AVO theory for non-welded contacts

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ABSTRACT

In this paper, a closed formulas of reflected and transmitted wave for incident compressional wave are carried out to analyse the AVO response to nonwelded interface. Based on the boundary conditions for the combined displacement and velocity discontinuity, the reflection coefficients of PP wave and PS wave are both complex function of frequency ω ,elastic parameters α , β , ρ across interface and fracture parameters C_x, C_z describing the interface. The closed formulas are function of elastic parameters contrasts across interface ($r_\alpha = \Delta\alpha/\alpha$, $r_\beta = \Delta\beta/\beta$ and $r_\rho = \Delta\rho/\rho$), average velocity ratio B_1 and B_2 of two layer respectively and new complex frequency dependent fracture parameters H_x and H_z . Different from welded interface, the reflection coefficients is not zero when the elastic parameters contrasts are zero. It shows that incident P wave propagating through a nonwelded interface in a single homogeneous medium (e.g., a crack, joint, or fracture) produces both frequency dependent reflected and transmitted P and S wave. The third order truncation approximation in series of fracture parameters and elastic parameter contrasts capture the exact reflected and transmitted coefficient at small incident angle at different frequency. The new closed formulas are convenient to analysis the effect of elastic parameters and fracture parameters on reflected and transmitted coefficients, and can be used to invert the elastic parameter contrasts and fracture parameters by use of linear inversion or series reversion method.

INTRODUCTION

The interface between two half spaces with elastic media is either in perfect well welded contacted or in non-welded contacted. When a compressional wave arrives at an interface between two layers, some of the energy reflects back to the surface and some is transmitted. AVO techniques that figures out The reflections and transmissions coefficient of a seismic wave propagating through a welded contact interface of the two half spaces at oblique angle have been used extensively in the industry with different approximation. Aki and Richards (1980) simplified the Zoeppritz equations to linear equation that describes the variation of seismic reflected and transmitted amplitudes as a function of incident angle and contrast of elastic parameters across interface. For different purpose, some varieties of linear AVO equation are derived. However observations from the borehole images, cores and outcrops show that there are not perfect welded contact interface underground, such as faults, joints and fractures. Picotti(2012) interpreted the abnormal amplitude reflected from ice layer across which there isn't appreciable vertical velocity variation as imperfect welded contact interface.

A suitable imperfect welded contact interface is necessary for forward modeling to describe its dynamic response. For perfect welded contact interface, the displacements and stresses are continuous across the interface of two half space with either isotropic or anisotropic media. Theories that consider imperfect welded contact interface were mainly based on the displacement discontinuity and stresses continuity model. The discontinuous displacement are a linear function of the stresses multiplied by compliance. (Schoenberg 1980) proposed a linear slip model for an imperfectly bonded interface between two elastic media. Reflection and transmission coefficients for harmonic plane wave incident at arbitrary angle upon a plane linear slip interface are computed in terms of the interface compliances. Those coefficients are frequency dependent. (Pyraknolte et al., 1990) presented a non-welded interface model based on the discontinuity of displacement and the particle velocity across the interface. Chaisri and Krebes (2000) expressed the exact formulas in terms of which is similar in form to the coefficient for the welded contract case plus a series of imaginary terms thoroughly due to nonwelded contrast. But the parameters in nonwelded terms are not consistent with those in welded terms which only include compressional wave contrast, shear wave contrast and density contrast. Cui (2013) deduced general closed approximation that can be used to invert not only the subsurface elastic parameters contrasts, but also eight parameters related to the fractured media. In this paper, we present a high order AVO approximation expressions and AVO series reversion solution.

BASIC THEORY

1. Theory of a Displacement Discontinuity

To theoretically model the effect of the fracture on propagation of plane seismic wave, the fracture is represented as a displacement discontinuity at the boundary between two elastic half spaces. It is assumed that the stress across the displacement discontinuity is continuous. The magnitude of the discontinuity in displacement is inversely proportional to the specific stiffness of the fracture. If the boundary between two elastic half spaces lies in the x-y plane, the boundary conditions for a compressional wave (P) and shear wave (SV) incident on the displacement discontinuity are (Pyraknolte et al., 1990)

$$[u_x] = u_{x1} - u_{x2} = \frac{1}{\kappa_x}\sigma_{zx} \tag{1}$$

$$[u_z] = u_{z1} - u_{z2} = \frac{1}{\kappa_z}\sigma_{zz} \tag{2}$$

$$\sigma_{zz1} = \sigma_{zz2} \tag{3}$$

$$\sigma_{zx1} = \sigma_{zx2} \tag{4}$$

where $\kappa_x^{-1} = Z_T$ and $\kappa_z^{-1} = Z_N$. Z_N and Z_T are normal and tangential compliance of fracture respectively, whose dimension is $1/stress$ (Hsu and Schoenberg 1993). The number 1 and 2 denotes upper and lower layer respectively.

For an incident shear wave with polarization in the x-y plane (SH wave) the boundary conditions are

$$[u_y] = u_{y1} - u_{y2} = \frac{1}{\kappa_y}\sigma_{zy} \tag{5}$$

$$\sigma_{zy1} = \sigma_{zy2} \tag{6}$$

The presence of liquid under saturated condition in joint or fracture will increase the specific stiffness of the displacement discontinuity for compressional waves and possibly for shear wave. Paullson (1983) observed an increase in shear wave transmission when quartz monzonite specimens, which contained micro-cracks, were saturated with water. The liquid may also introduce viscous coupling between the two surfaces of the fracture. Schoenberg(Johnson and Schoenberg 1980) derived a solution for an elastic wave propagated across a viscous interface for pure viscous slip in shear i.e., $Z_T = -i\eta/\omega$, $Z_N = 0$,and neglecting the stiffness of the interface. To investigate the increase in shear wave transmission theoretically, the fluid filled fractures (nonwelded interface) are represented as a discontinuity in particle displacement across the fracture which depend on the specific stiffness of the fracture as well as a discontinuity in particle velocity which depend on a specific viscosity.

Pyraknolte et al. (1988) combined the effects of specific stiffness and specific viscosity in shear on wave propagation across a fracture to yield discontinuities in both displacement and velocity while the stress across the discontinuity remained constant. For an incident P wave and SV wave, the boundary conditions for the combined displacement and velocity discontinuity are

$$\kappa_x[u_x] + \eta[v_x] = \sigma_{zx} \tag{7}$$

$$\kappa_z[u_z] = \sigma_{zz} \tag{8}$$

$$\sigma_{zz1} = \sigma_{zz2} \tag{9}$$

$$\sigma_{zx1} = \sigma_{zx2} \tag{10}$$

For an incident SH wave the boundary conditions are

$$\kappa_y[u_y] + \eta[v_y] = \sigma_{zy} \tag{11}$$

$$\sigma_{zy1} = \sigma_{zy2} \tag{12}$$

Where the parameter η is the viscosity of fluid. $[v_l]$ ($l = x, y$) is the velocity difference across the nonwelded interface.

Carcione (1998) extended the velocity discontinuity in shear developed by Pyraknolte (1988) to both shear and vertical on wave propagation across a fracture to yield discontinuities in both displacement and velocity while the stress across the discontinuity remained constant.

$$\kappa_x[u_x] + \eta_x[v_x] = \sigma_{zx} \tag{13}$$

$$\kappa_z[u_z] + \eta_z[v_z] = \sigma_{zz} \tag{14}$$

$$\sigma_{zz1} = \sigma_{zz2} \tag{15}$$

$$\sigma_{zx1} = \sigma_{zx2} \tag{16}$$

When either κ and η tend to infinity, the solution reverts to a welded interface solution. When κ tends to zero the solution gives the displacement discontinuity When η tend to zero the solution gives the particle velocity discontinuity model. When both κ and η tend to zero, the coefficients revert to those for free surface.

The non-welded interface is modelled by the discontinuity of the displacement and the particle velocity across the interface. The stress components are proportional the displacement and velocity discontinuity through the specific stiffness and specific viscosity, respectively. Displacement discontinuities conserve energy and yield frequency dependant reflection and transmission coefficient. On the other hand, velocity discontinuities imply an energy loss at the interface and frequency independent reflection and transmission coefficients. The specific viscosity accounts for the presence of a liquid under saturation conditions. The liquid introduces a viscous coupling between two surfaces of the fracture and enhances energy transmission, but at the same time this is reduced by viscous losses.

The imperfect bonding is described by four parameters: the normal and tangential specific stiffness, viscosities lame and shear constant of background medium. The stiffness account for frequency dependent and phase change effects, and the viscosity allows for damping in the interface response. In order to model a crack embed in a finely laminated background, Carcione(J. Carcione 1998) described the model by a transversely isotropic medium whose symmetry axis is perpendicular to the crack surface.

Assuming an incident compressional harmonic plane wave of the form

$$u = u_0 e^{-i(\omega t - k_x x - k_z z)} \tag{17}$$

The particle velocity v is the derivation of the displacement with time

$$v = \frac{du}{dt} = -i\omega u_0 e^{-i(\omega t - k_x x - k_z z)} \tag{18}$$

Substituting equation (17) and (18) into discontinuity equation (13) and (14), the follow equations are obtained(J. M. Carcione 1996)

$$(\kappa_x + i\omega\eta_x)[u_x] = \frac{1}{C_x}[u_x] = \sigma_{zx} \tag{19}$$

$$(\kappa_z + i\omega\eta_z)[u_z] = \frac{1}{C_z}[u_z] = \sigma_{zz} \tag{20}$$

2. Reflection and Transmission coefficients of Nonwelded interface for isotropic medium

Chaisri and Krebes (2000) present new and exact mathmatic formulas for P-SV particle displacement reflection and transmission coefficients for elastic plane waves incident upon a nonwelded contact interface separating two solid half-spaces. Consider a plane horizontal interface between two solids in nonwelded contact. The boundary conditions at the interface are such that the stress is continuous across the boundary but the displacement is not. The displacement discontinuity is proportional to the traction through specific fracture compliance. If the incident plane P-SV wave propagating in the x-z plane, with the interface being the plane z=0, then in consistent with equation (15), (16),(19) and (20) the boundary conditions at the interface are

$$[u_x] = u_{x2} - u_{x1} = C_x\sigma_{zx} \tag{21}$$

$$[u_z] = u_{z2} - u_{z1} = C_z\sigma_{zz} \tag{22}$$

$$\sigma_{zz1} = \sigma_{zz2} \tag{23}$$

$$\sigma_{zx1} = \sigma_{zx2} \tag{24}$$

For isotropic medium, the normal and tangent stress are

$$\sigma_{zx} = \mu\left[\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right] \tag{25}$$

$$\sigma_{zz} = \lambda\frac{\partial u_x}{\partial x} + [\lambda + 2\mu]\frac{\partial u_z}{\partial z} \tag{26}$$

Subscripts 1 and 2 denote the upper and lower media, respectively; σ_{zz} and σ_{zx} are the component of stress tensor parallel and normal to the interface; u_x and u_z are the component of displacement parallel and normal to the interface; λ and μ are Lame's constant of background medium; C_x and C_z are the specific fracture compliance parallel and normal to the interface respectively; $C_x = 1/(\kappa_x + i\omega\eta_x)$ and $C_z = 1/(\kappa_z + i\omega\eta_z)$. Specially, $C_x = 1/\kappa_x = Z_T$ and $C_z = 1/\kappa_z = Z_N$ for fracture model without water saturation. In equation (21) and (22), the stress component on the right hand sides can be evaluated in either medium 1 or 2, as the stress is continuous across the interface. Either choice leads to same solutions. Choosing medium 2, however, results in simpler equations to solve if the wave are incident from medium 1 and vice versa.

In applying the above boundary condition to the incident and scattered plane waves, Chaisri and Krebes (2000) used the conventions of Aki and Richards; that is the z axis points downward and x component of each unit polarization vector is positive (figure 1). Substitution of the usual mathmatic expressions $u = u_0 e^{-i(\omega t - s\cdot x)}$ for the displacement components of the incident and scatter single frequency plane waves in medium 1 and 2 into boundary conditions results in four equations for the four scattered wave amplitudes, which can be expressed in matrix form as

$$M\left[P_{1,}^{s}S_{1,}^{s}P_{2,}^{s}S_{2}^{s}\right]^{T}=N\left[P_{1,}^{I}S_{1,}^{I}P_{2,}^{I}S_{2}^{I}\right]^{T} \tag{27}$$

Where the column vector on the left and right contain the amplitudes of the scattered and incident waves denoted by superscript S and I respectively.

