Exact formulas for *qP*, *qS$_V$*, and *SH* group velocities in VTI media

Joe Wong

ABSTRACT

For analyzing traveltimes of *qP* and *qS* arrivals propagating through media with transversely isotropic or VTI symmetry, geophysicists often use the Thomsen linearized formulas for phase velocities to approximate group velocities. However, the Thomsen approximations are suitable only for weak anisotropy and for small angles relative to the VTI symmetry axis. When the anisotropy is strong and for larger angles, exact expressions for group velocity may be more appropriate. The exact group velocity formulas are not widely familiar to most applied geophysicists. This report summarizes the derivation of these exact formulas using the method of characteristics, and presents them in forms that facilitate calculation of traveltimes relevant to all the common acquisition geometries (surface reflection, VSP, and crosswell).

INTRODUCTION

A transversely isotropic medium with a vertical symmetry axis, commonly referred to as a VTI medium, possesses anisotropic velocities that are isotropic in any horizontal plane, but vary in any vertical direction with angle θ measured from the vertical symmetry axis. Such media support three types of elastic waves: the coupled quasi-pressure (*qP*) and quasi-shear (*qS$_V$*) waves with polarization within any vertical plane, and the transverse shear (*SH*) wave with horizontal polarization perpendicular to the propagation plane.

In real-world seismic surveys, event arrival times are important observable quantities that are associated with the group velocities of elastic waves propagating through isotropic and anisotropic geological media. In applied seismology, analysis of traveltimes associated with VTI media often use approximate formulas to quantify *qP* and *qS$_v$* group velocities. An example of such approximations is the linearized expressions for phase velocity involving the Thomsen parameters (Thomsen, 1986). However, the Thomsen approximations are suitable only for weak anisotropy and for small angles relative to the VTI symmetry axis. When strong anisotropy exists, or for large group angles, exact expressions are more suitable and should be used.

Exact expressions for the x- and z-components (v_{gx}, v_{gz}) of the group velocity in VTI media can be derived from the exact expressions for phase velocity using the method of characteristics described by Červený (2001). For each wave type *M* = *qP*, *qS$_V$*, or *SH*, the squared phase velocity function $V_M^2(\theta)$ has an exact functional form $f_M(\rho, C_{11}, C_{33}, C_{44}, C_{66}, C_{13},\ sin^2\theta,\ cos^2\theta)$, where ρ is the density and C_{mn} are the Voigt elastic parameters for VTI media. For each wave type, the characteristic function $G_M(p_x,\ p_z)$ can be formed from $V_M^2(\theta)$, where $(p_x,\ p_z)$ are slowness components. Group velocity components (v_{gx}, v_{gz}) are then calculated by taking the derivatives of $G_M(p_x,\ p_z)$ with p_x and p_z.

METHOD OF CHARACTERISTICS

Given an exact form for phase velocity function $V_M^2(\theta)$, the x- and z-components for group velocity are found by evaluating the following expressions (see Equations 3.6.28 and 3.6.30 in Červený, 2001):

$$V_M^2(\theta) = f_M(C_{11}, C_{33}, C_{44}, C_{66}, C_{13},\ sin^2\theta,\ cos^2\theta)\ , \tag{1.1}$$

$$G_M(p_x,\ p_z) = \ \frac{1}{2}\frac{f_M(C_{11}, C_{33}, C_{44}, C_{66}, C_{13}, sin^2\theta, cos^2\theta)}{V_M^2(\theta)}\ , \tag{1.2}$$

$$G_M(p_x,\ p_z) = 0.5 \cdot f_M(C_{11}, C_{33}, C_{44}, C_{66}, C_{13}, p_x^2, p_z^2\ )\ , \tag{1.3}$$

$$p_x = \sin\theta / V_M(\theta)\ \ , \tag{1.4}$$

$$p_z = \cos\theta / V_M(\theta)\ , \tag{1.5}$$

$$v_{gx}(\theta) = \frac{dx}{dT} = \ v_{gx} = \frac{1}{2}\frac{dG_m(p_1, p_3)}{dp_1}\ , \tag{1.6}$$

