P-S_V wave propagation in a radially symmetric vertically inhomogeneous TI medium with absorbing boundary conditions

P.F. Daley and E.S. Krebes

ABSTRACT

Finite integral transforms, which are a specific subset of pseudo – spectral methods are used to reduce the spatial dimensionality of the coupled $qP-qS_V$ wave propagation problem in a transversely isotropic (TI) medium to that in one spatial dimension, usually depth, and time. The introduction of an absorbing boundaries, at least at the model bottom is useful in the removal of spurious arrivals. The top model boundary is usually wanted in the numerical calculations and reflections from the model sides may be removed by a judicious choice of model parameters, which does not significantly increase the run time.

In this paper, a method based on that presented in Clayton and Engquist (1977) and is employed for the coupled $P-S_V$ wave propagation problem in a transversely isotropic medium at the model bottom. The medium considered here is assumed to be radially symmetric and finite Hankel transforms are used to remove the radial coordinate (r). The problem that remains is a coupled problem in depth (z) and time (t). The anisotropic parameters may arbitrarily vary with depth.

INTRODUCTION

This method is most often referred to as the pseudo-spectral method, but due to the extensive work done in this area by B.G. Mikhailenko and A.S. Alekseev it is sometimes referred to, in seismic applications, as the Alekseev-Mikhailenko Method (AMM), (Alekseev and Mikhailenko, 1980). It falls within the genetic class of pseudo-spectral methods, but is possibly more formal and rigorous in its development. However, much of their work is relatively physically inaccessible and a considerable number of the more significant contributions are in Russian. Other works of interest in this area are Gazdag (1973), Gazdag (1981) and Kosloff and Baysal (1982).

One numerical advantage of applying finite integral transforms is that the resultant FD problem is in one spatial variable and time and there are no cross derivative terms. These are differentials of the form $\partial/\partial x_i\left[c\left(x_1,x_2,x_3\right)\partial u_k/\partial x_j\right] \quad i,j,k=1,2,3:\quad i\neq j$. Several suggested approaches for dealing with these in a finite difference context may be found in Zahradník et al. (1993).

Apart from a number of other numerical considerations, the removal of spurious reflections from the pseudo model bottom is required. This is done here using the method described in Clayton and Engquist (1977). There and in other related papers, the following statement, or something similar appears:

*"Paraxial approximations for the elastic wave equation [*and elastic TI wave equation *a*s well as more complicated wave equations*] analogous to those of the scalar wave equation can also be found. We cannot, however, perform the analysis by considering expansions of the dispersion relation because the differential equations for vector fields are not uniquely specified from their dispersion relations. Instead, we use the scalar case to provide a hint as to the general form of the paraxial approximation and fit the coefficients by matching to the full elastic wave equation."* [Clayton and Engquist (1977)]

What is being said here is that if paraxial approximations are derived for the two coupled equations of particle motion they are a partial solution of the absorbing boundary problem. The full solution requires the integration of these scalar equations into a scalar equivalent of the two coupled equations of motion. It has been determined that using just the first part of the solution produces better than expected results.

THEORETICAL DEVELOPMENT

General Theory

Consider the problem of coupled $P-S_V$ wave propagation in a radially symmetric (no lateral inhomogeneities), vertically inhomogeneous transversely isotropic half space.

The equations of motion are defined by the elastodynamic equations (Martynov and Mikhailenko, 1984 Mikhailenko and Korneev, 1984, or Mikhailenko, 1985)

$$\rho\frac{\partial^2 U}{\partial t^2}=c_{11}\left[\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}-\frac{U}{r^2}\right]+c_{13}\frac{\partial^2 V}{\partial r\partial z}+\frac{\partial}{\partial z}\left[c_{55}\left(\frac{\partial U}{\partial z}+\frac{\partial V}{\partial r}\right)\right]+\rho F_r \qquad (1)$$

$$\rho\frac{\partial^2 V}{\partial t^2}=c_{55}\left[\frac{\partial}{\partial r}\left(\frac{\partial U}{\partial z}+\frac{\partial V}{\partial r}\right)+\frac{1}{r}\left(\frac{\partial U}{\partial z}+\frac{\partial V}{\partial r}\right)\right]+\frac{\partial}{\partial z}\left[c_{13}\left(\frac{\partial U}{\partial r}+\frac{U}{r}\right)+c_{33}\frac{\partial V}{\partial z}\right]+\rho F_z \qquad (2)$$

where the particle displacement vector $\mathbf{u}$ is of the form

$$\mathbf{u}\equiv\mathbf{u}\left(r,z,t\right)=\left(U\left(r,z,t\right),V\left(r,z,t\right)\right). \qquad (3)$$

