partial v}{\partial t}\bigg|_{t=0} = w\big|_{t=0} = \frac{\partial w}{\partial t}\bigg|_{t=0} = 0 \tag{7}$$

As previously mentioned, the medium has been chosen such that it is composed of plane parallel layers where the elastic parameters, stiffnesses and density $\left(C_{ij} \text{ and } \rho\right)$, do not vary in the horizontal directions. A consequence the $\left(x \text{ and } y\right)$ coordinates may be temporarily removed using finite Fourier transforms. The two dimensional finite Fourier transforms and inverses for the three components of displacement are defined as (Sneddon, 1995)

$$S\left(z,n,m,t\right) = \int_0^a dx \int_0^b dy\, u\left(z,x,y,t\right)\cos\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi x}{a}\right) \tag{8}$$

$$u\left(z,x,y,t\right) = \frac{4}{ab}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} S\left(z,n,m,t\right)\cos\left(\frac{m\pi y}{b}\right)\sin\left(\frac{n\pi x}{a}\right) \tag{9}$$

$$H\left(z,n,m,t\right) = \int_0^a dx \int_0^b dy\, v\left(z,x,y,t\right)\sin\left(\frac{m\pi y}{b}\right)\cos\left(\frac{n\pi x}{a}\right) \tag{10}$$

$$v\left(z,x,y,t\right) = \frac{4}{ab}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} H\left(z,n,m,t\right)\sin\left(\frac{m\pi y}{b}\right)\cos\left(\frac{n\pi x}{a}\right) \tag{11}$$

$$R\left(z,n,m,t\right) = \int_0^a dx \int_0^b dy\, w\left(z,x,y,t\right)\cos\left(\frac{m\pi y}{b}\right)\cos\left(\frac{n\pi x}{a}\right) \tag{12}$$

$$w\left(z,x,y,t\right) = \frac{4}{ab}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} R\left(z,n,m,t\right)\cos\left(\frac{m\pi y}{b}\right)\cos\left(\frac{n\pi x}{a}\right) \tag{13}$$

After applying the finite forward transforms given above to equations (1) – (3) the following result

$$\rho \frac{\partial^2 S}{\partial t^2} = -k_x^2 C_{11} S - k_x C_{13} \frac{\partial R}{\partial z} - k_y^2 C_{66} S - k_x k_y \left(C_{12} + C_{66} \right) H + \frac{\partial}{\partial z}\left(C_{55} \frac{\partial S}{\partial z} \right) - k_x \frac{\partial}{\partial z}\left(C_{55} R \right) + F_x \left(k_x, k_y, z, t \right) \tag{14}$$

$$\rho \frac{\partial^2 H}{\partial t^2} = \left(-k_x k_y C_{66} S - k_x^2 C_{66} H \right) - k_x k_y C_{12} S - k_y^2 C_{22} H - k_y C_{23} \frac{\partial R}{\partial z} + \frac{\partial}{\partial z}\left(C_{44} \frac{\partial H}{\partial z} \right) - k_y \frac{\partial}{\partial z}\left(C_{44} R \right) + F_y \left(k_x, k_y, z, t \right) \tag{15}$$

$$\rho \frac{\partial^2 R}{\partial t^2} = k_x C_{55} \frac{\partial S}{\partial z} + k_y C_{44} \frac{\partial H}{\partial z} - \left(k_y^2 C_{44} + k_x^2 C_{55} \right) R + k_x \frac{\partial}{\partial z}\left(C_{13} S \right) + k_y \frac{\partial}{\partial z}\left(C_{23} H \right) + \frac{\partial}{\partial z}\left(C_{33} \frac{\partial R}{\partial z} \right) + F_z \left(k_x, k_y, z, t \right) \tag{16}$$

where it is to be remembered that it has been assumed that the stiffnesses do not depend on the horizontal $\left(x \text{ and } y \right)$ coordinates. It will initially be assumed that the pseudo – boundaries introduced at $\left(x = a,\ y = b \right)$ are perfectly reflecting.

