Inversion of quasi-compressional ray travel time data for anisotropic parameters in a TI Medium using phase velocities

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ABSTRACT

Problems are encountered during the course of one's research on related topics. These are put on a list with promises made that one day, they will be revisited. The one discussed here was put on such a list four decades ago. It was visited a number of times during that time. However, there was always the problem of being unable rederive the results presented in the paper. The specific work dealt with in this report is: [*Section (4.3) Computation of the Coefficients of Elasticity, Using the* $\mathbf{w}$ *Surface (Gassmann, 1964)]*. As noted in the title, using ray travel times (the $\mathbf{w}$ surface) the phase velocity is used to invert for the anisotropic parameters in a transversely isotropic (*TI*) medium. The exact phase velocities may be used to this end, given that the measured ray travel times have been acquired with reasonable accuracy. Apart from the exact phase velocities, the problem is set up so that approximate and linearized forms of the phase velocity may also be used. It should be further mentioned that the author is now able to derive the equations in the abovementioned section of the paper. As this is a fairly complex and lengthily undertaking, it will not be included here. A proper resolution with the formulae in the above paper was achieved. Consequently the equations given in that paper will be used in a moderately modified form. Only *P*-wave travel times are considered, as the *qP* and qS_V wave fronts are coupled so that the anisotropic medium parameters for both may be obtained using only *P*-wave data.

INTRODUCTION

The medium of interest will initially be assumed to be a transversely isotropic (*TI*) homogeneous halfspace with the axis of anisotropy aligned with the model coordinates (Figure 1). A medium where the axis of anisotropy $(\boldsymbol{\eta}_z)$ is rotated with respect to the model axes, as shown in Figure (2) is often easier to deal with as both A_{11} and A_{33} may be obtained from ray travel times at the receiver locations. The medium is homogeneous so that the rays are straight line paths from the source(s) to the receiver(s). The receivers are located at the surface of the halfspace in the meridional (x_1, x_3) plane. The source(s) are also situated in the meridional (x_1, x_3) plane in a vertical borehole so that the source and receivers are in the same plane. It will be further assumed that there are a "sufficient" number of rays from the source(s) to the receivers to satisfy requirements to obtain a solution for the inversion of travel time data for the anisotropic parameters of the medium within the context of the method employed.

The inverse problem for the above described medium type and geometry, the problem may be stated in the following manner. Given the group velocities (ray travel times), both magnitude and direction, from a "sufficient" set of sources and receivers, invert the data for the anisotropic

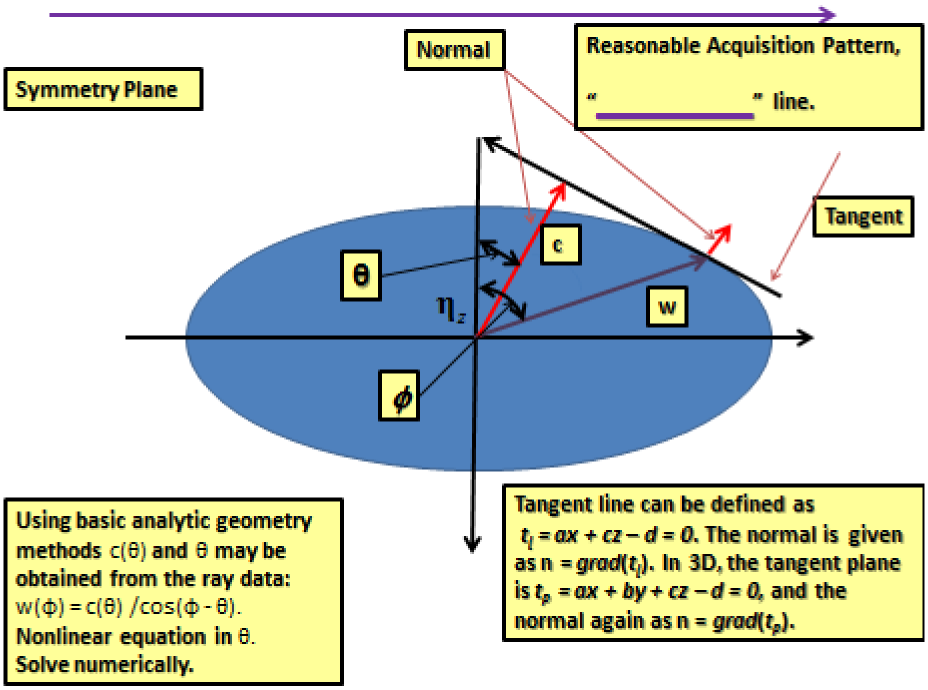

Fig. 1. Schematic of a situation where only the parameter A_{33} may be obtained from ray travel time measurements at the surface receivers.

parameters of the medium. The exact expression for the anisotropic phase velocity is employed in the inversion. Using a moderate anisotropic approximation or the linearized form of the phase velocity reduces the complexity; however, use of the exact expression is not overly cumbersome. It is required to solve for the A_{ij} that describe the medium from input data consisting of phase velocity travel times measured at specific angles, initially using the ray arrival times. It is assumed that the "coverage" of the input data is sufficient for this task.