$$v_{gz}(\theta) = \frac{dz}{dT} = \ v_{gz} = \frac{1}{2}\frac{dG_m(p_1, p_3)}{dp_3}\ , \tag{1.7}$$

$$\varphi_g(\theta) = arctan(v_{gz}/v_{gx})\ , \tag{1.8}$$

where $\varphi_g(\theta)$ is the group or ray angle. The group velocity components $\left(v_{gz}(\theta),\ v_{gx}(\theta)\right)$ and the group angle $\varphi_g(\theta)$ determine the speed and direction of energy propagation in the VTI symmetry plane. The group angle φ_g is different from the phase angle θ. All directions are measured relative to the vertical symmetry axis.

In Equations 1.1 to 1.8, the angle θ is a parametric variable, and all equations involving it are parametric equations.

EXACT EXPRESSIONS FOR GROUP VELOCITIES

We define five squared-velocity quantities that are related to the density ρ and Voigt elastic parameters C_{mn} for VTI media (Carcione, 2007):

$$V_{P90}^2 = C_{11}/\rho = C_{22}/\rho\ , \tag{2.1}$$

$$V_{P90}^2 = C_{11}/\rho = C_{22}/\rho\ , \tag{2.2}$$

$$V_{13}^2 = C_{13}/\rho\ , \tag{2.3}$$

$$V_{ST}^2 = C_{66}/\rho\ . \tag{2.4}$$

In the literature, the quantities C_{mn}/ρ are usually given variable names A_{mn}. We additionally define three auxiliary quantities : f_0, which is part of the analytic expressions defining the qP and qS_v squared phase velocities; and f_p, and f_s, obtained by taking the derivative of f_0 with respect to the slowness components p_x and p_z.

$$f_0 = [V_{P90}^2 p_x^2 - V_{P0}^2 p_z^2 + V_{S0}^2(p_z^2 - p_x^2)]^2 + (V_{13}^2 + V_{S0}^2)^2(p_z^2 + p_x^2 \ , \qquad (2.6)$$

$$f_p = sqrt\{f_s^2 + 4(V_{13}^2 + V_{S0}^2)^2 p_1^2 p_3^2\} \ , \qquad (2.7)$$

$$f_s = V_{P90}^2 p_1^2 - V_{P0}^2 p_3^2 + V_{S0}^2(p_3^2 - p_1^2 \ . \qquad (2.8)$$

The exact expressions for phase and group velocities can now be written.

For the *qP* wave:

$$V_{qP}^2(\theta) = \frac{1}{2} * [V_{P90}^2 sin^2\theta + V_{P0}^2 cos^2\theta + V_{S0}^2 - sqrt(f_0)] \ , \qquad (3.1)$$

$$p_x = \sin\theta / V_P(\theta) \ \ , \qquad (3.2)$$

$$p_z = \cos\theta / V_P(\theta) \ , \qquad (3.3)$$

$$v_{qPx} = (p_1/2)\{ V_{P90}^2 + V_{S0}^2 + (1/f_p)[f_s(V_{P90}^2 - V_{S0}^2) + 2(V_{13}^2 + V_{S0}^2)p_3^2] \} \ , \qquad (3.4)$$

$$v_{qPz} = (p_3/2)\{ V_{P0}^2 + V_{S0}^2 + (1/f_p)[f_s(-V_{P0}^2 + V_{S0}^2) + 2(V_{13}^2 + V_{S0}^2)p_1^2] \} \ . \qquad (3.5)$$

For the *qS_V* wave:

$$V_{qSv}^2(\theta) = \frac{1}{2} * [V_{P90}^2 sin^2\theta + V_{P0}^2 cos^2\theta + V_{S0}^2 - sqrt(f_0)] \ , \qquad (4.1)$$

$$p_x = \sin\theta / V_{qSx}(\theta) \ \ , \qquad (4.2)$$

$$p_z = \cos\theta / V_{qSx}(\theta) \ , \qquad (4.3)$$

$$v_{qSx} = (p_1/2)\{ V_{P90}^2 + V_{S0}^2 - (1/f_p)[f_s(V_{P90}^2 - V_{S0}^2) + 2(V_{13}^2 + V_{S0}^2)p_3^2] \} \ , \qquad (4.4)$$

$$v_{qSz} = (p_3/2)\{ V_{P0}^2 + V_{S0}^2 - (1/f_p)[f_s(-V_{P0}^2 + V_{S0}^2) + 2(V_{13}^2 + V_{S0}^2)p_1^2] \} \ . \qquad (4.5)$$