Here $U\left(r,z,t\right)$ and $V\left(r,z,t\right)$ are the radial (horizontal) and vertical components of vector particle displacement, the azimuthal component of displacement being zero for the coupled $P-S_V$ problem. The coordinates r and z are the radial and vertical coordinates in a cylindrical coordinate system, respectively, t is time In Voigt notation, the c_{ij} are the stiffness parameters of the medium and ρ is the density, all of which may be dependent on the vertical $\left(z\right)$ coordinate. The density normalized anisotropic parameters, $a_{ij}=c_{ij}/\rho$, having dimensions of velocity squared, may also be used at some points within this report.

The problem is solved subject to the initial conditions

$$\mathbf{u}\big|_{t=0} = \frac{\partial \mathbf{u}}{\partial t}\bigg|_{t=0} = 0 \tag{4}$$

and the free surface boundary conditions that are required to be satisfied are

$$\sigma_{zz}\big|_{z=0} = 0, \qquad \text{and} \qquad \sigma_{rz}\big|_{z=0} = 0. \tag{5}$$

That is, the normal stress and shear stress are zero at the free surface. These will be defined shortly for a transversely isotropic medium.

Two typical types of point sources, $\mathbf{F}(r,z,t)$, used in seismic applications are (Mikhailenko, 1980):

1. Vertical point force :

$$\mathbf{F}(x,y,z,t) = \delta(x-x_s)\delta(y-y_s)\delta(z-z_s)f(t)\mathbf{n_z}. \tag{6}$$

where $\mathbf{n_z}$ is a unit vector in the *z* (vertical downwards) direction.

2. Explosive point source of *P* waves:

$$\mathbf{F}(x,y,z,t) = \nabla\left[\delta(x-x_s)\delta(y-y_s)\delta(z-z_s)\right]f(t). \tag{7}$$

In the above, $\delta(\xi)$ is the Dirac delta function and $f(t)$ is some band limited source wavelet, about which more will be said later. In what follows, an explosive point source of *P* waves is assumed. The Green's function solution for this problem would require that, $f(t) = \delta(t-t_0)$ such that $(0 \le t_0 < t_{\max})$ for some finite time $t_{\max}$.

In terms of $U(r,z,t)$, $V(r,z,t)$ and the anisotropic stiffness coefficients, c_{ij}, the expressions for the normal and shear stresses at the free surface are given by

$$\sigma_{zz}\big|_{z=0} = \left[c_{13}\left(\frac{\partial U}{\partial r} - \frac{V}{r}\right) + c_{55}\frac{\partial V}{\partial z}\right] = 0 \tag{8}$$

$$\sigma_{rz}\big|_{z=0} = c_{55}\left(\frac{\partial U}{\partial z} - \frac{\partial V}{\partial r}\right) = 0 \tag{9}$$

Introducing the finite Hankel integral transforms and the vector designation $\mathbf{G}(\tilde{k}_i,k_i,z,t) = \left(S(\tilde{k}_i,z,t), R(k_i,z,t)\right)$ has

$$S\left(\tilde{k}_i, z, t\right) = \int_0^a U\left(r, z, t\right) J_1\left(k_i r\right) r\, dr \tag{10.a}$$

$$R\left(k_i, z, t\right) = \int_0^a V\left(r, z, t\right) J_0\left(\tilde{k}_i r\right) r\, dr \tag{10.b}$$

where the k_i and $\tilde{k}_i$ are the roots of the transcendental equations

$$J_0\left(\tilde{k}_i r\right) = 0 \tag{11}$$

and

$$J_1\left(k_i r\right) = 0, \tag{12}$$

respectively. Using the two formulations of the Hankel transforms discussed in Appendix A, it may be shown that both of the inverse series summations may be accomplished using only the roots of one of the Bessel function transcendental equation, $J_1\left(k_i r\right) = 0$, so that the inverse transforms are defined by

$$U\left(r, z, t\right) = \frac{2}{a^2} \sum_{i=1}^{\infty} \frac{S\left(k_i, z, t\right) J_1\left(k_i r\right)}{\left[J_0\left(k_i a\right)\right]^2} \tag{13}$$

$$V\left(r, z, t\right) = \frac{2}{a^2} \sum_{i=1}^{\infty} \frac{R\left(k_i, z, t\right) J_0\left(k_i r\right)}{\left[J_0\left(k_i a\right)\right]^2} \tag{14}$$