THEORY

The method derived for the determination of anisotropic parameters employs the phase or wave front normal velocity as its basis. In practice, given a buried source and receiver located at the surface plane in a homogeneous halfspace, what is measured is the travel time along the ray that transports energy from the source to the receiver. As the medium is assumed to be homogeneous the ray is a straight line path. Energy propagates along the ray with what is known as group or ray velocity. In general, the phase and group velocities are not the same for some given ray angles, ϕ_{ray}. There is, however, a one to one mapping of the ray angle to some phase angle θ_{phase}.

Before proceeding further some definitions of a number of quantities related to wave propagation in a general anisotropic media should be presented. (All presented in the Appendix together with other useful vector formulae). The ray or group velocity along which energy propagates from one point to another in a medium is denoted in a Cartesian system as $\mathbf{w}(\varphi)=(w_1,w_2,w_3)$, with $w_i = dx_i/dt$ being the group velocity component in the $e_i,(i=1,2,3)$

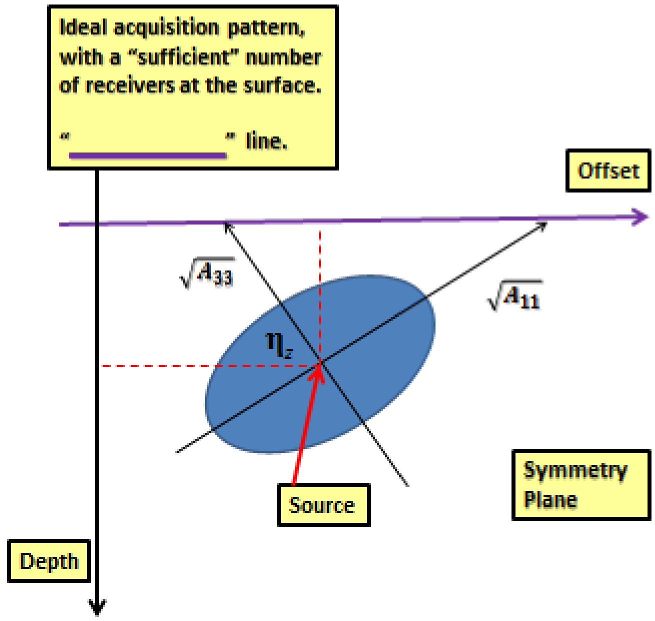

Fig. 2. Schematic of a geometry where the parameters A_{11} and A_{33} may be obtained from ray travel time measurements at the surface receivers.

direction, $\mathbf{e}\cdot\mathbf{e}=1$, $\mathbf{e}=(e_1,e_2,e_3)$, and ϕ is the ray angle. The loci of points taken at some equal time interval from the time origin, $t_0=0$, due to a disturbance at some point in the half space that causes a disturbance to propagate, is called the ray surface or wave front. In a 3D isotropic homogeneous medium, the wave front is a sphere. The same is not true in an anisotropic medium. The phase or wave front normal velocity associated with a particular ray is in the direction of the normal to the ray surface or wave front at the point where the ray comes in contact with the wave front (Figure 1). The magnitude of the phase velocity will be discussed after some other quantities are introduced and defined in the Appendix.

The exact eikonal for quasi-compressional (qP) wave propagation in a (rotationally invariant about $\boldsymbol{\eta}_z$) *T.I.* medium that is assumed to be homogeneous is (Gassmann, 1964)

$$G_{qP}\left(q_1^2,q_3^2\right)=A_{11}q_1^2+A_{33}q_3^2+\frac{A_\alpha\left(\sqrt{1+4\varepsilon_D}-1\right)}{2} \tag{1}$$

where q_1 and q_3 are the radial (horizontal) and vertical components of the slowness vector,

$$\mathbf{q} = (q_1, 0, q_3) = (q_1, q_3) \tag{2}$$

in a Cartesian system, defined relative to the unit vector, $\boldsymbol{\eta}_z$. The above quantities in (1) are specified by