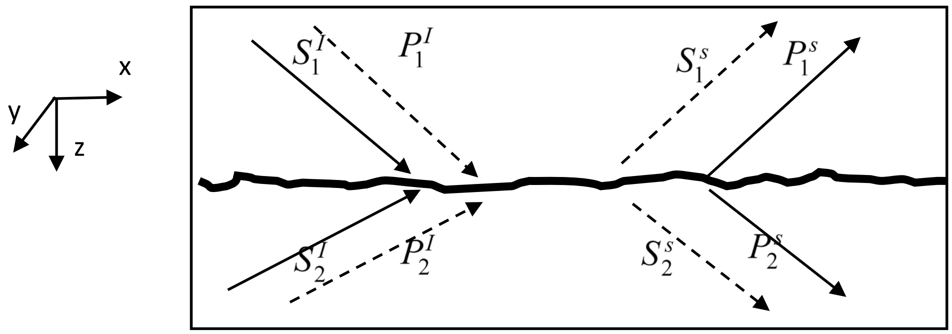

Figure 1. Scheme of incident and scatter wave for nonwelded interface

where

$$M=\begin{bmatrix} -\sin\theta_1 & -\cos\phi_1 & \sin\theta_2-i\omega C_x\chi_2\cos\theta_2 & \cos\phi_2-i\omega C_x\beta_2\gamma_2 \\ \cos\theta_1 & -\cos\phi_1 & \cos\theta_2-i\omega C_z\alpha_2\gamma_2 & -\cos\phi_2+i\omega C_z\chi_2\cos\phi_2 \\ \chi_1\cos\theta_1 & \beta_1\gamma_1 & \chi_2\cos\theta_2 & \beta_2\gamma_2 \\ -\alpha_1\gamma_1 & \chi_1\cos\phi_1 & \alpha_2\gamma_2 & -\chi_2\cos\phi_2 \end{bmatrix} \tag{28}$$

$$N=\begin{bmatrix} \sin\theta_1 & \cos\phi_1 & -\sin\theta_2-i\omega C_x\chi_2\cos\theta_2 & -\cos\phi_2-i\omega C_x\beta_2\gamma_2 \\ \cos\theta_1 & -\cos\phi_1 & \cos\theta_2+i\omega C_z\alpha_2\gamma_2 & -\cos\phi_2-i\omega C_z\chi_2\cos\phi_2 \\ \chi_1\cos\theta_1 & \beta_1\gamma_1 & \chi_2\cos\theta_2 & \beta_2\gamma_2 \\ \alpha_1\gamma_1 & -\chi_1\cos\phi_1 & -\alpha_2\gamma_2 & \chi_2\cos\phi_2 \end{bmatrix} \tag{29}$$

and

$$\chi_l=2\rho_l\beta_l^2 p \qquad \gamma_l=\rho_l\left(1-2\beta_l^2p^2\right)$$

Where p is ray parameter and l =1 or 2. In these equation, ρ is density, α is the P wave velocity, β is S wave velocity, θ and ϕ are angle of P and S wave respectively. The first column in N is used for the case of incident P wave in medium 1, the second column is used for the case of incident S wave in medium 1, the third column is used for the case of incident P wave in medium 2, the fourth column is used for the case of incident S wave in medium 2.

Equation (27) is identical to the Zoeppritz equation of isotropic medium for welded contact boundary conditions, expect for the additional frequency dependent terms in the matrix due to the presence of nonwelded contact. Equation (27) can be solved analytically to obtain closed form formulas for the reflection and transmission coefficients. Defining the variables of a, b, c, d, E, F, G and H in the same way as Aki and Richards. That is

$$a=\gamma_2-\gamma_1 \quad b=\gamma_2+\chi_1 p \quad c=\gamma_1+\chi_2 p \quad d=2\left(\rho_2\beta_2^2-\rho_1\beta_1^2\right)$$

$$e_l=\frac{\cos\theta_l}{\alpha_l} \quad n_l=\frac{\cos\phi_l}{\beta_l} \quad E=be_1+ce_2 \quad F=bn_1+cn_2 \quad G=a-de_1n_2$$

$$H=a-de_2n_1 \quad K_l=\gamma_l^2+\chi_l^2e_ln_l \quad L_l=\gamma_l^2-\chi_l^2e_ln_l$$

$$\begin{aligned} D=&EF+GHp^2-\omega^2C_xC_zK_1K_2 \\ &-i\omega C_x\left(\rho_1e_1K_2+\rho_2e_2K_1\right)-i\omega C_z\left(\rho_1n_1K_2+\rho_2n_2K_1\right) \end{aligned}$$

The closed form formulas for the reflection and transmission coefficients are obtained as follows.

For PP wave

$$R_{pp}=\left[\left(be_1-ce_2\right)F-\left(a+de_1n_2\right)Hp^2\right]D^{-1} \\ +\left[\omega^2C_xC_zK_2L_1+i\omega C_x\left(\rho_2e_2L_1-\rho_1e_1K_2\right)+i\omega C_z\left(\rho_1n_1K_2+\rho_2n_2L_1\right)\right]D^{-1} \tag{30}$$

For PS wave

$$R_{ps}=-e_1p\left(\alpha_1/\beta_1\right)\left(ab+cde_2n_2\right)D^{-1} \\ -2e_1\gamma_1\chi_1\left(\alpha_1/\beta_1\right)\left[\omega^2C_xC_zK_2+i\omega\rho_2\left(C_xe_2+C_zn_2\right)\right]D^{-1} \tag{31}$$

The first term in equation (30) an (31) is the reflection coefficient of PP and PS for welded interface in accordance with Aki and Richard equation. The second term represents the effects resulted from nonwelded contact.

3. High accuracy of the AVO series formula

If we consider a plane P wave obliquely incident upon a nonwelded interface, the equation (27) is simplified as following

$$M\begin{bmatrix}R_{pp}\\R_{ps}\\T_{pp}\\T_{ps}\end{bmatrix}=N \tag{32}$$

Where $M=\begin{bmatrix}-X & -\left(1-B^2X^2\right)^{\frac{1}{2}} & CX+Q1 & -\left(1-D^2X^2\right)^{\frac{1}{2}}+Q2\\ \left(1-X^2\right)^{\frac{1}{2}} & -BX & \left(1-C^2X^2\right)^{\frac{1}{2}}+Q3 & DX+Q4\\ 2B^2X\left(1-X^2\right)^{\frac{1}{2}} & B\left(1-B^2X^2\right) & 2AD^2X\left(1-C^2X^2\right)^{\frac{1}{2}} & -AD\left(1-2D^2X^2\right)\\ -\left(1-2B^2X^2\right) & 2B^2X\left(1-B^2X^2\right)^{\frac{1}{2}} & AC\left(1-2D^2X^2\right) & 2AD^2\left(1-D^2X^2\right)^{\frac{1}{2}}\end{bmatrix}$

$$N=\begin{bmatrix}X\\ \left(1-X^2\right)^{\frac{1}{2}}\\ 2B^2X\left(1-X^2\right)^{\frac{1}{2}}\\ 1-2B^2X^2\end{bmatrix}$$

$$Q1=-i\omega C_x\chi_2\cos\theta_2=i\omega C_x2\rho_2\beta_2^2p\cos\theta_2=-i\omega C_x2\rho_2\beta_2\frac{\beta_2}{\beta_1}\frac{\beta_1}{\alpha_1}\sin\theta_1\cos\theta_2$$

$$=-2H_xB_1\frac{\beta_2}{\beta_1}X\left(1-C^2X^2\right)^{\frac{1}{2}}$$

$$Q2=-i\omega C_x\beta_2\gamma_2=-i\omega C_x\beta_2\rho_2\left(1-2\beta_2^2p^2\right)=-i\omega C_x\rho_2\beta_2\left(1-2\left(\frac{\beta_2}{\beta_1}\right)^2\left(\frac{\beta_1}{\alpha_1}\right)^2\sin\theta_1^2\right)$$

$$=-H_x\left(1-2\left(\frac{\beta_2}{\beta_1}\right)^2 B_1^2X^2\right)$$

$$Q3=-i\omega C_z\alpha_2\gamma_2=-i\omega C_z\rho_2\alpha_2\left(1-2\beta_2^2p^2\right)=-i\omega C_z\rho_2\alpha_2\left(1-2\left(\frac{\beta_2}{\beta_1}\right)^2\left(\frac{\beta_1}{\alpha_1}\right)^2\sin\theta_1^2\right)$$

$$=-H_z\left(1-2\left(\frac{\beta_2}{\beta_1}\right)^2 B_1^2X^2\right)$$

$$Q4=i\omega C_z\chi_2\cos\phi_2=i\omega C_z2\rho_2\beta_2^2p\cos\phi_2=i\omega C_z2\rho_2\alpha_2\frac{\beta_2}{\alpha_2}\frac{\beta_2}{\beta_1}\frac{\beta_1}{\alpha_1}\sin\theta_1\cos\phi_2$$

$$=2H_zB_2B_1\frac{\beta_2}{\beta_1}X\left(1-\left(\frac{\beta_2}{\beta_1}\right)^2 B_1^2X^2\right)^{\frac{1}{2}}$$

and where $X=\sin\theta$ is sinusoidal function of incident angle θ and constant parameters A through D denote the elastic parameters ratios

$$A=\frac{\rho_2}{\rho_1},B_1=\frac{\beta_1}{\alpha_1},B_2=\frac{\beta_2}{\alpha_2},C=\frac{\alpha_2}{\alpha_1},D=\frac{\beta_2}{\alpha_1}=B_1\frac{\beta_2}{\beta_1}$$
$$H_x=i\omega C_x\rho_2\beta_2 \qquad H_z=i\omega C_z\rho_2\alpha_2 \tag{33}$$

The complex parameter H_x and H_z is the function of frequency ω, fracture parameter and elastic parameters. H_x represents the effect of tangent fracture parameter C_x and shear impedance $\rho_2\beta_2$ of lower medium. H_z represents the effect of normal fracture parameter C_z and compressional impedance $\rho_2\alpha_2$ of lower medium.

In order to understand the response of the reflection coefficient R_{pp} and R_{ps} to contrasts in elastic parameters across the interface, we rewrite elastic parameter ratios in terms of elastic parameter contrasts.

$$\frac{\rho_2}{\rho_1}=1+r_\rho+\frac{1}{2}r_\rho{}^2+\frac{1}{4}r_\rho{}^3+\cdots \tag{34}$$
$$\frac{\alpha_2}{\alpha_1}=1+r_\alpha+\frac{1}{2}r_\alpha{}^2+\frac{1}{4}r_\alpha{}^3+\cdots \tag{35}$$
$$\frac{\beta_2}{\beta_1}=1+r_\beta+\frac{1}{2}r_\beta{}^2+\frac{1}{4}r_\beta{}^3+\cdots \tag{36}$$

where $r_\alpha=\Delta\alpha/\alpha$ is the ratio of difference of P wave velocity to average of P wave velocity , $r_\beta=\Delta\beta/\beta$ is the ratio of difference of S wave velocity to average of S wave velocity and $r_\rho=\Delta\rho/\rho$ is the ratio of difference of density to average of density.

Because the parameter B_1 and B_2 are only the ratio of the incident medium parameter, we do not expand it. The square root terms i is also expanded by making use of the expression

$$(1-x)^{\frac{1}{2}}=1-\frac{1}{2}x^2-\frac{1}{8}x^4+\cdots \tag{37}$$

Any one of the four displacement reflection coefficients can be solved from the equations using Cramer's rule. Forming two auxiliary matrixes Mp and Ms by replacing the first and second columns of M with N, the solutions are obtained

$$R_{pp} = \frac{\det(Mp)}{\det(M)}, R_{ps} = \frac{\det(Ms)}{\det(M)} \tag{38}$$