The boundary conditions at the free surface:

1. Normal Stress:

$$\tau_{zz}\big|_{z=0} = C_{13} \frac{\partial u}{\partial x} + C_{23} \frac{\partial v}{\partial y} + C_{11} \frac{\partial w}{\partial z} = 0 \tag{17}$$

transformed

$$\tau_{zz}\big|_{z=0} = k_x C_{13} S + k_y C_{23} H + C_{11} \frac{\partial R}{\partial z} = 0 \tag{18}$$

finite difference analogue at the free surface:

$$R_{-1}^m = R_1^m + \Delta z k_x \frac{C_{13}}{C_{11}} S_0^m + \Delta z k_y \frac{C_{23}}{C_{11}} H_0^m . \tag{19}$$

2. Shear Stress, τ_{xz}:

$$\tau_{xz}\big|_{z=0} = C_{55} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) = 0 \tag{20}$$

transformed

$$\tau_{xz}\big|_{z=0} = C_{55} \left(\frac{\partial S}{\partial z} - k_x R\left(z, n, m, t \right) \right) = 0 \tag{21}$$

finite difference analogue at free surface:

$$S_{-1}^{m} = S_{1}^{m} - \Delta z k_x R_0^m \tag{22}$$

3. Shear Stress, τ_{yz} :

$$\tau_{yz}\Big|_{z=0} = C_{44}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0 \tag{23}$$

transformed

$$\tau_{yz}\Big|_{z=0} = C_{44}\left(\frac{\partial H}{\partial z} - k_y R\right) = 0 \tag{25}$$

finite difference analogue at free surface:

$$H_{-1}^{m} = H_{1}^{m} - \Delta z k_y R_0^m \tag{26}$$

The problem is solved with zero initial data so that

$$S\Big|_{t=0} = \frac{\partial S}{\partial t}\Big|_{t=0} = H\Big|_{t=0} = \frac{\partial H}{\partial t}\Big|_{t=0} = R\Big|_{t=0} = \frac{\partial R}{\partial t}\Big|_{t=0} = 0 \tag{27}$$

It will be assumed that at the first 3 grid points at the free surface the C_{ij} and volume density (ρ) are independent of the z spatial coordinate. Thus equations (14) – (16) may be written in this region as

$$\rho\frac{\partial^2 S}{\partial t^2} = -k_x^2 C_{11} S - k_x\left(C_{13}+C_{55}\right)\frac{\partial R}{\partial z} - k_y^2 C_{66} S - k_x k_y\left(C_{12}+C_{66}\right)H + C_{55}\frac{\partial^2 S}{\partial z^2} \tag{28}$$

$$\rho\frac{\partial^2 H}{\partial t^2} = -k_x k_y C_{66} S - k_x^2 C_{66} H - k_x k_y C_{12} S - k_y^2 C_{22} H - k_y\left(C_{23}+C_{44}\right)\frac{\partial R}{\partial z} + C_{44}\frac{\partial^2 H}{\partial z^2} \tag{29}$$

$$\rho\frac{\partial^2 R}{\partial t^2} = k_x\left(C_{13}+C_{55}\right)\frac{\partial S}{\partial z} + k_y\left(C_{23}+C_{44}\right)\frac{\partial H}{\partial z} - \left(k_x^2 C_{55} + k_y^2 C_{44}\right)R + + C_{33}\frac{\partial^2 R}{\partial z^2} \tag{30}$$

Employing equations (19), (22) and (25) the finite difference analogues of the above equations have the form

$$\begin{aligned} S_0^{m+1} &= 2S_0^m - S_0^{m-1} - \delta^2\left(k_x^2 A_{11} + k_y^2 A_{66}\right)S_0^m + \\ &\frac{\delta^2\left(A_{13}+A_{55}\right)}{2}\left(k_x^2\frac{A_{13}}{A_{11}}S_0^m + k_x k_y\frac{A_{23}}{A_{11}}H_0^m\right) - \\ &\delta^2 k_x k_y\left(A_{12}+A_{66}\right)H_0^m + \delta^2 A_{55}\left(\frac{2S_1^m - 2S_0^m}{h^2}\right) - \delta^2 A_{55}\left(\frac{k_x R_0^m}{h}\right) \end{aligned} \tag{31}$$