$$\varepsilon_D = \frac{A_D q_1^2 q_3^2}{A_\alpha^2} \tag{3}$$

$$A_\alpha = (A_{11} - A_{55}) q_1^2 + (A_{33} - A_{55}) q_3^2 \tag{4}$$

and

$$A_D = (A_{13} + A_{55})^2 - (A_{11} - A_{55})(A_{33} - A_{55}). \tag{5}$$

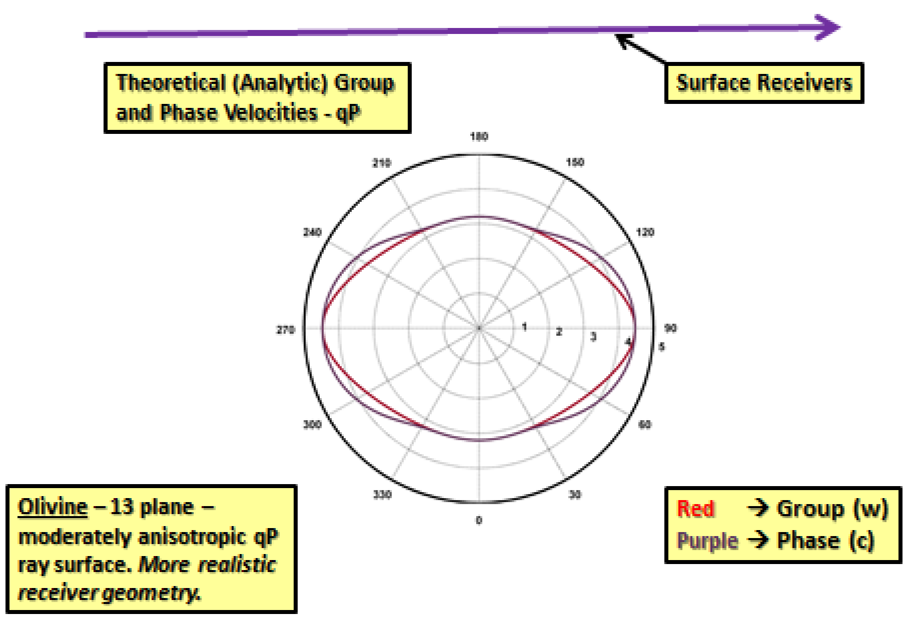

Fig. 3. Phase and group velocity surfaces in the 13-plane for Olivine.

The *moderate* anisotropic approximation (Schoenberg and Helbig, 1996) to the eikonal equation is obtained as

$$\frac{A_\alpha\left(\sqrt{1+4\varepsilon_D}-1\right)}{2} \approx \frac{A_\alpha\left(1+2\varepsilon_D-1\right)}{2} = A_\alpha\varepsilon_D = \frac{A_D q_1^2 q_3^2}{A_\alpha} \quad \left(\varepsilon_D << 1\right) \tag{6}$$

$$G_{qP}^{(app)}\left(q_1^2, q_3^2\right) \approx A_{11}q_1^2 + A_{33}q_3^2 + \frac{A_D q_1^2 q_3^2}{A_\alpha}. \tag{7}$$

For completeness, the linearized form of (1) may be written as

$$G_P^{(lin)}\left(q_1^2, q_3^2\right) \approx A_{11}q_1^2 + A_{33}q_3^2 + \frac{E_{13}q_1^2 q_3^2}{\left(q_1^2 + q_3^2\right)}. \tag{8}$$

where

$$E_{13} = 2\left(A_{13} + 2A_{55}\right) - \left(A_{11} + A_{33}\right). \tag{9}$$

From the theory of characteristics, given some initial conditionals on the slowness vector, $\mathbf{q}_0 = \mathbf{q}\left(t_0\right)$ and the position vector $\mathbf{x}_0 = \mathbf{x}\left(t_0\right)$, with t being time, the following initial value problems for the characteristics (rays) are obtained (Courant and Hilbert, 1962; Červený, 1972)

Exact:

$$\frac{dx_1}{dt} = \frac{1}{2}\frac{\partial G_P}{\partial p_1} = \frac{p_1}{2}\Bigg[\left(A_{11}+A_{55}\right) + \frac{\left[\left(\left(A_{11}-A_{55}\right)p_1^2 + \left(A_{33}-A_{55}\right)p_3^2\right)\left(A_{11}-A_{55}\right) + 2A_D p_3^2\right]}{\left[\left(\left(A_{11}-A_{55}\right)p_1^2 + \left(A_{33}-A_{55}\right)p_3^2\right)^2 + 4A_D p_1^2 p_3^2\right]^{1/2}}\Bigg] \tag{10}$$

$$\frac{dx_3}{dt} = \frac{1}{2}\frac{\partial G_P}{\partial p_3} = \frac{p_3}{2}\Bigg[\left(A_{33}+A_{55}\right) + \frac{\left[\left(\left(A_{11}-A_{55}\right)p_1^2 + \left(A_{33}-A_{55}\right)p_3^2\right)\left(A_{33}-A_{55}\right) + 2A_D p_1^2\right]}{\left[\left(\left(A_{11}-A_{55}\right)p_1^2 + \left(A_{33}-A_{55}\right)p_3^2\right)^2 + 4A_D p_1^2 p_3^2\right]^{1/2}}\Bigg] \tag{11}$$

$$\frac{dq_1}{dt} = -\frac{1}{2}\frac{\partial G_P}{\partial x_1} = 0 \tag{12}$$

$$\frac{dq_3}{dt} = -\frac{1}{2}\frac{\partial G_P}{\partial x_3} = 0 \tag{13}$$

Moderate anisotropy:

$$\frac{dx_1}{dt} = \frac{1}{2}\frac{\partial G_P^{(app.)}}{\partial q_1} = q_1\left[A_{11} + \frac{A_D q_3^4\left(A_{33} - A_{55}\right)}{A_\alpha^2}\right] \tag{14}$$

$$\frac{dx_3}{dt} = \frac{1}{2}\frac{\partial G_P^{(app.)}}{\partial q_3} = q_3\left[A_{33} + \frac{A_D q_1^4\left(A_{11} - A_{55}\right)}{A_\alpha^2}\right] \tag{15}$$

$$\frac{dq_1}{dt} = -\frac{1}{2}\frac{\partial G_P^{(app.)}}{\partial x_1} = 0 \tag{16}$$

$$\frac{dq_3}{dt} = -\frac{1}{2}\frac{\partial G_P^{(app.)}}{\partial x_3} = 0 \tag{17}$$