When the all of the expand series from equation (34) through (37) are further substitute into the coefficient matrix M and auxiliary matrixes Mp and Ms, all the elements are directly expressed as series in powers of the elastic constants. The determinant of the matrix is a linear combination of the elements of the matrix. Therefore, if the elements of the matrix are series in third orders of the elastic contrasts, reflection coefficient of PP and PS wave in equation (38) can be recast in terms of elastic contrast as following in same way

$$R = R^{w} + R^{nonw} \tag{39}$$

$$R^{w} = R_1^{w} + R_2^{w} + R_3^{w} \tag{40}$$

$$R^{nonw} = R_0^{nonw} + R_1^{nonw} + R_2^{nonw} + R_3^{nonw} \tag{41}$$

$$R_1^{w} = \Gamma_{\alpha 1} r_a + \Gamma_{\beta 1} r_\beta + \Gamma_{\rho 1} r_\rho \tag{42}$$

$$R_2^{w} = \Gamma_{\alpha 2} r_\alpha^2 + \Gamma_{\beta 2} r_\beta^2 + \Gamma_{\rho 2} r_\rho^2 + \Gamma_{\alpha 1\beta 1} r_\alpha r_\beta + \Gamma_{\alpha 1\rho 1} r_\alpha r_\rho + \Gamma_{\beta 1\rho 1} r_\beta r_\rho \tag{43}$$

$$\begin{aligned} R_3^{w} = {} & \Gamma_{\alpha 3} r_\alpha^3 + \Gamma_{\beta 3} r_\beta^3 + \Gamma_{\rho 3} r_\rho^3 + \Gamma_{\alpha 2\beta 1} r_\alpha^2 r_\beta + \Gamma_{\alpha 2\rho 1} r_\alpha^2 r_\rho \\ & + \Gamma_{\beta 2\alpha 1} r_\beta^2 r_\alpha + \Gamma_{\beta 2\rho 1} r_\beta^2 r_\rho + \Gamma_{\rho 2\alpha 1} r_\rho^2 r_\alpha + \Gamma_{\rho 2\beta 1} r_\rho^2 r_\beta + \Gamma_{\alpha 1\beta 1\rho 1} r_\alpha r_\beta r_\rho \end{aligned} \tag{44}$$

$$R_0^{nonw} = \sum_{i=0}^{M}\sum_{j=0}^{N} \Gamma_0^{x_i z_j} H_x^i H_z^j \tag{45}$$

$$R_1^{nonw} = \sum_{i=0}^{M}\sum_{j=0}^{N} \Gamma_{\alpha_1}^{x_i z_j} H_x^i H_z^j r_\alpha + \sum_{i=0}^{M}\sum_{j=0}^{N} \Gamma_{\beta_1}^{x_i z_j} H_x^i H_z^j r_\beta + \sum_{i=0}^{M}\sum_{j=0}^{N} \Gamma_{\rho_1}^{x_i z_j} H_x^i H_z^j r_\rho \tag{46}$$

$$\begin{aligned} R_2^{nonw} = {} & \sum_{i=0}^{M}\sum_{j=0}^{N} \Gamma_{\alpha_2}^{x_i z_j} H_x^i H_z^j r_\alpha^2 + \sum_{i=0}^{M}\sum_{j=0}^{N} \Gamma_{\beta_2}^{x_i z_j} H_x^i H_z^j r_\beta^2 + \sum_{i=0}^{M}\sum_{j=0}^{N} \Gamma_{\rho_2}^{x_i z_j} H_x^i H_z^j r_\rho^2 \\ & + \sum_{i=0}^{M}\sum_{j=0}^{N} \Gamma_{\alpha_1\beta_1}^{x_i z_j} H_x^i H_z^j r_\alpha r_\beta + \sum_{i=0}^{M}\sum_{j=0}^{N} \Gamma_{\alpha_1\rho_1}^{x_i z_j} H_x^i H_z^j r_\alpha r_\rho + \sum_{i=0}^{M}\sum_{j=0}^{N} \Gamma_{\beta_1\rho_1}^{x_i z_j} H_x^i H_z^j r_\beta r_\rho \end{aligned} \tag{47}$$

$$\begin{aligned} R_3^{nonw} = {} & \sum_{i=0}^{M}\sum_{j=0}^{N} \Gamma_{\alpha_3}^{x_i z_j} H_x^i H_z^j r_\alpha^3 + \sum_{i=0}^{M}\sum_{j=0}^{N} \Gamma_{\beta_3}^{x_i z_j} H_x^i H_z^j r_\beta^3 + \sum_{i=0}^{M}\sum_{j=0}^{N} \Gamma_{\rho_3}^{x_i z_j} H_x^i H_z^j r_\rho^3 \\ & + \sum_{i=0}^{M}\sum_{j=0}^{N} \Gamma_{\alpha_2\beta_1}^{x_i z_j} H_x^i H_z^j r_\alpha^2 r_\beta + \sum_{i=0}^{M}\sum_{j=0}^{N} \Gamma_{\alpha_2\rho_1}^{x_i z_j} H_x^i H_z^j r_\alpha^2 r_\rho + \sum_{i=0}^{M}\sum_{j=0}^{N} \Gamma_{\beta_2\alpha_1}^{x_i z_j} H_x^i H_z^j r_\beta^2 r_\alpha + \sum_{i=0}^{M}\sum_{j=0}^{N} \Gamma_{\beta_2\rho_1}^{x_i z_j} H_x^i H_z^j r_\beta^2 r_\rho \\ & + \sum_{i=0}^{M}\sum_{j=0}^{N} \Gamma_{\rho_2\alpha_1}^{x_i z_j} H_x^i H_z^j r_\rho^2 r_\alpha + \sum_{i=0}^{M}\sum_{j=0}^{N} \Gamma_{\rho_2\beta_1}^{x_i z_j} H_x^i H_z^j r_\rho^2 r_\beta + \sum_{i=0}^{M}\sum_{j=0}^{N} \Gamma_{\alpha_1\rho_1\beta_1}^{x_i z_j} H_x^i H_z^j r_\alpha r_\beta r_\rho \end{aligned} \tag{48}$$

Where R^{w} denotes the reflection coefficient of PP and PS wave for a welded interface as expressed by Aki and Richard. R^{nonw} denotes the reflection coefficient of PP and PS wave with respected to nonwelded interface. R_i^{w} represents the i th order combination of elastic contrast parameters r_α, r_β and r_ρ. R_i^{nonw} represents the i th order combination of H_x, H_z, r_α, r_β and r_ρ. The coefficients Γ, which generally are functions of incident angle, velocity ratio B_1 and B_2, within the current approximation are provided in Appendix.

For welded interface, the coefficients of series don't include zero order of elastic parameter contrast. But for nonwelded interface, the coefficients of series include zero order of elastic contrast. The coefficients of zero order just are the function of fracture parameter and frequency.

It shows that incident P wave upon nonwelded interface in a single homogeneous medium (e.g., a crack, joint, or fracture) produces both reflected and transmitted P and S wave.

For isotropic single homogeneous medium, substitution of $\alpha_1 = \alpha_2 = \alpha$ and $\beta_1 = \beta_2 = \beta$ into the definition of variables r_α , r_β and r_ρ produces

$$R = R_0^{nonw} = \sum_{i=0}^{M}\sum_{j=0}^{N}\Gamma_0^{x_i z_j} H_x^i H_z^j$$
$$= \Gamma_0^x H_x + \Gamma_0^z H_z + \Gamma_0^{xz} H_x H_z + \Gamma_0^{x2} H_x{}^2 + \Gamma_0^{z2} H_z{}^2 + \Gamma_0^{x3} H_x{}^3 + \Gamma_0^{x2z1} H_x{}^2 H_z^1 + \Gamma_0^{x1z2} H_x{}^1 H_z^2 + \Gamma_0^{z3} H_z{}^3 + \cdots \quad (49)$$

It means that in the case of welded contract, C_x and C_z are zero, leading to the results $R_{pp} = 0$, $R_{ps} = 0$, $T_{ps} = 0$ and $T_{pp} = 1$. In other words, the incident wave propagates through the interface without any changes. In the case of nonwelded contract, C_x and C_z aren't zero, leading to the results of reflected and transmitted coefficient. It is conceivable that the results could help to explain observations that are occasionally reported of anomalous seismic reflection in zones where the impedance contract is zeros or small.

Experimental analysis

To take a glance at the influence of high order term and the accuracy of the series AVO approximation, the comparison of each truncated term with the exact solution obtained from equation (32) is demonstrated. In general, the normal compliance of fracture is range from 10^{-10} to 10^{-13} m^2/N . In this example, the fluid in the fracture is pure fluid, i.e. the viscosity of the fluid is zero.

Figure 1 and 2 show the comparison of approximation of absolute R_{pp} at frequency 30Hz and 60Hz respectively with different order in fracture parameters C_x=5×10^{-10} and C_z=2.5×10^{-10} m^2/N in the case of fracture embed in homogeneous medium. The exact solution for R_{pp} is calculated from equation (32) in black line. Approximation in series of fracture parameters is denoted by solid red line, solid green line, solid blue line, solid pure line, point dash green line and point dash blue line respectively. For series order equation, the third order truncation appears to capture the exact R_{pp} curve. It means that the approximation in series of fracture parameters is enough to characterize the reflection coefficients at third order truncation.

Figure 3 illustrates the comparison of approximation of absolute R_{pp} for nonwelded interface at frequency 30Hz with different order in fracture parameters C_x =5×10^{-10} and C_z =2.5×10^{-10} m^2/N .The model with parameters P wave velocity, S wave velocity and density $[\alpha, \beta, \rho]$ are [3000m/s, 1500m/s,2.0g/cm^3] for upper media and [36000m/s,1700m/s, 2.1g/cm^3] for lower media respectively. The exact solution of R_{pp} for welded interface is calculated from Zoeppritz equation in dash black line. The exact solution of R_{pp} for nonwelded interface is calculated from equation (32) in black line. Approximation in series of fracture parameters is denoted by dash red line, dash green line, dash blue line, dash pure line respectively. The order of elastic parameter keep third order. For this model, the effect of discontinuity on reflection coefficient is larger. The intercept increases from 0.12 for welded interface to 0.19 for nonwelded interface. It demonstrates that the effect of discontinuity of nonwelded interface must be introduced to interpret abnormal amplitude, especially in the case of difference between two layers across the interface is close to zero. For series order equation, when the order of elastic parameters keep third order, the reflection coefficient curve moves toward exact R_{pp} curve with increasing order of fracture parameters. It approves again that the third order truncation approximation in series of fracture parameters and elastic parameters is enough to characterize the reflection coefficients.

Figure 4 shows comparison of approximation of absolute R_{pp} for nonwelded interface at different frequency with third order in fracture parameters and elastic parameters (called 3E3C) that is same as figure 3. The exact solution of R_{pp} for nonwelded interface in black line is calculated from equation (32). Approximation is denoted by dash red line. The approximation captures the exact R_{pp} curve at low frequency of less than 60Hz for the model and deviates from the exact curve. At small incident angle of less than 30 degrees, the approximation at third order for welded interface is almost the same as the exact solution. So the deviation at high frequency for nonwelded interface is resulted from the truncation of fracture parameters.

SERIES REVERSION

According to series reversion theory (e.g., Innanen 2013), we expand each instance of r_α , r_β and r_ρ in a new series:

$$r_\alpha = r_{\alpha 1} + r_{\alpha 2} + r_{\alpha 3} + \cdots \tag{50}$$

$$r_\beta = r_{\beta 1} + r_{\beta 2} + r_{\beta 3} + \cdots \tag{51}$$

$$r_\rho = r_{\rho 1} + r_{\rho 2} + r_{\rho 3} + \cdots \tag{52}$$

$$H_x = H_{x1} + H_{x2} + H_{x3} + \cdots \tag{53}$$

$$H_z = H_{z1} + H_{z2} + H_{z3} + \cdots \tag{54}$$

Put those expansions into equation 23 through 28 and further into equation (39) and (48). Consequently equating like orders, we solve for each $r_{\alpha n}$, $r_{\beta n}$, $r_{\rho n}$, making use of all the results for $r_{\alpha i}$, $r_{\beta i}$, $r_{\rho i}$ $(i < n)$.