For the *SH* wave:

$$V_{SH}^2(\theta) = V_{ST}^2 sin^2\theta + V_{S0}^2 cos^2\theta \ , \qquad (5.1)$$

$$p_x = \sin\theta / V_S(\theta) \ \ , \qquad (5.2)$$

$$p_z = \cos\theta / V_S(\theta) \ , \qquad (5.3)$$

$$v_{SHx} = p_1 V_{ST}^2 \ , \qquad (5.4)$$

$$v_{SHz} = p_3 V_{S0}^2 \ , \qquad (5.5)$$

Equations 5.4 and 5.5 are equivalent to

$$v_{SHx}/V_{ST} = \ p_1/V_{ST}^{-1} = p_1/s_T \ = \sin\theta \ , \qquad (5.6)$$

$$v_{SHz}/V_{S0} = \ p_3/V_{S0}^{-1} = p_3/s_0 = \cos\theta \ , \qquad (5.7)$$

Finally,

$$\frac{v_{SHx}^2}{V_{ST}^2} + \frac{v_{SHz}^2}{V_{S0}^2} \ = \frac{p_1^2}{s_T^2} + \frac{p_2^2}{s_0^2} \ = \ 1 \ . \qquad (5.8)$$

Equation 5.8 shows that the group velocity surface and the slowness surface are both elliptical in shape (cf. Carcione, 2007; Equation 1.181).

We have restricted our discussion to a transversely isotropic (VTI) medium with a vertical symmetry axis. The extension to media with tilted or horizontal symmetry axes (commonly referred to as TTI or HTI media) is straightforward, as TTI propagation angles can be found from the VTI angles through rotation by an angle equal to that between the symmetry and vertical axes.

EXAMPLES

We wrote MATLAB software based on the above equations to generate examples of the qP, qS_v, and SH phase and group velocities. These examples are shown on Figures 1 to 3 for three VTI substances: beryl, ice, and olivine. Values for the densities and the five elastic constants C_{11}, C_{33}, C_{44}, C_{66}, and C_{13} for these crystalline materials were taken from Musgrave (2003). To check the accuracy of the derived expressions, selected points were taken from the plots for beryl in Musgrave's book and plotted on Figure 1 in direct comparison with the curves produced by the MATLAB code. We can see that the curves for the qP, qS_v, and SH group velocities agree very well with the points taken from Musgrave.

On the examples, we can see clearly the differences in angular dependence of the qS_v and SH group velocities. These differences are the origin of shear-wave splitting. For the olivine example, the Voigt elastic constants are such that the calculated qS_v group velocities show triplication at the cusp angles. These triplications and cusps are predicted for materials with pure crystal structures, and are quite subtle. Such purity does not exist for the vast majority of geological media, and so triplications and cusps are unlikely to be observed in seismic field data. Because of this, the approximate formulas for group velocities presented by Thomsen (1986) and by Byun et al. (1989) have found more utility in the analysis of real-world seismic reflection data. In my opinion, the Byun approximation for qP group velocity, summarized in Appendix B, is to be preferred over the Thomsen linear approximation because it is more accurate for large group angles, and since it avoids the use of the rather nebulous Thomsen parameter δ, it is more intuitive.

CONCLUSION

We have used the method of characteristics (Červený, 2001) to derive closed-form expressions for exact qP, qS_v, and SH group velocities from the exact equations for phase velocities in VTI anisotropic media. Berryman (1979) and Crampin (1981) have given an alternative technique for calculating exact group velocities from the exact expressions for the phase velocities. This technique is summarized in Appendix A.

The exact expressions are not much more complicated to use in calculations than the Thomsen linear approximations for weak anisotropy. In cases where the anisotropy is strong, or for analyzing seismograms with large source-receiver offsets, use of the exact formulas may result in more accurate analyses of seismic data for all the common acquisition geometries (surface reflection, VSP, and crosswell).