Linearized:

$$\frac{dx_1}{dt} = \frac{1}{2}\frac{\partial G_P^{(lin.)}}{\partial q_1} = q_1\left[A_{11} + \frac{E_{13} q_3^4}{\left(q_1^2 + q_3^2\right)^2}\right] \tag{18}$$

$$\frac{dx_3}{dt} = \frac{1}{2}\frac{\partial G_P^{(lin.)}}{\partial q_3} = q_3\left[A_{33} + \frac{E_{13} q_1^4}{\left(q_1^2 + q_3^2\right)^2}\right] \tag{19}$$

$$\frac{dq_1}{dt} = -\frac{1}{2}\frac{\partial G_P^{(lin.)}}{\partial x_1} = 0 \tag{20}$$

$$\frac{dq_3}{dt} = -\frac{1}{2}\frac{\partial G_P^{(lin.)}}{\partial x_3} = 0 \tag{21}$$

The last two equations in each of the above three sets of equations both being equal to zero is a result of the assumption of homogeneity, so that along a straight ray both q_1 and q_3 are constant. The ray (group) velocity is given in all 3 cases by

$$w(\phi) = \left[\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_3}{dt}\right)^2\right]^{1/2} \tag{22}$$

As previously stated, t has been taken to be time. The angle that the ray makes with the vertical anisotropic axis, ϕ, is defined through the relation

$$\tan\phi = \left[\frac{dx_1/dt}{dx_3/dt}\right] = \frac{q_1}{q_3} F\left(A_{ij}, q_1, q_3\right) = \tan\theta F\left(A_{ij}, q_1, q_3\right) \tag{23}$$

From equations (10) and (11) (the exact case)

$$F\left(A_{ij}, q_1, q_3\right) = \frac{F_1\left(A_{ij}, q_1, q_3\right)}{F_2\left(A_{ij}, q_1, q_3\right)} \tag{24}$$

$$F_1\left(A_{ij}, q_1, q_3\right) = \left(A_{11} + A_{55}\right) + \frac{\left[\left(\left(A_{11} - A_{55}\right)q_1^2 + \left(A_{33} - A_{55}\right)q_3^2\right)\left(A_{11} - A_{55}\right) + 2A_D q_3^2\right]}{\left[\left(\left(A_{11} - A_{55}\right)q_1^2 + \left(A_{33} - A_{55}\right)q_3^2\right)^2 + 4A_D q_1^2 q_3^2\right]^{1/2}} \tag{25}$$

$$F_2\left(A_{ij}, q_1, q_3\right) = \left(A_{33} + A_{55}\right) + \frac{\left[\left(\left(A_{11} - A_{55}\right)q_1^2 + \left(A_{33} - A_{55}\right)q_3^2\right)\left(A_{33} - A_{55}\right) + 2A_D q_1^2\right]}{\left[\left(\left(A_{11} - A_{55}\right)q_1^2 + \left(A_{33} - A_{55}\right)q_3^2\right)^2 + 4A_D q_1^2 q_3^2\right]^{1/2}} \tag{26}$$

where θ is the wave front normal or phase angle and the formulae for the vertical and horizontal components of the slowness vector, $q_1 = \sin\theta / c(\theta)$ and $q_2 = \cos\theta / c(\theta)$, $c(\theta)$ being the phase velocity which will be discussed next.

The exact expression for the phase (wave front normal) velocity for the quasi-compressional, qP, case of wave propagation in a transversely isotropic medium follows from the eikonal equations with the substitutions for q_1 and q_3, defined above, as

$$c^2(\theta) = c_e^2(\theta) + \frac{A_\alpha\left(\sqrt{1+4\varepsilon_D} - 1\right)}{2} \quad (\textbf{exact}) \tag{27}$$

$$c^2(\theta) \approx c_e^2(\theta) + \frac{A_D \sin^2\theta\cos^2\theta}{A_\alpha} \quad (\textbf{approximate}) \tag{28}$$

$$c^2(\theta) \approx c_e^2(\theta) + E_{13}\sin^2\theta\cos^2\theta \quad (\textbf{linearized}) \tag{29}$$