for first order:

$$R = \Gamma_{\alpha 1} r_{\alpha 1} + \Gamma_{\beta 1} r_{\beta 1} + \Gamma_{\rho 1} r_{\rho 1} + \Gamma_0^{x1} H_{x1} + \Gamma_0^{z1} H_{z1} \tag{55}$$

for second order

$$-\Delta R_2 = \Gamma_{\alpha 1} r_{\alpha 2} + \Gamma_{\beta 1} r_{\beta 2} + \Gamma_{\rho 1} r_{\rho 2} + \Gamma_0^{x1} H_{x2} + \Gamma_0^{z1} H_{z2} \tag{56}$$

$$\begin{aligned} \Delta R_2 = {} & \Gamma_{\alpha 2} r_{\alpha 1}{}^2 + \Gamma_{\beta 2} r_{\beta 1}{}^2 + \Gamma_{\rho 2} r_{\rho 1}{}^2 + \Gamma_{\alpha 1 \beta 1} r_{\alpha 1} r_{\beta 1} + \Gamma_{\alpha 1 \rho 1} r_{\alpha 1} r_{\rho 1} + \Gamma_{\beta 1 \rho 1} r_{\beta 1} r_{\rho 1} \\ & + \Delta_0^2 + \Delta_{\alpha 1}^1 r_{\alpha 1} + \Delta_{\beta 1}^1 r_{\beta 1} + \Delta_{\rho 1}^1 r_{\rho 1} \end{aligned} \tag{57}$$

for third order

$$-\Delta R_3 = \Gamma_{\alpha 1} r_{\alpha 3} + \Gamma_{\beta 1} r_{\beta 3} + \Gamma_{\rho 1} r_{\rho 3} + \Gamma_0^{x1} H_{x3} + \Gamma_0^{z1} H_{z3} \tag{58}$$

$$\begin{aligned} \Delta R_3 = {} & \Gamma_{\alpha 3} r_{\alpha 1}{}^3 + \Gamma_{\beta 3} r_{\beta 1}{}^3 + \Gamma_{\rho 3} r_{\rho 1}{}^3 + \Gamma_{\alpha 2 \beta 1} r_{\alpha 1}{}^2 r_{\beta 1} + \Gamma_{\alpha 2 \rho 1} r_{\alpha 1}{}^2 r_{\rho 1} \\ & + \Gamma_{\beta 2 \alpha 1} r_{\beta 1}{}^2 r_{\alpha 1} + \Gamma_{\beta 2 \rho 1} r_{\beta 1}{}^2 r_{\rho 1} + \Gamma_{\rho 2 \alpha 1} r_{\rho 1}{}^2 r_{\alpha 1} + \Gamma_{\rho 2 \beta 1} r_{\rho 1}{}^2 r_{\beta 1} \\ & + \Gamma_{\alpha 2} 2 r_{a1} r_{a2} + \Gamma_{\beta 2} 2 r_{\beta 1} r_{\beta 2} + \Gamma_{\rho 2} 2 r_{\rho 1} r_{\rho 2} + \Gamma_{\alpha 1 \beta 1} \left(r_{a_1} r_{\beta 2} + r_{\alpha 2} r_{\beta 1} \right) \\ & + \Gamma_{\alpha 1 \rho 1} \left(r_{\alpha 1} r_{\rho 2} + r_{\alpha 2} r_{\rho 1} \right) + \Gamma_{\beta 1 \rho 1} \left(r_{\beta 1} r_{\rho 2} + r_{\beta 2} r_{\rho 1} \right) + \Gamma_{\alpha 1 \beta 1 \rho 1} r_{a1} r_{\beta 1} r_{\rho 1} \\ & + \Delta_0^3 + \Delta_{\alpha 1}^2 r_{\alpha 1} + \Delta_{\alpha 1}^1 r_{\alpha 2} + \Delta_{\beta 1}^2 r_{\beta 1} + \Delta_{\beta 1}^1 r_{\beta 2} + \Delta_{\rho 1}^2 r_{\rho 1} + \Delta_{\rho 1}^1 r_{\rho 2} \\ & + \Delta_{\alpha 2}^1 r_{\alpha 1}^2 + \Delta_{\beta 2}^1 r_{\beta 1}^2 + \Delta_{\rho 2}^1 r_{\rho 1}^2 + \Delta_{\alpha 1 \beta 1}^1 r_{\alpha 1} r_{\beta 1} + \Delta_{\alpha 1 \rho 1}^1 r_{\alpha 1} r_{\rho 1} + \Delta_{\beta 1 \rho 1}^1 r_{\beta 1} r_{\rho 1} \end{aligned} \tag{59}$$

Where the R represents PP wave or PS wave respectively. Δ_{MK}^L is the combined L order coefficient with respect to fracture parameters of K order elastic parameter M . In general, for multi-angle and multi-frequency prestack seismic data, the AVO series reversion is carried out with four steps (Chen,2015).

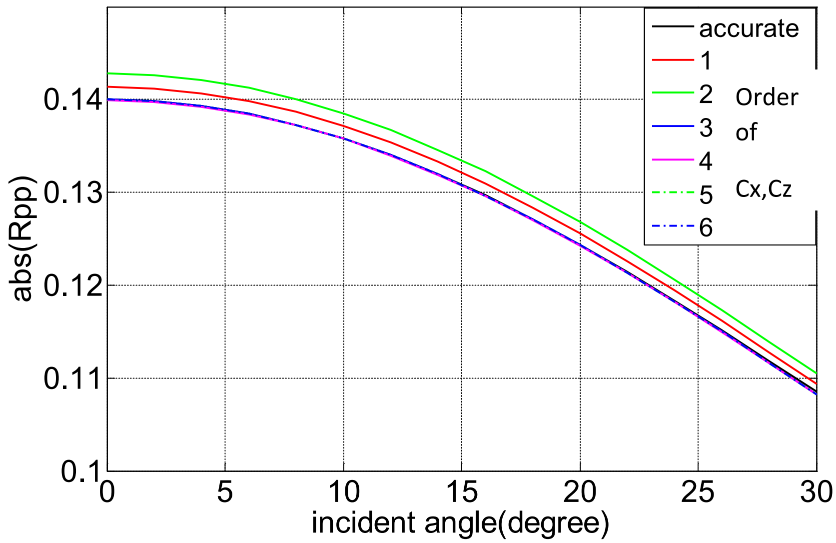

Figure 1. Comparison of approximation of absolute Rpp at frequency 30Hz with different order in fracture parameters $C_x=5\times10^{-10}$ and $C_z=2.5\times10^{-10}$ in the case of fracture embed in homogeneous medium. The exact solution for Rpp is calculated from equation (32) in black line. Approximation in series of fracture parameters is denoted by solid red line, solid green line, solid blue line, solid pure line, point dash green line and point dash blue line respectively.

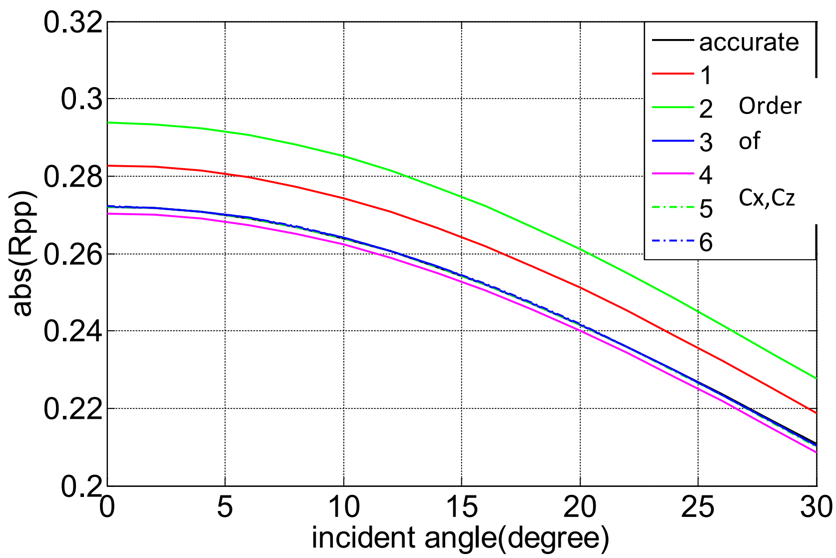

Figure 2. Comparison of approximation of absolute Rpp at frequency 60Hz with different order in fracture parameters $C_x=5\times10^{-10}$ and $C_z=2.5\times10^{-10}$ in the case of fracture embed in homogeneous medium. The exact solution for Rpp is calculated from equation (32) in black line. Approximation in series of fracture parameters is denoted by solid red line, solid green line, solid blue line, solid pure line, point dash green line and point dash blue line respectively.

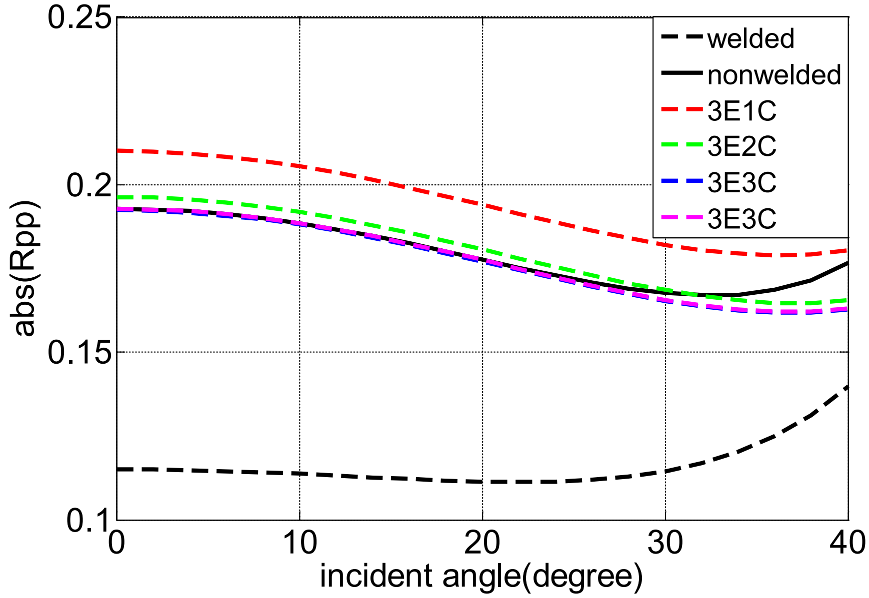

Figure 3. Comparison of approximation of absolute Rpp for nonwelded interface at frequency 30Hz with different order in fracture parameters $C_x=5\times10^{-10}$ and $C_z=2.5\times10^{-10}$.The exact solution of Rpp for nonwelded interface is calculated from equation (32) in black line. Approximation in series of fracture parameters is denoted by dash red line, dash green line, dash blue line, dash pure line respectively. The order of elastic parameter keep third order "3E".

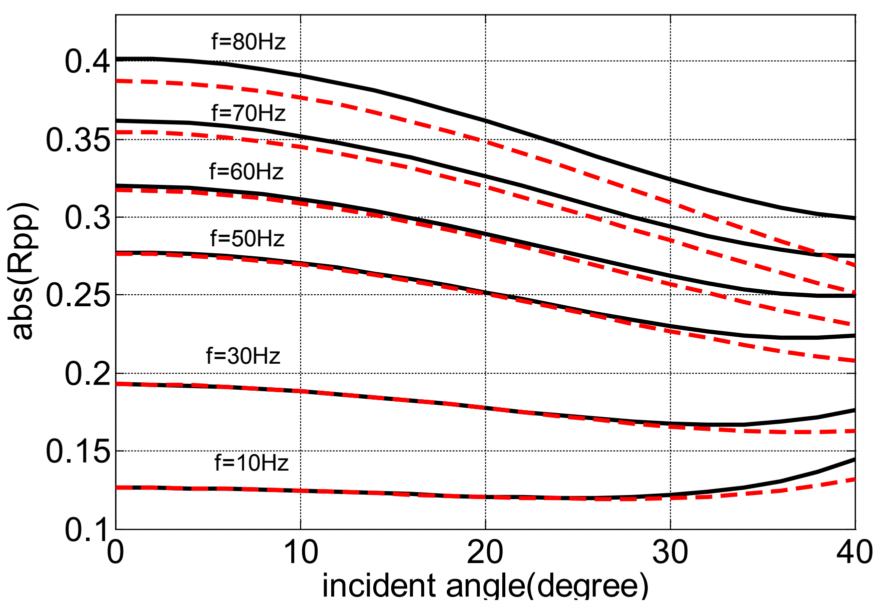

Figure 4. Comparison of approximation of absolute Rpp for nonwelded interface at different frequency with third order in fracture parameters $C_x=5\times10^{-10}$ and $C_z=2.5\times10^{-10}$ and elastic parameters (called 3E3C). The exact solution of Rpp for nonwelded interface is calculated from equation (32) in black line. Approximation is denoted by dash red line.

CONCLUSION

In this paper, we present a high order AVO approximation expressions of reflection coefficient and AVO series reversion solution for nonwelded interface. Based on the boundary conditions for the combined displacement and velocity discontinuity, the reflection coefficients of PP wave and PS wave are both complex function of frequency ω ,elastic parameters α , β , ρ across interface and fracture parameters C_x, C_z describing the interface. For saturated fracture, the fracture parameter are recast as function of inverse normal and tangential compliance of fracture (κ_x and κ_z) and viscosity η of fluid. When either κ and η tend to infinity, the solution reverts to a welded interface solution. When κ tends to zero the solution gives the displacement discontinuity. When η tend to zero the solution gives the particle velocity discontinuity model. When both κ and η tend to zero, the coefficients revert to those for free surface.

In order to understand the responses of the reflection coefficient R_{pp} and R_{ps} to elastic parameters contrasts across the interface and fracture parameters, we deduced new closed approximation that can be expressed as the function of elastic parameters contrasts across interface ($r_\alpha = \Delta\alpha/\alpha$, $r_\beta = \Delta\beta/\beta$ and $r_\rho = \Delta\rho/\rho$), average velocity ratio B_1 and B_2 of two layer respectively and new complex frequency dependent fracture parameters H_x and H_z .The complex parameter H_x and H_z is the function of frequency ω , fracture parameter and elastic parameters. H_x represents the effect of tangent fracture parameter C_x and shear impedance $\rho_2\beta_2$ of lower medium. H_z represents the effect of normal fracture parameter C_z and compressional impedance $\rho_2\alpha_2$ of lower medium.

Different from welded interface, the reflection coefficients include zero order of elastic contrast for nonwelded interface. The coefficients of zero order just are the function of fracture parameter and frequency. It shows that incident P wave propagating through a nonwelded interface in a single homogeneous medium (e.g., a crack, joint, or fracture) produces both frequency dependent reflected and transmitted P and S wave.