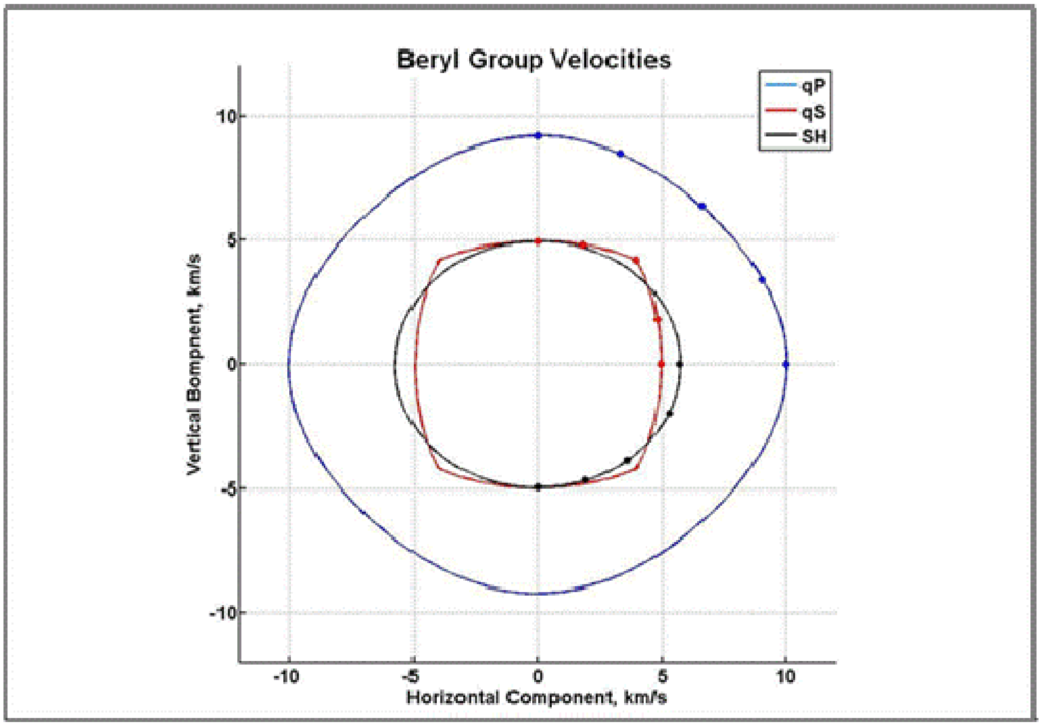

FIG. 1: Group velocities for beryl: ρ = 2850kg/m^3, C_{11} = 287.3, C_{33} = 241.8, C_{55} = 70.2, C_{66} = 94.2, and C_{13} = 72.8 (C_{mn} units = GPa). Lines are values calculated values using the exact formulas in this report; solid dots are values take from Musgrave's diagram for beryl.

ACKNOWLEDGEMENTS

We thank the industrial sponsors of CREWES and the Natural Sciences and Engineering Research Council (NSERC) of Canada for their support of this research.

REFERENCES

Berryman, J.G., 1979. Long-wave elastic anisotropy in transversely anisotropic media: Geophysics, **44**, 896-917.

Byun, B.S., Corrigan, D., and Gaider, J.E., 1989. Anisotropic velocity analysis for lithology discrimination: Geophysics, **54**, 1566-74.

Carcione, J.M., 2007. Wave Fields in Real Media, Handbook of Geophysical Exploration (Section 1: Seismic Exploration, Vol. 38), K. Helbig and S. Treitel, ed., Elsevier, Amsterdam.

Červený, V., 2001. Seismic Ray Theory, Cambridge University Press, Cambridge.

Crampin, S., 1981. A review of wave motion in anisotropic and cracked elastic media: Wave Motion, **3**, 343-391.

Kumar, D., Sen, M.K., and Ferguson, R.J., 2004. Traveltime calculation and prestack depth migration in tilted transversely isotropic media: Geophysics, **69**, 37-44.

Musgrave, M.J.P., 2003. Crystal Acoustics, Acoustical Society of America, American Institute of Physics.