Quantities requiring definition are the phase velocity for the degenerate ellipsoidal (elliptical) case

$$c_e^2(\theta) = A_{11}\sin^2\theta + A_{33}\cos^2\theta \tag{30}$$

together with the marginally redefined values of A_α and ε_D

$$A_\alpha = \left(A_{11} - A_{55}\right)\sin^2\theta + \left(A_{33} - A_{55}\right)\cos^2\theta$$
$$= A_{11}\sin^2\theta + A_{33}\cos^2\theta - A_{55} \quad (31)$$
$$= c_e^2\left(\theta\right) - A_{55}$$

$$\varepsilon_D = \frac{A_D \sin^2\theta\cos^2\theta}{A_\alpha^2} \quad (32)$$

and as before

$$A_D = \left(A_{13} + A_{55}\right)^2 - \left(A_{11} - A_{55}\right)\left(A_{33} - A_{55}\right) \quad (33)$$

and

$$E_{13} = 2\left(A_{13} + 2A_{55}\right) - \left(A_{11} + A_{33}\right) \ . \quad (34)$$

Values of A_{11} and A_{33} are known:

Assume that the profile gradients, $\mathbf{p}$ and $\tilde{\mathbf{p}}$ have been obtained from the two ray velocities, $w\left(\phi\left(\theta\right)\right)$ and $\tilde{w}\left(\tilde{\phi}\left(\tilde{\theta}\right)\right)$. Using the relationships between the profile gradients, $\mathbf{p}$ and $\tilde{\mathbf{p}}$, and the slownesses, $\mathbf{q}$ and $\tilde{\mathbf{q}}$, and their relation to the phase velocities $c\left(\theta\right)$ and $\tilde{c}\left(\tilde{\theta}\right)$ the following equations in the two unknowns, A_{55} and A_D (actually $\left(A_{13} + A_{55}\right)^2$ from which A_D may be obtained) may be obtained using the values of the known quantities A_{33}, A_{11} and A_{55} .

Defining

$$\xi = \tan^2\theta, \quad \tilde{\xi} = \tan^2\tilde{\theta} \quad (35)$$

the value of A_{55} may be obtained employing two of the measured phase angles and velocities. After a fairly substantial exercise in algebra it may be determined that

$$A_{55} = \frac{U_1 A_{11} + U_2 A_{33} + U_3}{\xi\tilde{\xi} A_{11} - A_{33} - U_1 - U_2} \quad (36)$$

given that

$$U_1 = \frac{\xi\tilde{\xi}\left[c^2(\theta)(1+\xi) - c^2(\tilde{\theta})(1+\tilde{\xi})\right]}{\xi - \tilde{\xi}} \tag{37}$$

$$U_2 = \frac{c^2(\theta)(1+\xi)\tilde{\xi} - c^2(\tilde{\theta})(1+\tilde{\xi})\xi}{\xi - \tilde{\xi}} \tag{38}$$

$$U_3 = \frac{c^4(\theta)(1+\xi)^2\tilde{\xi} - c^4(\tilde{\theta})(1+\tilde{\xi})^2\xi}{\xi - \tilde{\xi}} \tag{39}$$

The results are shown in Figure 4, where all possible combinations of 2 angles have been used.

The second term to be solved for is $(A_{13} + A_{55})^2$. In this case the phase angle and velocity related to only one ray is required. Using the exact eikonal equation, after a less substantial algebraic manipulation than for A_{55}, the following expression is obtained

$$\begin{aligned}(A_{13} + A_{55})^2 = \frac{c^4(\theta)(1+\xi)^2}{\xi} - c^2(\theta)(1+\xi)(A_{11} + A_{55}) - \\ \frac{c^2(\theta)(1+\xi)(A_{33} + A_{55})}{\xi} + \xi A_{11}A_{55} + A_{11}A_{33} + A_{55}^2\end{aligned} \tag{40}$$

All of the recorded ray angles are used in this process, with the results shown in Figure 5.
A_D may now determined, as it has been assumed that A_{11} and A_{33} are known and A_{55} has been determined previously.

Value of A_{11} is unknown with A_{33} known:

Assume that the phase velocity $\hat{c}(\hat{\theta})$ is known at some third angle, $\hat{\theta}$. Rearrange the expression for the phase velocity for qP wave propagation in a transversely isotropic medium, with the object of isolating the variable A_{11}, which has been assumed unknown, and the following quadratic equation in A_{11} is obtained. All possible combinations of the determined phase angles and velocities are use here.

Let $\hat{\zeta} = \tan^2\hat{\theta}$ so that the following quadratic equation for A_{11} is obtained.

$$S_1 = \xi\hat{\xi}U_1 - \xi\tilde{\xi}V_1 \tag{41}$$

$$S_2 = \xi\hat{\xi}U_2 - U_1 - \xi\tilde{\xi}V_2 + V_1 \tag{42}$$

$$S_3 = V_2 - U_2 \tag{43}$$

$$S_4 = \xi\hat{\xi}U_3 - \left(V_1 + V_2\right)U_1 - \xi\tilde{\xi}V_3 + \left(U_1 + U_2\right)V_1 \tag{44}$$

$$S_5 = -U_3 - \left(V_1 + V_2\right)U_2 + V_3 + \left(U_1 + U_2\right)V_2 \tag{45}$$

$$S_6 = \left(U_1 + U_2\right)V_3 - \left(V_1 + V_2\right)U_3 \tag{46}$$

where

$$V_1 = \frac{\xi\hat{\xi}\left[c^2(\theta)(1+\xi) - c^2\left(\hat{\theta}\right)\left(1+\hat{\xi}\right)\right]}{\xi - \hat{\xi}} \tag{47}$$