Experimental analyses approve that the third order truncation approximation in series of fracture parameters and elastic parameter contrasts is enough to characterize the reflection coefficients and can be used to invert the elastic parameter contrasts and fracture parameters by use of series reversion method.

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APPENDIX

The coefficients Γ in the expansion for PP and PS wave in equation 42 through 48 are expressed in detail as following.

For PP wave

$$\Gamma_0^{x} = -2B1^3 X^2$$
$$\Gamma_0^{z} = 1/2 + \left(-2B1^2 + 1/4\right) X^2$$
$$\Gamma_0^{xz} = 1/4 + \left(B1^2 B2 - 2B1^2 + 1/4\right) X^2$$
$$\Gamma_0^{x2} = -B1^3 X^2$$
$$\Gamma_0^{z2} = 1/4 + \left(B1^2 B2 - 2B1^2 + 1/4\right) X^2$$
$$\Gamma_0^{x3} = -1/2\, B1^3 X^2$$
$$\Gamma_0^{x2z1} = 1/2\, B1^2 \left(B1 - B2\right) X^2$$
$$\Gamma_0^{x1z2} = 1/2B1^2 \left(B1 - B2\right) X^2$$
$$\Gamma_0^{z3} = -1/2\, B1^3 X^2$$

$$\Gamma_{a_1}^{x} = 2B1^3 X^2$$
$$\Gamma_{a_1}^{z} = 1/2 X^2$$
$$\Gamma_{a_1}^{xz} = 0$$
$$\Gamma_{a_1}^{x2} = B1^3 X^2$$
$$\Gamma_{a_1}^{z2} = -1/8 + \left(B1^2 + 1/4\right) X^2$$
$$\Gamma_{\alpha 1}^{x3} = 1/2\, B1^3 X^2$$
$$\Gamma_{a1}^{x2z1} = 0$$
$$\Gamma_{\alpha 1}^{x1z2} = 1/2B1^2 \left(-B1 + B2\right) X^2$$

$$\Gamma_{\alpha 1}^{z3} = -1/8 + \left(-1/2\,B1^2\,B2 + 3/2\,B1^2 + 1/16\right)X^2$$

$$\Gamma_{\beta 1}^{x} = -2B1^3\,X^2$$
$$\Gamma_{\beta 1}^{z} = 2\,B1^2\left(-2 + B2 + B1\right)X^2$$
$$\Gamma_{\beta 1}^{xz} = 1/2\,B1^2\left(B1 - B2\right)X^2$$
$$\Gamma_{\beta 1}^{x2} = -1/2\,B1^3\,X^2$$
$$\Gamma_{\beta 1}^{z2} = 1/2\,B1^2\left(-6 + 5\,B2\right)X^2$$
$$\Gamma_{\beta 1}^{x3} = 0$$
$$\Gamma_{\beta 1}^{x2z1} = 0$$
$$\Gamma_{\beta 1}^{x1z2} = 1/2B1^2\left(B1 - B2\right)X^2$$
$$\Gamma_{\beta 1}^{z3} = 1/4\,B1^2\left(-2\,B1 + 8\,B2 - 8\right)X^2$$

$$\Gamma_{\rho 1}^{x} = B1^2\left(2\,B1 - 1\right)X^2$$
$$\Gamma_{\rho 1}^{z} = 1/2\,B1\left(2\,B1^2 + 2\,B1\,B2 - B1 - B2\right)X^2$$
$$\Gamma_{\rho 1}^{xz} = 1/4\,B1\left(4\,B1\,B2 - B1 - B2\right)X^2$$
$$\Gamma_{\rho 1}^{x2} = 1/2\,B1^2\left(-1 + 3\,B1\right)X^2 \qquad \Gamma_{\rho 1}^{z2} = -1/8 + \left(1/2\,B1^2\,B2 + B1^2 - 1/2\,B1\,B2 - 1/8\right)X^2$$
$$\Gamma_{\rho 1}^{x3} = 1/8\,B1^2\left(8\,B1 - 2\right)X^2$$
$$\Gamma_{\rho 1}^{x2z1} = 1/16\,B1\left(-4\,B1^2 + 12\,B1\,B2 - 2\,B1 - 2\,B2\right)X^2$$
$$\Gamma_{\rho 1}^{x1z2} = 1/8\,B1\left(-4\,B1^2 + 8\,B1\,B2 - 2\,B2\right)X^2$$
$$\Gamma_{\rho 1}^{z3} = -1/8 + \left(-1/4\,B1^3 - 1/4\,B1^2\,B2 + \frac{13\,B1^2}{8} - 3/8\,B1\,B2 - 3/16\right)X^2$$

$$\Gamma_{\alpha_2}^{x} = -1/2\,B1^3\,X^2$$
$$\Gamma_{\alpha 2}^{z} = -1/8 + \left(1/2\,B1^2 + 3/16\right)X^2$$
$$\Gamma_{\alpha 2}^{xz} = 1/4B1^2\left(-3B1 + B2\right)X^2$$
$$\Gamma_{\alpha_2}^{x2} = -\frac{B1^3}{4}X^2$$
$$\Gamma_{\alpha 2}^{z2} = -1/16 + \left(-1/4\,B1^2\,B2 + 1/2\,B1^2 - 1/16\right)X^2$$
$$\Gamma_{\alpha 2}^{x3} = -1/8\,B1^3\,X^2$$
$$\Gamma_{\alpha 2}^{x2z1} = -1/16\,B1^2\left(6\,B1 - 2\,B2\right)X^2$$
$$\Gamma_{\alpha 2}^{x1z2} = -1/16\,B1^2\left(4\,B1 - 2\,B2\right)X^2$$
$$\Gamma_{\alpha 2}^{z3} = \left(-1/4\,B1^2\,B2 - 1/8\right)X^2$$

$$\Gamma_{\beta 2}^{x} = -B1^3\,X^2$$
$$\Gamma_{\beta 2}^{z} = B1^2\left(B1 + 3\,B2 - 4\right)X^2$$
$$\Gamma_{\beta 2}^{xz} = 0$$
$$\Gamma_{\beta 2}^{x2} = 0$$

$$\Gamma_{\beta 2}^{z2} = -1/2\, B1^2 \left(2\,B1 - 6\,B2 + 6\right) X^2$$
$$\Gamma_{\beta 2}^{x3} = 1/8\, B1^3\, X^2$$
$$\Gamma_{\beta 2}^{x2z1} = 1/8\, B1^2 \left(-B1 + B2\right) X^2$$
$$\Gamma_{\beta 2}^{x1z2} = -1/8 B1^2 \left(-2B1 + B2\right) X^2$$
$$\Gamma_{\beta 2}^{z3} = -1/4 B1^2 \left(5B1 - 9B2 + 8\right) X^2$$

$$\Gamma_{\rho 2}^{x} = 1/8\, B1\left(4\,B1 - 1\right) X^2$$
$$\Gamma_{\rho 2}^{z} = -1/8 + \left(-B1^3 + 3/2\, B1^2 - B1/4 - 1/16\right) X^2$$
$$\Gamma_{\rho 2}^{xz} = -1/16\, B1\left(16\,B1^2 - 10\,B1 - 2\,B2 + 2\right) X^2$$
$$\Gamma_{\rho 2}^{x2} = 1/16\, B1\left(-8B1^2 + 8B1 - 1\right) X^2$$
$$\Gamma_{\rho 2}^{z2} = -\frac{1}{16} + \left(-\frac{3}{4} B1^2\, B2 + \frac{5}{4} B1^2 - \frac{3}{4} B1^3 + \frac{1}{4} B1\, B2 - \frac{3}{16} B1 - \frac{1}{16}\right) X^2$$
$$\Gamma_{\rho 2}^{x3} = -\frac{1}{32} B1\left(20B1^2 - 12B1 + 1\right) X^2$$
$$\Gamma_{\rho 2}^{x2z1} = -1/16B1\left(8B1^2 + 4B1\,B2 - 6B1 - 2B2 + 1\right) X^2$$
$$\Gamma_{\rho 2}^{x1z2} = -\frac{1}{32} B1\left(12B1^2 + 16B1\,B2 - 12B1 - 8B2 + 3\right) X^2$$
$$\Gamma_{\rho 2}^{z3} = -1/8B1\left(2B1^2 + 6B1\,B2 - 3B1 - 3B2 + 1\right) X^2$$

$$\Gamma_{\alpha 1\beta 1}^{x} = 2\,B1^3\, X^2$$
$$\Gamma_{\alpha 1\beta 1}^{z} = -2\,B1^2 \left(B1 - 1\right) X^2$$
$$\Gamma_{\alpha 1\beta 1}^{xz} = \frac{1}{2} B1^3\, X^2$$
$$\Gamma_{\alpha 1\beta 1}^{x2} = \frac{1}{2} B1^3\, X^2$$
$$\Gamma_{\alpha 1\beta 1}^{z2} = -B1^2 \left(B1 + B2 - 3\right) X^2$$
$$\Gamma_{\alpha 1\beta 1}^{x3} = 0$$
$$\Gamma_{\alpha 1\beta 1}^{x2z1} = 1/4\, B1^3\, X^2$$
$$\Gamma_{\alpha 1\beta 1}^{x1z2} = -1/4B1^2 \left(B1 - B2\right) X^2$$
$$\Gamma_{\alpha 1\beta 1}^{z3} = -1/4\, B1^2 \left(7\,B2 - 12\right) X^2$$

$$\Gamma_{\alpha 1\rho 1}^{x} = -B1^3\, X^2$$
$$\Gamma_{\alpha 1\rho 1}^{z} = -1/4 + \left(-B1^3 + 2\,B1^2 - 1/2\,B1\,B2 - 3/8\right) X^2$$
$$\Gamma_{\alpha 1\rho 1}^{xz} = -1/4B1\left(6B1^2 - 2B1\,B2 - 2B1 + B2\right) X^2$$
$$\Gamma_{\alpha 1\rho 1}^{x2} = -B1^3\, X^2$$
$$\Gamma_{\alpha 1\rho 1}^{z2} = -\frac{1}{8} + \left(-\frac{1}{2} B1^3 - B1^2\, B2 + \frac{7}{4} B1^2 - \frac{1}{4} B1\, B2 - \frac{1}{2}\right) X^2$$
$$\Gamma_{\alpha 1\rho 1}^{x3} = -3/4\, B1^3\, X^2$$

$$\Gamma^{x2z1}_{\alpha1\rho1} = -1/8\,B1\left(6B1^2 - 2B1\,B2 - 2B1 + B2\right)X^2$$
$$\Gamma^{x1z2}_{\alpha1\rho1} = -1/8\,B1\left(4B1^2 + 2B1\,B2 - 3B1 + B2\right)X^2$$
$$\Gamma^{z3}_{\alpha1\rho1} = -1/8\left(8\,B1^2\,B2 - 3\,B1^2 + 3\right)X^2$$

$$\Gamma^{x}_{\beta1\rho1} = 1/2\,B1^2\left(6\,B1 - 1\right)X^2$$
$$\Gamma^{z}_{\beta1\rho1} = -1/4\,B1\left(6\,B1^2 - 6\,B1\,B2 - 5\,B1 + B2\right)X^2$$
$$\Gamma^{xz}_{\beta1\rho1} = -\frac{1}{4}B1^2\left(-4B2 + 1\right)X^2$$
$$\Gamma^{x2}_{\beta1\rho1} = \frac{3}{2}B1^3\,X^2$$
$$\Gamma^{z2}_{\beta1\rho1} = -1/4\,B1\left(8\,B1^2 - 11\,B1 + B2\right)X^2$$
$$\Gamma^{x3}_{\beta1\rho1} = 1/8\,B1^2\left(4\,B1 + 1\right)X^2$$
$$\Gamma^{x2z1}_{\beta1\rho1} = -1/16\,B1\left(2\,B1^2 - 6\,B1\,B2 + B1 - B2\right)X^2$$
$$\Gamma^{x1z2}_{\beta1\rho1} = -1/8\,B1^2\left(4\,B1 - 8\,B2 + 1\right)X^2$$
$$\Gamma^{z3}_{\beta1\rho1} = -1/16\,B1\left(18\,B1^2 + 18\,B1\,B2 - 47\,B1 + 3\,B2\right)X^2$$

$$\Gamma^{x}_{\alpha3} = 0$$
$$\Gamma^{z}_{\alpha3} = 0$$
$$\Gamma^{xz}_{\alpha3} = \frac{1}{4}B1^3\,X^2$$
$$\Gamma^{x2}_{\alpha3} = 0$$
$$\Gamma^{z2}_{\alpha3} = 1/32 + \left(-1/4\,B1^2 - 1/16\right)X^2$$
$$\Gamma^{x3}_{\alpha3} = 0$$
$$\Gamma^{x2z1}_{\alpha3} = 1/8\,B1^3\,X^2$$
$$\Gamma^{x1z2}_{\alpha3} = 1/8\,B1^2\left(3\,B1 - B2\right)X^2$$
$$\Gamma^{z3}_{\alpha3} = 1/32 + 1/8\left(B1^2\,B2 - 3B1^2\,X^2 - \frac{1}{8}\right)X^2$$