$$\begin{aligned} H_0^{m+1} &= 2H_0^m - H_0^{m-1} - \delta^2\left(k_y^2 A_{22} + k_x^2 A_{66}\right)H_0^m - \delta^2 k_x k_y\left(A_{12}+A_{66}\right)S_0^m + \\ &\frac{\delta^2\left(A_{23}+A_{44}\right)}{2}\left(k_x k_y\frac{A_{13}}{A_{11}}S_0^m + k_y^2\frac{A_{23}}{A_{11}}H_0^m\right) + \delta^2 A_{44}\left(\frac{2H_1^m - 2H_0^m - hk_y R_0^m}{h^2}\right) \end{aligned} \tag{32}$$

$$\begin{aligned} R_0^{m+1} &= 2R_0^m - R_0^{m-1} + \frac{\delta^2 k_x^2\left(A_{13}+A_{55}\right)}{2}R_0^m - \frac{\delta^2 k_y^2\left(A_{23}+A_{44}\right)}{2}R_0^m - \\ &\delta^2\left(k_x^2 A_{55} + k_y^2 A_{44}\right)R_0^m - \delta^2 A_{33}\left(\frac{R_0^m}{h^2} + \frac{k_x A_{13}}{hA_{11}}S_0^m + \frac{k_y A_{23}}{hA_{11}}H_0^m\right) \end{aligned} \tag{33}$$

At an interior point, where the stiffnesses and density may depend on the spatial coordinate z the finite difference analogues have the more complex form

$$\begin{aligned} S_j^{m+1} &= \left[2 - \frac{k_x^2\delta^2\left(\bar{C}_{11}\right)_j}{\rho_j} - \frac{\delta^2}{h^2\rho_j}\left[\left(\bar{C}_{55}\right)_{j+1/2} + \left(\bar{C}_{55}\right)_{j-1/2}\right] - \frac{k_y^2\delta^2\left(\bar{C}_{66}\right)_j}{\rho_j}\right]S_j^m - \\ &\frac{k_x\delta^2}{2h\rho_j}\left[\left(\bar{C}_{13}\right)_{j+1/2} + \left(\bar{C}_{55}\right)_{j+1/2}\right]R_{j+1}^m + \frac{k_x\delta^2}{2h\rho_j}\left[\left(\bar{C}_{13}\right)_{j-1/2} + \left(\bar{C}_{55}\right)_{j-1/2}\right]R_{j-1}^m - \\ &\frac{k_x k_y\delta^2}{\rho_j}\left(\bar{C}_{12} + \bar{C}_{66}\right)_j H_j^m + \frac{\delta^2}{h^2\rho_j}\left[\left(\bar{C}_{55}\right)_{j+1/2}S_{j+1}^m + \left(\bar{C}_{55}\right)_{j-1/2}S_{j-1}^m\right] + \\ &\frac{k_x\delta^2}{2h\rho_j}\left[\left[\left(\bar{C}_{55}\right)_{j+1/2} - \left(\bar{C}_{55}\right)_{j-1/2}\right]R_j^m\right] - S_j^{m-1} \end{aligned} \tag{34}$$