$$V_2 = \frac{c^2(\theta)(1+\xi)\hat{\xi} - c^2\left(\hat{\theta}\right)\left(1+\hat{\xi}\right)\xi}{\xi - \hat{\xi}} \tag{48}$$

$$V_3 = \frac{c^4(\theta)(1+\xi)^2\hat{\xi} - c^4\left(\hat{\theta}\right)\left(1+\hat{\xi}\right)^2\xi}{\xi - \hat{\xi}} \tag{49}$$

leading to

$$S_1A_{11}^2 + \left(S_2A_{33} + S_4\right)A_{11} + \left(S_3A_{33}^2 + S_5A_{33} + S_6\right) = 0 \tag{50}$$

The numerically computed results for A_{11} are shown in Figure 6.

To check the inverted values of the quantities A_{55}, A_D and A_{11} obtained a check may be made by selecting another angle to obtain a quadratic equation for A_{33} in terms of the computed values of A_{55}, A_D and A_{11} which were all computed under the assumption that A_{33} was known. Again, all combinations of phase angles and velocities are used.

$$S_3A_{33}^2 + \left(S_2A_{11} + S_5\right)A_{33} + \left(S_1A_{11}^2 + S_4A_{33} + S_6\right) = 0\,. \tag{51}$$

The equation in the unknown A_{11} is a function of the two secondary unknowns A_{55} and A_D .as well as A_{33}. Using an iterative process in which an initial estimates of A_{11} and A_{33} are used to solve for A_{55} and A_D, updated estimates of A_{11} and A_{33} are obtained and the process repeated until convergence is obtained. The values for A_{33} appear in Figure 6.

NUMERICAL RESULTS

In a homogeneous *TI* medium the anisotropic parameters may be obtained from the group velocity travel times using the exact formula for the phase velocity. In any inversion process, at least one quantity is assumed to be known, or a good estimate. Here it is the square of the vertical velocity in a *TI* medium $\alpha_0^2 = A_{33}$. As the *qP* and *qSv* wave (ray) fronts are coupled in this case it is also possible to obtain all anisotropic parameters related to the shear (*qSv*) wavefront. As a consequence, the first parameter that will be sought is the *isotopic* shear wave velocity $\beta_0^2 = A_{55}$. This process requires that two (ray) data points on the ray surface be employed to solve for A_{55}. Determining A_{11} or A_{33} also requires two (ray) data points, while $\left(A_{13} + A_{55}\right)^2$ requires only one (ray) data point. The model parameters used are given in Table 1.

The derivations are moderately algebraically complex and, as a consequence, are not included here. There are numerous ways to compute the values of A_{mn} or combinations thereof. The better the coverage (number of data points) the better. This method may be extended to 3D. However, the trade – off between analytic derivation and accuracy might be questionable.

CONCLUSIONS

A method for recovering anisotropic parameters in a TI medium using ray travel times as input and exact expressions for phase velocities in the inversion process. There a number of other manners for accomplishing the same result. However, as mentioned earlier in the report this method has been left without attention for several decades and it was only investigated to determine if it actually works. Exact ray travel times with some random noise introduced were used. The resulting data was then smoothed. It is clear that more noise should have been included. However, the object of this experiment was to determine if this method actually worked. Sensitivity studies are the next matters which could be addressed.

Anisotropic Parameters	A_{11} km^2/s^2	A_{33} km^2/s^2	A_{55} km^2/s^2	A_D km^4/s^4
	20.00	10.25	2.34	2.073

Table 1. Anisotropic parameters for *TI* wave propagation in the 13-plane of Olivine.

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APPENDIX A

Formulae required for converting ray directions (angles) and velocities to phase angles and velocities. It required that a nonlinear type of equation be solve for this and that it is required that it be done numerically. Newton's or Brent's Methods are both equally accurate.