$$\Gamma^{x}_{\beta3} = -\frac{1}{2}B1^3\,X^2$$
$$\Gamma^{z}_{\beta3} = -1/2B1^2\left(-B1 - 5B2 + 6\right)X^2$$
$$\Gamma^{xz}_{\beta3} = 0$$
$$\Gamma^{x2}_{\beta3} = 0$$
$$\Gamma^{z2}_{\beta3} = -1/4\,B1^2\left(6\,B1 - 10\,B2 + 9\right)X^2$$
$$\Gamma^{x3}_{\beta3} = 0$$
$$\Gamma^{x2z1}_{\alpha3} = 0$$
$$\Gamma^{x1z2}_{\alpha3} = 1/8\,B1^3\,X^2$$
$$\Gamma^{z3}_{\alpha3} = -1/8\,B1^2\left(13\,B1 - 15\,B2 + 12\right)X^2$$

$$\Gamma_{\rho 3}^{x} = -1/4B1^2\left(2B1-1\right)X^2$$
$$\Gamma_{\rho 3}^{z} = -1/8B1\left(2B1^2+2B1B2-B1-B2\right)X^2$$
$$\Gamma_{\rho 3}^{xz} = -1/16B1\left(-4B1^2+4B1B2+3B1-B2-1\right)X^2$$
$$\Gamma_{\rho 3}^{x2} = -\frac{1}{32}B1\left(8B1^2-1\right)X^2$$
$$\Gamma_{\rho 3}^{z2} = \frac{1}{32}+\frac{1}{8}\left(3B1^3+-B1^2B2-5B1^2+B1B2+\frac{3}{4}B1+\frac{1}{4}\right)X^2$$
$$\Gamma_{\rho 3}^{x3} = -1/32B1\left(4B1-1\right)X^2$$
$$\Gamma_{\rho 3}^{x2z1} = 1/16B1\left(6B1^2-2B1B2-4B1+1\right)X^2$$
$$\Gamma_{\rho 3}^{x1z2} = 1/32B1\left(16B1^2-4B1B2-12B1+3\right)X^2$$
$$\Gamma_{\rho 3}^{z3} = 1/32+\frac{1}{2}\left(B1^3+\frac{1}{2}B2B1^2-\frac{7B1^2}{4}+\frac{1}{4}B1+\frac{3}{32}\right)X^2$$

$$\Gamma_{\alpha 2\beta 1}^{x} = -\frac{1}{2}B1^3X^2$$
$$\Gamma_{\alpha 2\beta 1}^{z} = -1/2B1^2\left(-B1+B2-2\right)X^2$$
$$\Gamma_{\alpha 2\beta 1}^{xz} = +1/8B1^2\left(-7B1+B2\right)X^2$$
$$\Gamma_{\alpha 2\beta 1}^{x2} = -\frac{B1^3}{8}X^2$$
$$\Gamma_{\alpha 2\beta 1}^{z2} = +1/8B1^2\left(8B1-5B2\right)X^2$$
$$\Gamma_{\alpha 2\beta 1}^{x3} = 0$$
$$\Gamma_{\alpha 2\beta 1}^{x2z1} = -1/4B1^3X^2$$
$$\Gamma_{\alpha 2\beta 1}^{x1z2} = -1/8B1^2\left(4B1-B2\right)X^2$$
$$\Gamma_{\alpha 2\beta 1}^{z3} = 1/8B1^2\left(6B1-B2-8\right)X^2$$

$$\Gamma_{\alpha 2\rho 1}^{x} = +1/4B1^2\left(-2B1+1\right)X^2$$
$$\Gamma_{\alpha 2\rho 1}^{z} = \left(\frac{1}{4}B1^3+\frac{1}{8}B1^2-\frac{1}{4}B1^2B2-\frac{1}{8}B1B2-\frac{1}{2}\right)X^2$$
$$\Gamma_{\alpha 2\rho 1}^{xz} = \frac{1}{16}B1\left(12B1^2-4B1B2+B1-B2\right)X^2$$
$$\Gamma_{\alpha 2\rho 1}^{x2} = \frac{1}{8}B1^2\left(-B1+1\right)X^2$$
$$\Gamma_{\alpha 2\rho 1}^{z2} = \frac{1}{8}+\left(\frac{1}{2}B1^3-\frac{11}{8}B1^2-\frac{1}{8}B1^2B2+\frac{1}{8}B1B2-\frac{1}{4}\right)X^2$$
$$\Gamma_{\alpha 2\rho 1}^{x3} = 1/16B1^2X^2$$
$$\Gamma_{\alpha 2\rho 1}^{x2z1} = 1/32B1\left(18B1^2-6B1B2+B1-B2\right)X^2$$
$$\Gamma_{\alpha 2\rho 1}^{x1z2} = 1/16B1\left(20B1^2-10B1B2-3B1+B2\right)X^2$$
$$\Gamma_{\alpha 2\rho 1}^{z3} = \frac{1}{8}+\frac{1}{16}\left(6B1^3+10B1^2B2-32B1^2+3B1B2+1\right)X^2$$

$$\Gamma^{x}_{\beta 2\alpha 1} = B1^3 X^2$$
$$\Gamma^{z}_{\beta 2\alpha 1} = -B1^2\left(3B1-2\right)X^2$$
$$\Gamma^{xz}_{\beta 2\alpha 1} = \frac{1}{2}B1^3 X^2$$
$$\Gamma^{x2}_{\beta 2\alpha 1} = 0$$
$$\Gamma^{z2}_{\beta 2\alpha 1} = \frac{1}{2}B1^2\left(-B1-3B2+6\right)X^2$$
$$\Gamma^{x3}_{\beta 2\alpha 1} = -1/8\,B1^3 X^2$$
$$\Gamma^{x2z1}_{\beta 2\alpha 1} = 1/8\,B1^3 X^2$$
$$\Gamma^{x1z2}_{\beta 2\alpha 1} = 0$$
$$\Gamma^{z3}_{\beta 2\alpha 1} = 1/4\,B1^2\left(4\,B1-9\,B2+12\right)X^2$$

$$\Gamma^{x}_{\beta 2\rho 1} = 2B1^3 X^2$$
$$\Gamma^{z}_{\beta 2\rho 1} = -\frac{1}{2}B1^2\left(6B1-2B2-3\right)X^2$$
$$\Gamma^{xz}_{\beta 2\rho 1} = -\frac{1}{16}B1\left(-4B1^2-4B1\,B2+B1-B2\right)X^2$$
$$\Gamma^{x2}_{\beta 2\rho 1} = \frac{1}{8}B1^2\left(3B1+1\right)X^2$$
$$\Gamma^{z2}_{\beta 2\rho 1} = -\frac{1}{8}B1^2\left(20B1+5B2-22\right)X^2$$
$$\Gamma^{x3}_{\beta 2\rho 1} = -1/16\,B1^2\left(4\,B1-1\right)X^2$$
$$\Gamma^{x2z1}_{\beta 2\rho 1} = 1/32\,B1\left(6\,B1^2-6\,B1\,B2+B1+B2\right)X^2$$
$$\Gamma^{x1z2}_{\beta 2\rho 1} = -1/16\,B1\left(4\,B1^2-6\,B1\,B2+B1-B2\right)X^2$$
$$\Gamma^{z3}_{\beta 2\rho 1} = -1/8\,B1^2\left(8\,B1+13\,B2-23\right)X^2$$

$$\Gamma^{x}_{\rho 2\alpha 1} = -\frac{1}{8}B1\left(8B1^2-4B1+1\right)X^2$$
$$\Gamma^{z}_{\rho 2\alpha 1} = \frac{1}{8}\left(2B1^2-4B1^2B2+2B1B2-B1-1\right)X^2$$
$$\Gamma^{xz}_{\rho 2\alpha 1} = \frac{1}{16}B1\left(12\,B1^2-8\,B1\,B2-2\,B1+4\,B2-1\right)X^2$$
$$\Gamma^{x2}_{\rho 2\alpha 1} = -\frac{1}{16}B1\left(4B1^2-4B1+1\right)X^2$$
$$\Gamma^{z2}_{\rho 2\alpha 1} = \frac{1}{8}+\frac{1}{8}\left(8B1^3-14B1^2-2B1^2B2+4B1\,B2+1\right)X^2$$
$$\Gamma^{x3}_{\rho 2\alpha 1} = 1/32\,B1\left(4\,B1^2+4\,B1-1\right)X^2$$
$$\Gamma^{x2z1}_{\rho 2\alpha 1} = 1/32\,B1\left(24\,B1^2-12\,B1\,B2-6\,B1+6\,B2-1\right)X^2$$
$$\Gamma^{x1z2}_{\rho 2\alpha 1} = 1/16\,B1\left(22\,B1^2-10\,B1\,B2-9\,B1+5\,B2\right)X^2$$
$$\Gamma^{z3}_{\rho 2\alpha 1} = 1/8+\left(B1^3+3/4\,B2\,B1^2-5/2\,B1^2+3/8\,B1\,B2+1/16\,B1+\frac{5}{16}\right)X^2$$

$$\Gamma^{x}_{\rho 2\beta 1} = -\frac{1}{4}B1^{2}\left(4B1-3\right)X^{2}$$
$$\Gamma^{z}_{\rho 2\beta 1} = -\frac{1}{8}B1\left(18B1^{2}+6B1\,B2-17B1-B2+1\right)X^{2}$$
$$\Gamma^{xz}_{\rho 2\beta 1} = -\frac{1}{8}B1\left(8B1^{2}+4B1\,B2-4B1-B2\right)X^{2}$$
$$\Gamma^{x2}_{\rho 2\beta 1} = -\frac{1}{32}B1\left(44B1^{2}-12B1-1\right)X^{2}$$
$$\Gamma^{z2}_{\rho 2\beta 1} = -\frac{1}{32}B1\left(4B1^{2}+48B1B2-28B1-8B2+3\right)X^{2}$$
$$\Gamma^{x3}_{\rho 2\beta 1} = -1/32\,B1\left(32\,B1^{2}-2\,B1-1\right)X^{2}$$
$$\Gamma^{x2z1}_{\rho 2\beta 1} = -1/32\,B1\left(6\,B1^{2}+18\,B1\,B2-5\,B1-B2-1\right)X^{2}$$
$$\Gamma^{x1z2}_{\rho 2\beta 1} = -1/16\,B1\left(6\,B1^{2}+12\,B1\,B2-5\,B1-2\,B2\right)X^{2}$$
$$\Gamma^{z3}_{\rho 2\beta 1} = 1/32\,B1\left(30\,B1^{2}-34\,B1\,B2-15\,B1+9\,B2-2\right)X^{2}$$

$$\Gamma^{x}_{\alpha 1\beta 1\rho 1} = -2B1^{3}X^{2}$$
$$\Gamma^{z}_{\alpha 1\beta 1\rho 1} = -\frac{1}{4}B1\left(6B1^{2}+4B1\,B2-12B1+B2\right)X^{2}$$
$$\Gamma^{xz}_{\alpha 1\beta 1\rho 1} = -\frac{1}{4}B1^{2}\left(7B1-B2-1\right)X^{2}$$
$$\Gamma^{x2}_{\alpha 1\beta 1\rho 1} = -\frac{5}{4}B1^{3}X^{2}$$
$$\Gamma^{z2}_{\alpha 1\beta 1\rho 1} = \frac{1}{8}B1\left(14\,B1^{2}-16\,B1\,B2+7\,B1-B2\right)X^{2}$$
$$\Gamma^{x3}_{\alpha 1\beta 1\rho 1} = -1/2\,B1^{3}X^{2}$$
$$\Gamma^{x2z1}_{\alpha 1\beta 1\rho 1} = -1/16\,B1\left(10\,B1^{2}-B2\right)X^{2}$$
$$\Gamma^{x1z2}_{\alpha 1\beta 1\rho 1} = -1/4\,B1^{2}\left(3\,B1+B2-1\right)X^{2}$$
$$\Gamma^{z3}_{\alpha 1\beta 1\rho 1} = 1/16\,B1^{2}\left(40\,B1-18\,B2-23\right)X^{2}$$
$$B1=\beta_1/\alpha_1 \qquad B2=\beta_2/\alpha_2 \qquad X=\sin^2\theta$$
$$H_x = i\omega C_x\rho_2\beta_2 \qquad H_z = i\omega C_z\rho_2\alpha_2$$