$$
\begin{aligned}
H_j^{m+1} = & \left[2 - \frac{k_x^2 \delta^2}{\rho_j} \left(\bar{C}_{66} \right)_j - \frac{\delta^2}{h^2 \rho_j} \left(\left(\bar{C}_{44} \right)_{j+1/2} + \left(\bar{C}_{44} \right)_{j-1/2} \right) - \frac{k_y^2 \delta^2}{\rho_j} \left(\bar{C}_{22} \right)_j \right] H_j^m + \\
& \frac{\delta^2}{h^2 \rho_j} \left[\left(\bar{C}_{44} \right)_{j+1/2} H_{j+1}^m + \left(\bar{C}_{44} \right)_{j-1/2} H_{j-1}^m \right] - \frac{k_x k_y \delta^2}{\rho_j} \left[\left(\bar{C}_{12} \right)_j - \left(\bar{C}_{66} \right)_j \right] S_j^m - \\
& \frac{k_y \delta^2}{2h\rho_j} \left[\left(\bar{C}_{23} \right)_{j+1/2} + \left(\bar{C}_{44} \right)_{j+1/2} \right] R_{j+1}^m + \frac{k_y \delta^2}{2h\rho_j} \left[\left(\bar{C}_{23} \right)_{j-1/2} + \left(\bar{C}_{44} \right)_{j-1/2} \right] R_{j-1}^m + \\
& \frac{k_y \delta^2}{2h\rho_j} \left[\left(\bar{C}_{44} \right)_{j+1/2} - \left(\bar{C}_{44} \right)_{j-1/2} \right] R_j^m - H_j^{m-1}
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
R_j^{m+1} = & \left[2 - \frac{\delta^2}{\rho_j} \left[k_y^2 \left(\bar{C}_{44} \right)_j + k_x^2 \left(\bar{C}_{55} \right)_j \right] - \frac{\delta^2}{h^2 \rho_j} \left[\left(\bar{C}_{33} \right)_{j+1/2} + \left(\bar{C}_{33} \right)_{j-1/2} \right] \right] R_j^m + \\
& \frac{k_x \delta^2}{2h\rho_j} \left[\left(\bar{C}_{13} \right)_{j+1/2} + \left(\bar{C}_{55} \right)_{j+1/2} \right] S_{j+1}^m - \frac{k_x \delta^2}{2h\rho_j} \left[\left(\bar{C}_{13} \right)_{j-1/2} + \left(C_{55} \right)_{j-1/2} \right] S_{j-1}^m + \\
& \frac{k_y \delta^2}{2h\rho_j} \left[\left(\bar{C}_{23} \right)_{j+1/2} + \left(\bar{C}_{44} \right)_{j+1/2} \right] H_{j+1}^m - \frac{k_y \delta^2}{2h\rho_j} \left[\left(\bar{C}_{23} \right)_{j-1/2} + \left(C_{44} \right)_{j-1/2} \right] H_{j-1}^m - \\
& \frac{k_x \delta^2}{2h\rho_j} \left[\left(\bar{C}_{13} \right)_{j+1/2} - \left(\bar{C}_{13} \right)_{j-1/2} \right] S_j^m - \frac{k_y \delta^2}{2h\rho_j} \left[\left(\bar{C}_{23} \right)_{j+1/2} - \left(\bar{C}_{23} \right)_{j-1/2} \right] H_j^m + \\
& \frac{\delta^2}{h^2 \rho_j} \left[\left(\bar{C}_{33} \right)_{j+1/2} R_{j+1}^m + \left(\bar{C}_{33} \right)_{j-1/2} R_{j-1}^m \right] - R_j^{m-1}
\end{aligned}
\tag{36}
$$

In the above, an over scored quantity indicates the harmonic average, δ is the time step, while $h = \Delta z$ is the spatial depth step. The source term has not been included, however, once a source type has been decided upon the vector components of it may be added to equations (34) – (36).

DISCUSSION AND CONCLUSIONS

Finite difference analogues, accurate to second order in space and time, for a plane parallel orthorhombic $(3D)$ media in which dependence on the horizontal Cartesian coordinates have been removed by finite Fourier transforms have been presented. For the type of elastic medium discussed here, the simplest source type to incorporate is a vertical point source located the origin of the Cartesian system so that $F(z,x,y,t) = \delta(z) f(t) \mathbf{e}_z$ where $\mathbf{e}_z$ is a unit vector in the z direction and $f(t)$ is the time dependence of the source wavelet. This wavelet is most often assumed to be band limited, as the range of its power spectrum in the frequency domain is linearly related to the number of terms required to approximate the two infinite Fourier series summations. The particle displacement may be recovered by applying inverse series summations, also specified above. In the early 1980's programs of this type were written in cooperation with Professor Boris G. Mikhailenko during his working visit at the University of Alberta in Edmonton, Alberta. At that time "supercomputers" of the Cray-1 and CDC205 type were required to produce numerical results. At the end of Professor Mikhailenko's stay, all source code and program listings were destroyed for unknown reasons. The formulae presented here are an accurate reproduction (the final equations being checked several times) and could be introduced into some modeling package if there was any interest. At this time, a Linux cluster would probably be the optimal operating platform, due to the parallel nature of the problem presented in the text.

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