1. For some vector $\boldsymbol{\xi}$, define $\boldsymbol{\xi}\cdot\boldsymbol{\xi}=\xi^2$ and $|\boldsymbol{\xi}|=\xi$.
2. Ray (group) velocity - $\mathbf{w}(\phi)$, where ϕ is the ray angle.
3. Profile gradient - $\mathbf{p}(\phi)$, $\mathbf{p}=\dfrac{\mathbf{w}}{w^2}$, $\mathbf{w}=\dfrac{\mathbf{p}}{p^2}$, $\mathbf{w}\cdot\mathbf{p}=1$
4. Normal (phase) velocity - $\mathbf{c}(\theta)$ where θ is the phase angle.
5. Slowness vector - $\mathbf{q}=\dfrac{\mathbf{c}}{c^2}$, $\mathbf{c}=\dfrac{\mathbf{q}}{q^2}$, $\mathbf{c}\cdot\mathbf{q}=1$.
6. Relationships between group and phase quantities: $\mathbf{p}\cdot\mathbf{q}=p^2$, $\mathbf{c}\cdot\mathbf{w}=c^2$.
7. Let $\boldsymbol{\eta}_z$ be a unit vector along the axis about which the transversely isotropic medium is rotationally invariant. ($\pm\boldsymbol{\eta}_z$ may be used. However, apart from very special cases, it is best to be consistent and choose only one.)
8. Plot $\mathbf{w},\mathbf{c}$ and $\boldsymbol{\eta}_z$ from a common point, usually the origin of the ray surface. Then, it may be seen that $|\mathbf{w}|=w(\phi)=\dfrac{c(\theta)}{\cos(\theta-\phi)}$, $\mathbf{w}=g_1\mathbf{c}-g_2\boldsymbol{\eta}_z$ or $\mathbf{c}=\dfrac{\mathbf{w}}{g_1}+\dfrac{g_2\boldsymbol{\eta}_z}{g_1}$, $|\mathbf{c}|=c(\theta)$. From this it may be inferred that the orientation of $\boldsymbol{\eta}_z$ has no effect on the relation between the phase and group velocities or the difference between the phase and group propagation angles.
9. The above point is equivalent to $g_1=\dfrac{w\sin\phi}{c\sin\theta}$ and $g_2=\dfrac{w\sin(\phi-\theta)}{\sin\theta}$.
10. Given the above relations, together with the source and receiver locations and the ray travel times, the ray angles and velocities and hence the phase angles and velocities may be determined.

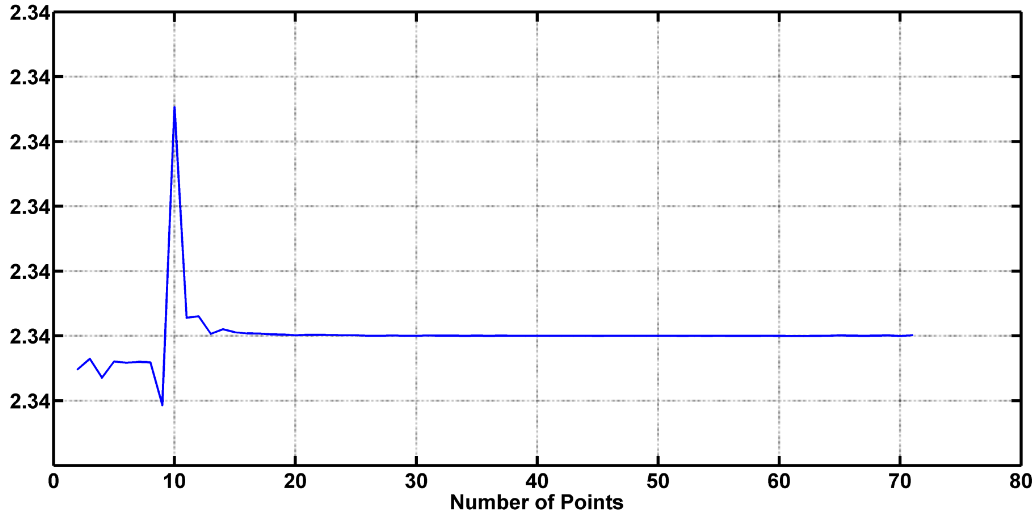

Fig. 4. A_{55} in the 13-plane for Olivine. Two data points are required to be used concurrently to obtain this value. The error is approximately 1.0e-07.

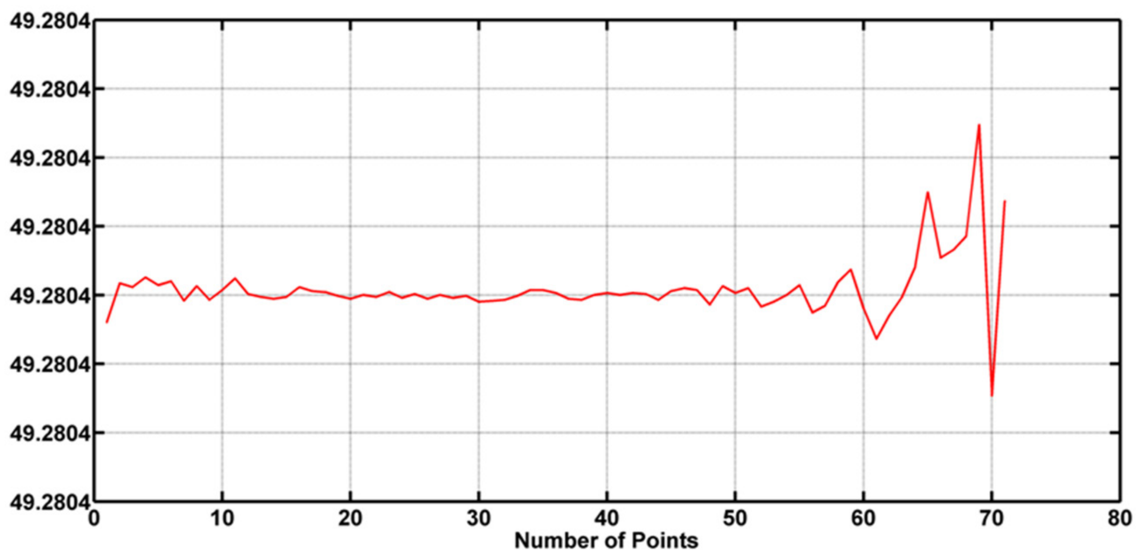

Fig. 5. $(A_{13}+A_{55})^2$ in the 13-plane for Olivine. One point is required for the determination. The error is again about 1.0e-07. Once this quantity has been determined, the deviation from the elliptical, A_D may be computed. ($A_D = (A_{13}+A_{55})^2-(A_{11}-A_{55})(A_{33}-A_{55})$)