For PS wave

$$\Gamma^{x}_{0} = -2B1^{3}X^{2}$$
$$\Gamma^{z}_{0} = 1/2+\left(-2B1^{2}+1/4\right)X^{2}$$
$$\Gamma^{xz}_{0} = 1/4+\left(B1^{2}B2-2B1^{2}+1/4\right)X^{2}$$
$$\Gamma^{x2}_{0} = -B1^{3}X^{2}$$
$$\Gamma^{z2}_{0} = 1/4+\left(B1^{2}B2-2B1^{2}+1/4\right)X^{2}$$
$$\Gamma^{x3}_{0} = -1/2\,B1^{3}X^{2}$$
$$\Gamma^{x2z1}_{0} = 1/2\,B1^{2}\left(B1-B2\right)X^{2}$$
$$\Gamma^{x1z2}_{0} = 1/2B1^{2}\left(B1-B2\right)X^{2}$$

$$\Gamma_0^{z3} = -1/2\,B1^3 X^2$$

$$\Gamma_{a_1}^{x} = 2B1^3 X^2$$
$$\Gamma_{a_1}^{z} = 1/2X^2$$
$$\Gamma_{a_1}^{xz} = 0$$
$$\Gamma_{a_1}^{x2} = B1^3 X^2$$
$$\Gamma_{a_1}^{z2} = -1/8 + \left(B1^2 + 1/4\right) X^2$$
$$\Gamma_{\alpha 1}^{x3} = 1/2\,B1^3 X^2$$
$$\Gamma_{a1}^{x2z1} = 0$$
$$\Gamma_{\alpha 1}^{x1z2} = 1/2B1^2\left(-B1 + B2\right) X^2$$
$$\Gamma_{\alpha 1}^{z3} = -1/8 + \left(-1/2\,B1^2 B2 + 3/2\,B1^2 + 1/16\right) X^2$$

$$\Gamma_{\beta 1}^{x} = -2B1^3 X^2$$
$$\Gamma_{\beta 1}^{z} = 2\,B1^2\left(-2 + B2 + B1\right) X^2$$
$$\Gamma_{\beta 1}^{xz} = 1/2\,B1^2\left(B1 - B2\right) X^2$$
$$\Gamma_{\beta 1}^{x2} = -1/2\,B1^3 X^2$$
$$\Gamma_{\beta 1}^{z2} = 1/2\,B1^2\left(-6 + 5\,B2\right) X^2$$
$$\Gamma_{\beta 1}^{x3} = 0$$
$$\Gamma_{\beta 1}^{x2z1} = 0$$
$$\Gamma_{\beta 1}^{x1z2} = 1/2B1^2\left(B1 - B2\right) X^2$$
$$\Gamma_{\beta 1}^{z3} = 1/4\,B1^2\left(-2\,B1 + 8\,B2 - 8\right) X^2$$

$$\Gamma_{\rho 1}^{x} = B1^2\left(2\,B1 - 1\right) X^2$$
$$\Gamma_{\rho 1}^{z} = 1/2\,B1\left(2\,B1^2 + 2\,B1\,B2 - B1 - B2\right) X^2$$
$$\Gamma_{\rho 1}^{xz} = 1/4\,B1\left(4\,B1\,B2 - B1 - B2\right) X^2$$
$$\Gamma_{\rho 1}^{x2} = 1/2\,B1^2\left(-1 + 3\,B1\right) X^2 \qquad \Gamma_{\rho 1}^{z2} = -1/8 + \left(1/2\,B1^2 B2 + B1^2 - 1/2\,B1\,B2 - 1/8\right) X^2$$
$$\Gamma_{\rho 1}^{x3} = 1/8\,B1^2\left(8\,B1 - 2\right) X^2$$
$$\Gamma_{\rho 1}^{x2z1} = 1/16\,B1\left(-4\,B1^2 + 12\,B1\,B2 - 2\,B1 - 2\,B2\right) X^2$$
$$\Gamma_{\rho 1}^{x1z2} = 1/8\,B1\left(-4\,B1^2 + 8\,B1\,B2 - 2\,B2\right) X^2$$
$$\Gamma_{\rho 1}^{z3} = -1/8 + \left(-1/4\,B1^3 - 1/4\,B1^2 B2 + \frac{13\,B1^2}{8} - 3/8\,B1\,B2 - 3/16\right) X^2$$

$$\Gamma_{\alpha_2}^{x} = -1/2\,B1^3 X^2$$
$$\Gamma_{\alpha 2}^{z} = -1/8 + \left(1/2\,B1^2 + 3/16\right) X^2$$
$$\Gamma_{\alpha 2}^{xz} = 1/4B1^2\left(-3B1 + B2\right) X^2$$
$$\Gamma_{\alpha_2}^{x2} = -\frac{B1^3}{4} X^2$$

$$\Gamma_{\alpha 2}^{z2} = -1/16 + \left(-1/4\,B1^2\,B2 + 1/2\,B1^2 - 1/16\right)X^2$$
$$\Gamma_{\alpha 2}^{x3} = -1/8\,B1^3\,X^2$$
$$\Gamma_{\alpha 2}^{x2z1} = -1/16\,B1^2\left(6\,B1 - 2\,B2\right)X^2$$
$$\Gamma_{\alpha 2}^{x1z2} = -1/16\,B1^2\left(4\,B1 - 2\,B2\right)X^2$$
$$\Gamma_{\alpha 2}^{z3} = \left(-1/4\,B1^2\,B2 - 1/8\right)X^2$$

$$\Gamma_{\beta 2}^{x} = -B1^3\,X^2$$
$$\Gamma_{\beta 2}^{z} = B1^2\left(B1 + 3\,B2 - 4\right)X^2$$
$$\Gamma_{\beta 2}^{xz} = 0$$
$$\Gamma_{\beta 2}^{x2} = 0$$
$$\Gamma_{\beta 2}^{z2} = -1/2\,B1^2\left(2\,B1 - 6\,B2 + 6\right)X^2$$
$$\Gamma_{\beta 2}^{x3} = 1/8\,B1^3\,X^2$$
$$\Gamma_{\beta 2}^{x2z1} = 1/8\,B1^2\left(-B1 + B2\right)X^2$$
$$\Gamma_{\beta 2}^{x1z2} = -1/8B1^2\left(-2B1 + B2\right)X^2$$
$$\Gamma_{\beta 2}^{z3} = -1/4B1^2\left(5B1 - 9B2 + 8\right)X^2$$

$$\Gamma_{\rho 2}^{x} = 1/8\,B1\left(4\,B1 - 1\right)X^2$$
$$\Gamma_{\rho 2}^{z} = -1/8 + \left(-B1^3 + 3/2\,B1^2 - B1/4 - 1/16\right)X^2$$
$$\Gamma_{\rho 2}^{xz} = -1/16\,B1\left(16\,B1^2 - 10\,B1 - 2\,B2 + 2\right)X^2$$
$$\Gamma_{\rho 2}^{x2} = 1/16\,B1\left(-8B1^2 + 8B1 - 1\right)X^2$$
$$\Gamma_{\rho 2}^{z2} = -\frac{1}{16} + \left(-\frac{3}{4}B1^2B2 + \frac{5}{4}B1^2 - \frac{3}{4}B1^3 + \frac{1}{4}B1\,B2 - \frac{3}{16}B1 - \frac{1}{16}\right)X^2$$
$$\Gamma_{\rho 2}^{x3} = -\frac{1}{32}B1\left(20B1^2 - 12B1 + 1\right)X^2$$
$$\Gamma_{\rho 2}^{x2z1} = -1/16B1\left(8B1^2 + 4B1\,B2 - 6B1 - 2B2 + 1\right)X^2$$
$$\Gamma_{\rho 2}^{x1z2} = -\frac{1}{32}B1\left(12B1^2 + 16B1\,B2 - 12B1 - 8B2 + 3\right)X^2$$
$$\Gamma_{\rho 2}^{z3} = -1/8B1\left(2B1^2 + 6B1\,B2 - 3B1 - 3B2 + 1\right)X^2$$

$$\Gamma_{\alpha 1\beta 1}^{x} = 2\,B1^3\,X^2$$
$$\Gamma_{\alpha 1\beta 1}^{z} = -2\,B1^2\left(B1 - 1\right)X^2$$
$$\Gamma_{\alpha 1\beta 1}^{xz} = \frac{1}{2}B1^3\,X^2$$
$$\Gamma_{\alpha 1\beta 1}^{x2} = \frac{1}{2}B1^3\,X^2$$
$$\Gamma_{\alpha 1\beta 1}^{z2} = -B1^2\left(B1 + B2 - 3\right)X^2$$
$$\Gamma_{\alpha 1\beta 1}^{x3} = 0$$
$$\Gamma_{\alpha 1\beta 1}^{x2z1} = 1/4\,B1^3\,X^2$$

$$\Gamma_{\alpha 1\beta 1}^{x1z2} = -1/4B1^2\left(B1 - B2\right)X^2$$
$$\Gamma_{\alpha 1\beta 1}^{z3} = -1/4\,B1^2\left(7\,B2 - 12\right)X^2$$

$$\Gamma_{\alpha 1\rho 1}^{x} = -B1^3 X^2$$
$$\Gamma_{\alpha 1\rho 1}^{z} = -1/4 + \left(-B1^3 + 2\,B1^2 - 1/2\,B1\,B2 - 3/8\right)X^2$$
$$\Gamma_{\alpha 1\rho 1}^{xz} = -1/4B1\left(6B1^2 - 2B1\,B2 - 2B1 + B2\right)X^2$$
$$\Gamma_{\alpha 1\rho 1}^{x2} = -B1^3 X^2$$
$$\Gamma_{\alpha 1\rho 1}^{z2} = -\frac{1}{8} + \left(-\frac{1}{2}B1^3 - B1^2 B2 + \frac{7}{4}B1^2 - \frac{1}{4}B1\,B2 - \frac{1}{2}\right)X^2$$
$$\Gamma_{\alpha 1\rho 1}^{x3} = -3/4\,B1^3 X^2$$
$$\Gamma_{\alpha 1\rho 1}^{x2z1} = -1/8\,B1\left(6B1^2 - 2B1\,B2 - 2B1 + B2\right)X^2$$
$$\Gamma_{\alpha 1\rho 1}^{x1z2} = -1/8\,B1\left(4B1^2 + 2B1\,B2 - 3B1 + B2\right)X^2$$
$$\Gamma_{\alpha 1\rho 1}^{z3} = -1/8\left(8\,B1^2 B2 - 3\,B1^2 + 3\right)X^2$$

$$\Gamma_{\beta 1\rho 1}^{x} = 1/2\,B1^2\left(6\,B1 - 1\right)X^2$$
$$\Gamma_{\beta 1\rho 1}^{z} = -1/4\,B1\left(6\,B1^2 - 6\,B1\,B2 - 5\,B1 + B2\right)X^2$$
$$\Gamma_{\beta 1\rho 1}^{xz} = -\frac{1}{4}B1^2\left(-4B2 + 1\right)X^2$$
$$\Gamma_{\beta 1\rho 1}^{x2} = \frac{3}{2}B1^3 X^2$$
$$\Gamma_{\beta 1\rho 1}^{z2} = -1/4\,B1\left(8\,B1^2 - 11\,B1 + B2\right)X^2$$
$$\Gamma_{\beta 1\rho 1}^{x3} = 1/8\,B1^2\left(4\,B1 + 1\right)X^2$$
$$\Gamma_{\beta 1\rho 1}^{x2z1} = -1/16\,B1\left(2\,B1^2 - 6\,B1\,B2 + B1 - B2\right)X^2$$
$$\Gamma_{\beta 1\rho 1}^{x1z2} = -1/8\,B1^2\left(4\,B1 - 8\,B2 + 1\right)X^2$$
$$\Gamma_{\beta 1\rho 1}^{z3} = -1/16\,B1\left(18\,B1^2 + 18\,B1\,B2 - 47\,B1 + 3\,B2\right)X^2$$

$$\Gamma_{\alpha 3}^{x} = 0$$
$$\Gamma_{\alpha 3}^{z} = 0$$
$$\Gamma_{\alpha 3}^{xz} = \frac{1}{4}B1^3 X^2$$
$$\Gamma_{\alpha 3}^{x2} = 0$$
$$\Gamma_{\alpha 3}^{z2} = 1/32 + \left(-1/4\,B1^2 - 1/16\right)X^2$$
$$\Gamma_{\alpha 3}^{x3} = 0$$
$$\Gamma_{\alpha 3}^{x2z1} = 1/8\,B1^3 X^2$$
$$\Gamma_{\alpha 3}^{x1z2} = 1/8\,B1^2\left(3\,B1 - B2\right)X^2$$
$$\Gamma_{\alpha 3}^{z3} = 1/32 + 1/8\left(B1^2 B2 - 3B1^2 X^2 - \frac{1}{8}\right)X^2$$