Thomsen, L., 1986. Weak elastic anisotropy: Geophysics, **51**, 1954-1966.

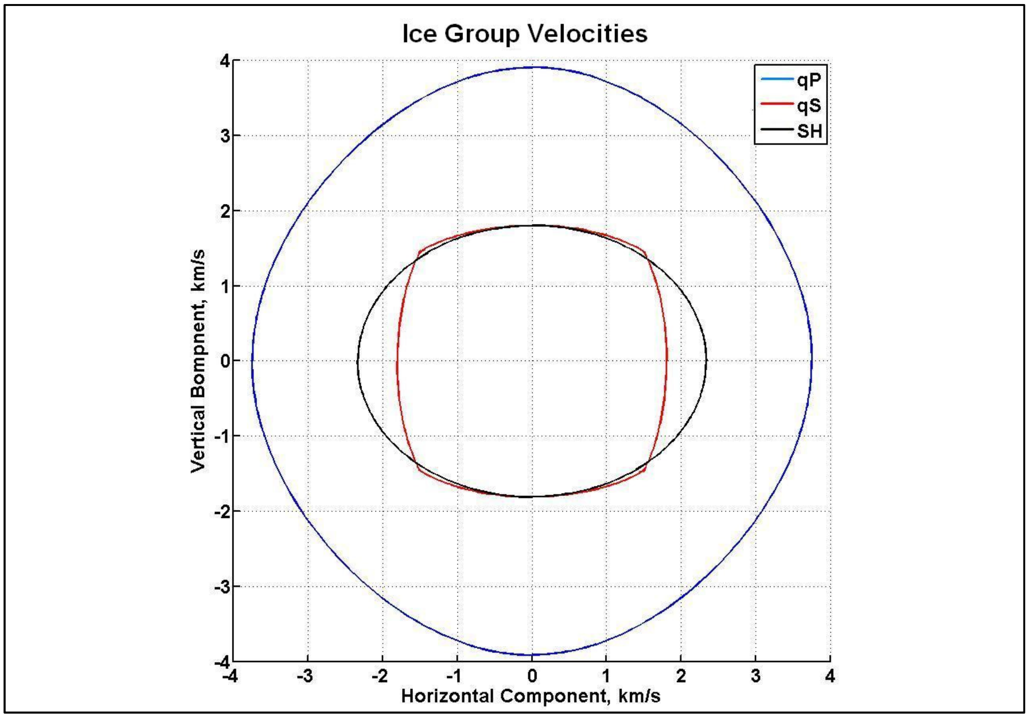

FIG. 2: Group velocities for ice: ρ = 980kg/m^3, C_{11} = 13.8, C_{33} = 15.0, C_{55}, = 3.2, C_{66} = 5.35, and C_{13} = 5.8 (C_{mn} units = GPa).