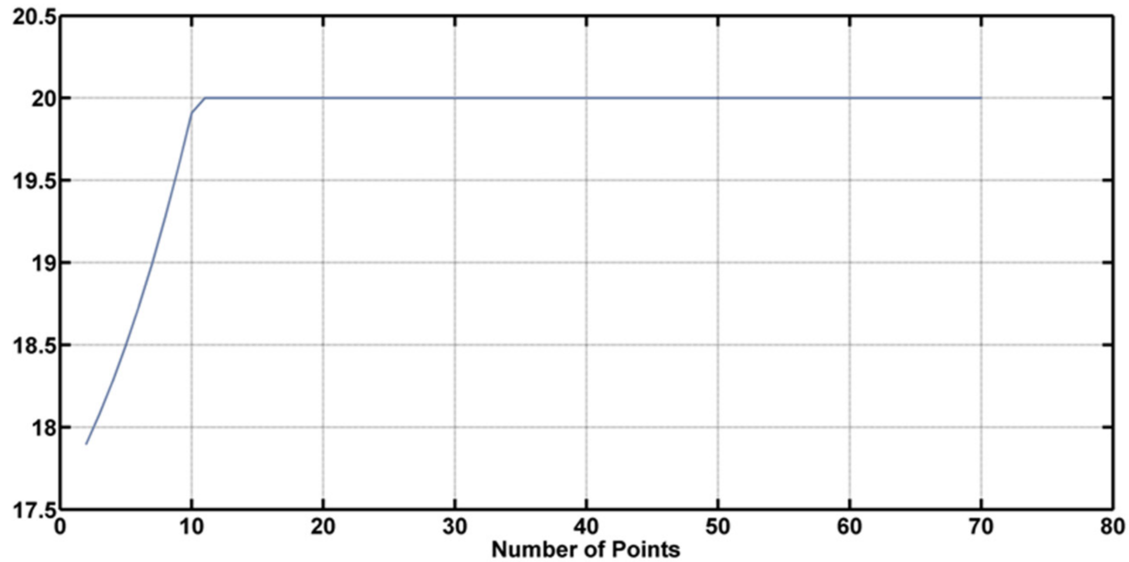

Fig. 6. A_{11} in the 13-plane for Olivine. Two points are required for the determination. The error is about 1.0e-07.

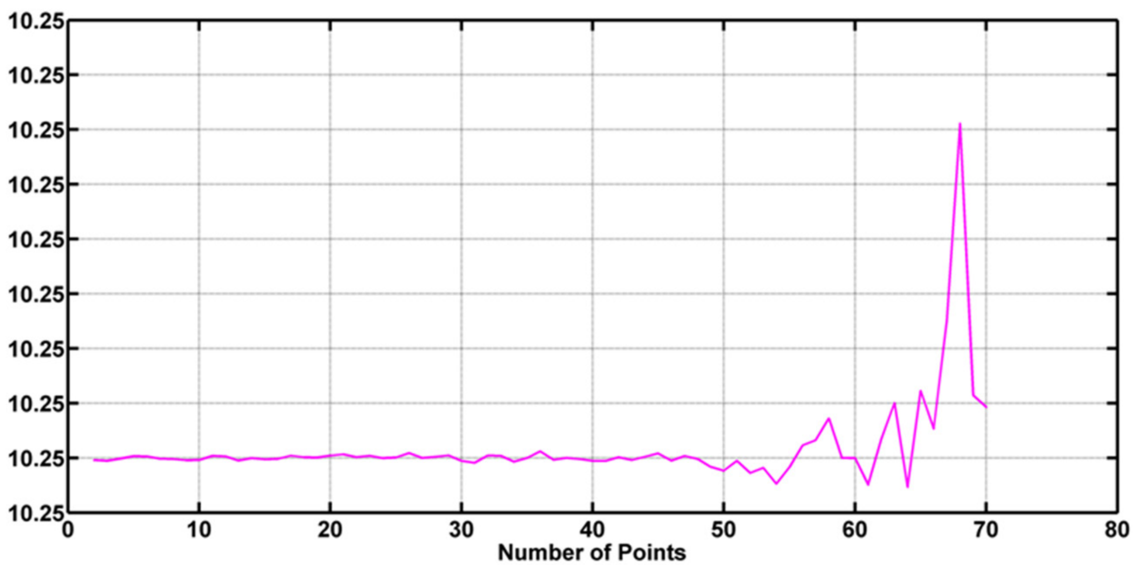

Fig.7. Check on A_{33} in the 13-plane for Olivine. Two points are required for the determination.