$$\Gamma^{x}_{\beta 3} = -\frac{1}{2}B1^3 X^2$$
$$\Gamma^{z}_{\beta 3} = -1/2B1^2\left(-B1 - 5B2 + 6\right)X^2$$
$$\Gamma^{xz}_{\beta 3} = 0$$
$$\Gamma^{x2}_{\beta 3} = 0$$
$$\Gamma^{z2}_{\beta 3} = -1/4\,B1^2\left(6\,B1 - 10\,B2 + 9\right)X^2$$
$$\Gamma^{x3}_{\beta 3} = 0$$
$$\Gamma^{x2z1}_{\alpha 3} = 0$$
$$\Gamma^{x1z2}_{\alpha 3} = 1/8\,B1^3 X^2$$
$$\Gamma^{z3}_{\alpha 3} = -1/8\,B1^2\left(13\,B1 - 15\,B2 + 12\right)X^2$$

$$\Gamma^{x}_{\rho 3} = -1/4B1^2\left(2B1 - 1\right)X^2$$
$$\Gamma^{z}_{\rho 3} = -1/8B1\left(2B1^2 + 2B1\,B2 - B1 - B2\right)X^2$$
$$\Gamma^{xz}_{\rho 3} = -1/16B1\left(-4B1^2 + 4B1\,B2 + 3B1 - B2 - 1\right)X^2$$
$$\Gamma^{x2}_{\rho 3} = -\frac{1}{32}B1\left(8B1^2 - 1\right)X^2$$
$$\Gamma^{z2}_{\rho 3} = \frac{1}{32} + \frac{1}{8}\left(3B1^3 + -B1^2 B2 - 5B1^2 + B1\,B2 + \frac{3}{4}B1 + \frac{1}{4}\right)X^2$$
$$\Gamma^{x3}_{\rho 3} = -1/32\,B1\left(4\,B1 - 1\right)X^2$$
$$\Gamma^{x2z1}_{\rho 3} = 1/16\,B1\left(6\,B1^2 - 2\,B1\,B2 - 4\,B1 + 1\right)X^2$$
$$\Gamma^{x1z2}_{\rho 3} = 1/32\,B1\left(16\,B1^2 - 4\,B1\,B2 - 12\,B1 + 3\right)X^2$$
$$\Gamma^{z3}_{\rho 3} = 1/32 + \frac{1}{2}\left(B1^3 + \frac{1}{2}B2\,B1^2 - \frac{7\,B1^2}{4} + \frac{1}{4}B1 + \frac{3}{32}\right)X^2$$

$$\Gamma^{x}_{\alpha 2\beta 1} = -\frac{1}{2}B1^3 X^2$$
$$\Gamma^{z}_{\alpha 2\beta 1} = -1/2B1^2\left(-B1 + B2 - 2\right)X^2$$
$$\Gamma^{xz}_{\alpha 2\beta 1} = +1/8\,B1^2\left(-7\,B1 + B2\right)X^2$$
$$\Gamma^{x2}_{\alpha 2\beta 1} = -\frac{B1^3}{8}X^2$$
$$\Gamma^{z2}_{\alpha 2\beta 1} = +1/8B1^2\left(8B1 - 5B2\right)X^2$$
$$\Gamma^{x3}_{\alpha 2\beta 1} = 0$$
$$\Gamma^{x2z1}_{\alpha 2\beta 1} = -1/4\,B1^3 X^2$$
$$\Gamma^{x1z2}_{\alpha 2\beta 1} = -1/8\,B1^2\left(4\,B1 - B2\right)X^2$$
$$\Gamma^{z3}_{\alpha 2\beta 1} = 1/8\,B1^2\left(6\,B1 - B2 - 8\right)X^2$$

$$\Gamma^{x}_{\alpha 2\rho 1} = +1/4B1^2\left(-2B1 + 1\right)X^2$$
$$\Gamma^{z}_{\alpha 2\rho 1} = \left(\frac{1}{4}B1^3 + \frac{1}{8}B1^2 - \frac{1}{4}B1^2 B2 - \frac{1}{8}B1\,B2 - \frac{1}{2}\right)X^2$$

$$\Gamma^{xz}_{\alpha 2\rho 1} = \frac{1}{16}B1\left(12B1^2 - 4B1\,B2 + B1 - B2\right)X^2$$

$$\Gamma^{x2}_{\alpha 2\rho 1} = \frac{1}{8}B1^2\left(-B1 + 1\right)X^2$$

$$\Gamma^{z2}_{\alpha 2\rho 1} = \frac{1}{8} + \left(\frac{1}{2}B1^3 - \frac{11}{8}B1^2 - \frac{1}{8}B1^2 B2 + \frac{1}{8}B1\,B2 - \frac{1}{4}\right)X^2$$

$$\Gamma^{x3}_{\alpha 2\rho 1} = 1/16\,B1^2 X^2$$

$$\Gamma^{x2z1}_{\alpha 2\rho 1} = 1/32\,B1\left(18\,B1^2 - 6\,B1\,B2 + B1 - B2\right)X^2$$

$$\Gamma^{x1z2}_{\alpha 2\rho 1} = 1/16\,B1\left(20\,B1^2 - 10\,B1\,B2 - 3\,B1 + B2\right)X^2$$

$$\Gamma^{z3}_{\alpha 2\rho 1} = \frac{1}{8} + \frac{1}{16}\left(6B1^3 + 10B1^2 B2 - 32B1^2 + 3B1\,B2 + 1\right)X^2$$

$$\Gamma^{x}_{\beta 2\alpha 1} = B1^3 X^2$$

$$\Gamma^{z}_{\beta 2\alpha 1} = -B1^2\left(3B1 - 2\right)X^2$$

$$\Gamma^{xz}_{\beta 2\alpha 1} = \frac{1}{2}B1^3 X^2$$

$$\Gamma^{x2}_{\beta 2\alpha 1} = 0$$

$$\Gamma^{z2}_{\beta 2\alpha 1} = \frac{1}{2}B1^2\left(-B1 - 3B2 + 6\right)X^2$$

$$\Gamma^{x3}_{\beta 2\alpha 1} = -1/8\,B1^3 X^2$$

$$\Gamma^{x2z1}_{\beta 2\alpha 1} = 1/8\,B1^3 X^2$$

$$\Gamma^{x1z2}_{\beta 2\alpha 1} = 0$$

$$\Gamma^{z3}_{\beta 2\alpha 1} = 1/4\,B1^2\left(4\,B1 - 9\,B2 + 12\right)X^2$$

$$\Gamma^{x}_{\beta 2\rho 1} = 2B1^3 X^2$$

$$\Gamma^{z}_{\beta 2\rho 1} = -\frac{1}{2}B1^2\left(6B1 - 2B2 - 3\right)X^2$$

$$\Gamma^{xz}_{\beta 2\rho 1} = -\frac{1}{16}B1\left(-4B1^2 - 4B1\,B2 + B1 - B2\right)X^2$$

$$\Gamma^{x2}_{\beta 2\rho 1} = \frac{1}{8}B1^2\left(3B1 + 1\right)X^2$$

$$\Gamma^{z2}_{\beta 2\rho 1} = -\frac{1}{8}B1^2\left(20B1 + 5B2 - 22\right)X^2$$

$$\Gamma^{x3}_{\beta 2\rho 1} = -1/16\,B1^2\left(4\,B1 - 1\right)X^2$$

$$\Gamma^{x2z1}_{\beta 2\rho 1} = 1/32\,B1\left(6\,B1^2 - 6\,B1\,B2 + B1 + B2\right)X^2$$

$$\Gamma^{x1z2}_{\beta 2\rho 1} = -1/16\,B1\left(4\,B1^2 - 6\,B1\,B2 + B1 - B2\right)X^2$$

$$\Gamma^{z3}_{\beta 2\rho 1} = -1/8\,B1^2\left(8\,B1 + 13\,B2 - 23\right)X^2$$

$$\Gamma^{x}_{\rho 2\alpha 1} = -\frac{1}{8}B1\left(8B1^2 - 4B1 + 1\right)X^2$$

$$\Gamma^{z}_{\rho 2\alpha 1} = \frac{1}{8}\left(2B1^2 - 4B1^2 B2 + 2B1B2 - B1 - 1\right)X^2$$

$$\Gamma^{xz}_{\rho 2\alpha 1} = \frac{1}{16} B1\left(12\,B1^2 - 8\,B1\,B2 - 2\,B1 + 4\,B2 - 1\right) X^2$$
$$\Gamma^{x2}_{\rho 2\alpha 1} = -\frac{1}{16} B1\left(4B1^2 - 4B1 + 1\right) X^2$$
$$\Gamma^{z2}_{\rho 2\alpha 1} = \frac{1}{8} + \frac{1}{8}\left(8B1^3 - 14B1^2 - 2B1^2 B2 + 4B1\,B2 + 1\right) X^2$$
$$\Gamma^{x3}_{\rho 2\alpha 1} = 1/32\,B1\left(4\,B1^2 + 4\,B1 - 1\right) X^2$$
$$\Gamma^{x2z1}_{\rho 2\alpha 1} = 1/32\,B1\left(24\,B1^2 - 12\,B1\,B2 - 6\,B1 + 6\,B2 - 1\right) X^2$$
$$\Gamma^{x1z2}_{\rho 2\alpha 1} = 1/16\,B1\left(22\,B1^2 - 10\,B1\,B2 - 9\,B1 + 5\,B2\right) X^2$$
$$\Gamma^{z3}_{\rho 2\alpha 1} = 1/8 + \left(B1^3 + 3/4\,B2\,B1^2 - 5/2\,B1^2 + 3/8\,B1\,B2 + 1/16\,B1 + \frac{5}{16}\right) X^2$$

$$\Gamma^{x}_{\rho 2\beta 1} = -\frac{1}{4} B1^2\left(4B1 - 3\right) X^2$$
$$\Gamma^{z}_{\rho 2\beta 1} = -\frac{1}{8} B1\left(18B1^2 + 6B1\,B2 - 17B1 - B2 + 1\right) X^2$$
$$\Gamma^{xz}_{\rho 2\beta 1} = -\frac{1}{8} B1\left(8B1^2 + 4B1\,B2 - 4B1 - B2\right) X^2$$
$$\Gamma^{x2}_{\rho 2\beta 1} = -\frac{1}{32} B1\left(44B1^2 - 12B1 - 1\right) X^2$$
$$\Gamma^{z2}_{\rho 2\beta 1} = -\frac{1}{32} B1\left(4B1^2 + 48B1B2 - 28B1 - 8B2 + 3\right) X^2$$
$$\Gamma^{x3}_{\rho 2\beta 1} = -1/32\,B1\left(32\,B1^2 - 2\,B1 - 1\right) X^2$$
$$\Gamma^{x2z1}_{\rho 2\beta 1} = -1/32\,B1\left(6\,B1^2 + 18\,B1\,B2 - 5\,B1 - B2 - 1\right) X^2$$
$$\Gamma^{x1z2}_{\rho 2\beta 1} = -1/16\,B1\left(6\,B1^2 + 12\,B1\,B2 - 5\,B1 - 2\,B2\right) X^2$$
$$\Gamma^{z3}_{\rho 2\beta 1} = 1/32\,B1\left(30\,B1^2 - 34\,B1\,B2 - 15\,B1 + 9\,B2 - 2\right) X^2$$

$$\Gamma^{x}_{\alpha 1\beta 1\rho 1} = -2B1^3 X^2$$
$$\Gamma^{z}_{\alpha 1\beta 1\rho 1} = -\frac{1}{4} B1\left(6B1^2 + 4B1\,B2 - 12B1 + B2\right) X^2$$
$$\Gamma^{xz}_{\alpha 1\beta 1\rho 1} = -\frac{1}{4} B1^2\left(7B1 - B2 - 1\right) X^2$$
$$\Gamma^{x2}_{\alpha 1\beta 1\rho 1} = -\frac{5}{4} B1^3 X^2$$
$$\Gamma^{z2}_{\alpha 1\beta 1\rho 1} = \frac{1}{8} B1\left(14\,B1^2 - 16\,B1\,B2 + 7\,B1 - B2\right) X^2$$
$$\Gamma^{x3}_{\alpha 1\beta 1\rho 1} = -1/2\,B1^3 X^2$$
$$\Gamma^{x2z1}_{\alpha 1\beta 1\rho 1} = -1/16\,B1\left(10\,B1^2 - B2\right) X^2$$
$$\Gamma^{x1z2}_{\alpha 1\beta 1\rho 1} = -1/4\,B1^2\left(3\,B1 + B2 - 1\right) X^2$$
$$\Gamma^{z3}_{\alpha 1\beta 1\rho 1} = 1/16\,B1^2\left(40\,B1 - 18\,B2 - 23\right) X^2$$