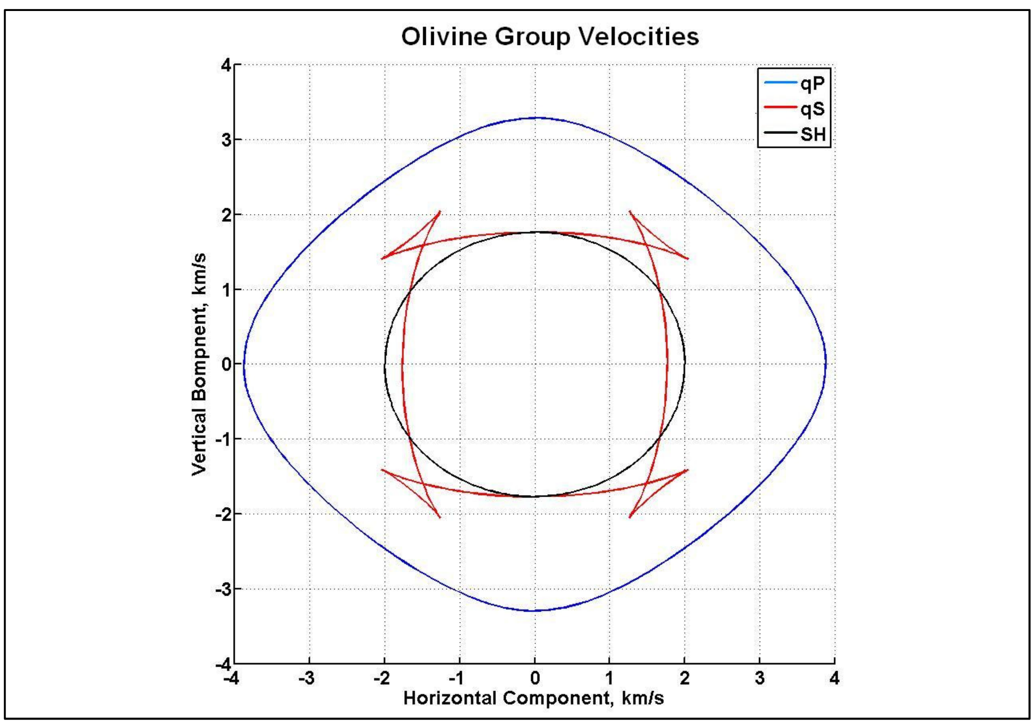

FIG. 3: Group velocities for olivine: A_{11} = 15.06, A_{33} = 10.84, A_{55} = 3.12, $A_{6\ 6}$ = 4.00, A_{13} = 1.64 (A_{mn} units = [km/s]^2; from Daley et al., 2010).

APPENDIX A: BERRYMAN'S CALCUALTION PROCEDURE

Berryman (1979) showed that the group velocities can be determined by taking derivatives of the squared phase velocities with respect to the phase angle θ. The following expressions , adapted from Crampin (1981), summarize Berryman's procedure:

$$\frac{\partial V_M(\theta)}{\partial \theta} = \frac{1}{2V_M(\theta)} \cdot \frac{\partial V_M^2(\theta)}{\partial \theta}, \tag{A1}$$

$$v_{gx}(\theta) = V_M(\theta)\cos\theta - \sin\theta \frac{\partial V_M(\theta)}{\partial \theta}, \tag{A2}$$

$$v_{gz}(\theta) = V_M(\theta)\sin\theta + \cos\theta \frac{\partial V_M(\theta)}{\partial \theta}. \tag{A3}$$

$$\varphi_g(\theta) = arctan(v_{gz}/v_{gx}) \ , \tag{A4}$$

where $v_{gx}(\theta)$ and $v_{gz}(\theta)$ are group velocity components, and $\varphi_g(\theta)$ is the group or ray angle. For M = qP, qS_V, and SH, respectively, the exact square phase velocities $V_M^2(\theta)$ are given by Equations 3.1, 4.1, and 5.1. All angles are measured from the VTI symmetry axis.

The analytic forms of the derivatives $\partial V_M(\theta)/\partial\theta$ and $\partial V_M^2(\theta)/\partial\theta$ are obtained by simple algebraic manipulation of Equations 3.1, 4.1, and 5.1. However, in writing software to calculate values of the group velocity components $\left(v_{gz}(\theta),\ v_{gx}(\theta)\right)$ and the group angle $\varphi_g(\theta)$, it is often more convenient to use a numerical approximation to $\partial V_M(\theta)/\partial\theta$ rather than the analytic forms indicated by Equation A1.

APPENDIX B: BYUN-KUMAR APPROXIMATIONS FOR *qP* GROUP VELOCITY IN VTI/TTI MEDIA

The exact expressions for group velocities in VTI media are appropriate for pure crystalline materials. Even though they may exhibit elastic anisotropy, geological media generally do not possess pure crystalline structure. Consequently, in most geophysical analysis of seismic propagation through VTI-like media, approximate formulas often are used to quantify *qP* and qS_v group velocities. Examples are the linearized approximate expressions for phase velocity involving the Thomsen parameters (Thomsen, 1986). Byun et al. (1989) and Kumar et al. (2004) have advocated a different approximation for the *qP* group velocity in VTI/TTI media, summarized in the following equations:

$$v_g^{-2}(\varphi) = a_0 + a_1 cos^2(\varphi + \vartheta) - a_2 cos^4(\varphi + \vartheta) \ , \tag{B1}$$

$$a_0 = V_H^{-2} \ , \tag{B2}$$

$$a_1 = 4V_{45}^{-2} - 3V_H^{-2} - V_V^{-2} \ , \tag{B3}$$

$$a_2 = 4V_{45}^{-2} - 2V_H^{-2} - 2V_V^{-2} \ , \tag{B4}$$

where φ is the angle measured from the symmetry axis of the VTI medium. V_H, V_H, and V_{45} are the velocities in the horizontal, vertical, and 45° directions relative to the symmetry axis. The angle ϑ is the angle between the symmetry axis and the coordinate vertical axis. If ϑ is 0°, we have a VTI medium. For $\vartheta = 90$°, we have an HTI medium. A TTI medium is defined when ϑ has a value between 0° and 90°.

Figure B1 compares the exact phase and group qP velocities for ice with values calculated using the Byun-Kumar approximation. We see that the approximate values are in close agreement with the exact group velocity. According to this figure, the exact phase velocity and the exact group velocity are not much different from each other. That they are so close is the reason that, for geophysical analysis of data through VTI/TTI media, values of phase velocity determined from the linearized Thomsen formulas often are suitable approximations for group velocities. However, the Thomsen approximations are appropriate only for weak anisotropy and at small propagation angles.

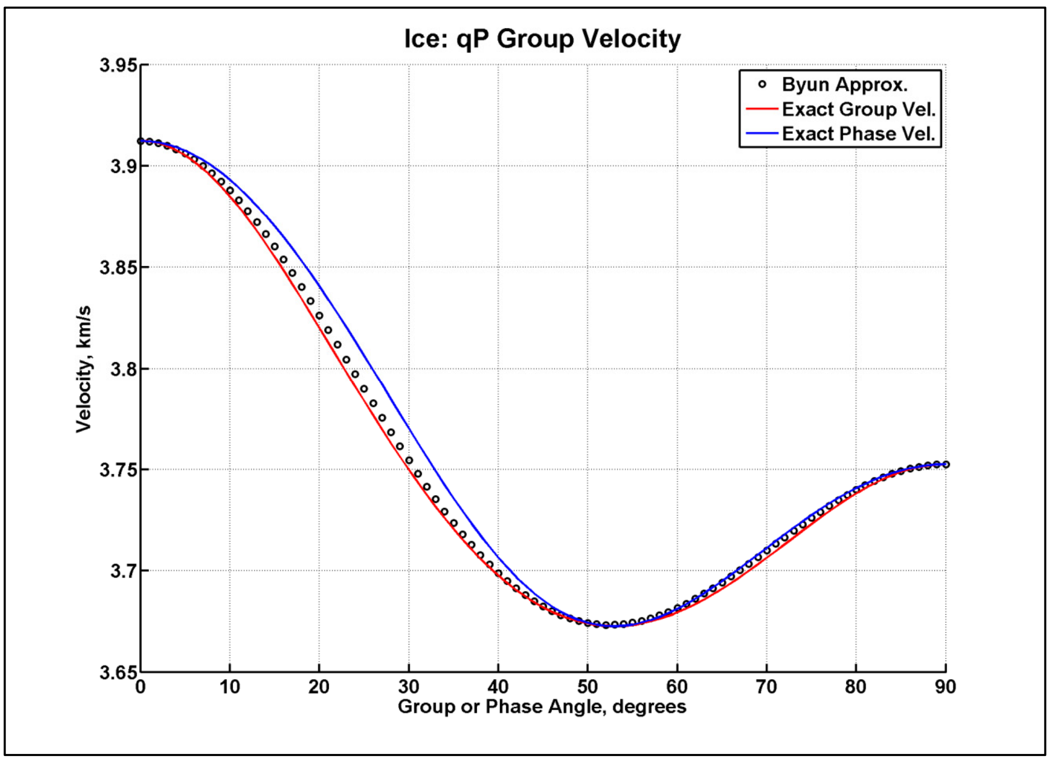

FIG. B1: Comparison of group velocities for ice, calculated by exact formulas, and by the Byun-Kumar approximation. The parameters values used in Equations B1 to B4 are [V_V, V_H, V_{45}] = [3.912, 3.753, 3.682] km/s, with tilt angle = 0°.