Thus both inverse series summations may be taken over the roots of one rather than two transcendental equations and as a consequence, $\mathbf{G}\left(k_i, z, t\right) = \left(S\left(k_i, z, t\right), R\left(k_i, z, t\right)\right)$. It is shown there that an earlier assumption that the source wavelet be band limited is significant in this determination. As the only spatial direction in which a finite difference is used is the *z* direction the most economical manner to introduce a damping conditions at the lower *z* boundary, i.e., $\gamma\left(z\right)\partial R/\partial t$ and $\gamma\left(z\right)\partial S/\partial t$. A safe estimate for the length of this damping region is of the order of 1 wavelength *(WL)* but $2WL$ are commonly used (B.G. Mikhailenko, 1980).

Applying the appropriate Hankel transforms to equations (1) and (2) results in

$$\rho \frac{\partial^2 S}{\partial t^2} = \frac{\partial}{\partial z}\left(c_{55}\frac{\partial S}{\partial z} - k_i c_{55} R\right) - k_i c_{13}\frac{\partial R}{\partial z} - k_i^2 c_{11} S + \rho \hat{F}_r \tag{15}$$

$$\rho \frac{\partial^2 R}{\partial t^2} = \frac{\partial}{\partial z}\left(c_{33}\frac{\partial R}{\partial z} + k_i c_{13} S\right) + k_i c_{55}\frac{\partial S}{\partial z} - k_i^2 c_{55} R + \rho \hat{F}_r \tag{16}$$

while the transforms of the shear and normal stresses at the free surface, which is assumed to be planar, have the form

$$\left[c_{55}\left(\frac{\partial S}{\partial z}-k_i R\right)\right]\Bigg|_{z=0}=0 \tag{17}$$

$$\left(c_{33}\frac{\partial R}{\partial z}+k_i c_{13} S\right)\Bigg|_{z=0}=0\,. \tag{18}$$

The transformed initial conditions at $t=0$ are

$$S\left(z,k_i,t\right)\big|_{t=0}=0;\quad R\left(z,k_i,t\right)\big|_{t=0}=0;\quad \frac{\partial S}{\partial t}\bigg|_{t=0}=\frac{\partial S}{\partial t}\bigg|_{t=0}=0. \tag{19}$$

where k_i are the roots of the transcendental equation $J_1\left(k_i a\right)=0$ which requires additional boundary conditions at $r=a$ (pseudo boundary such that)

$$U\big|_{r=a}=\frac{\partial V}{\partial r}\bigg|_{r=a}=0 \tag{20}$$

The pseudo boundary is placed at some distance $r=a$ so that no spurious reflections from this boundary are present in the synthetic traces. Care is required in choosing this distance, as the number of terms in the inverse series summation depends on it in a linear fashion.

If it is assumed that the anisotropic parameters (stiffness coefficients) are spatially independent the Hankel transformed equations take on the simplified forms given below. For convenience, it is assumed that the first two grid points in $z\left(z_0 \text{ and } z_1\right)$, at the free surface are of this form so that equations (15) and (16) may be written there as

$$\frac{\partial^2 S}{\partial t^2}=a_{55}\frac{\partial^2 S}{\partial z^2}-k_i\left(a_{13}+a_{55}\right)\frac{\partial R}{\partial z}-k_i^2 a_{11} S \tag{21}$$

$$\frac{\partial^2 R}{\partial t^2}=a_{33}\frac{\partial^2 R}{\partial z^2}+k_i\left(a_{13}+a_{55}\right)\frac{\partial S}{\partial z}-k_i^2 a_{55} R \tag{22}$$

and the Hankel transformed shear and normal stresses required at the free surface as boundary conditions have been given in equations (17) and (18).

An explicit finite difference scheme can be introduced into the transformed equations in depth and time $\left(z \text{ and } t\right)$. Equal grid spacing of h in the z direction and δ in time so that an arbitrary depth and time point are specified by $z_k=nh$ and $t_m=m\delta$. The order of accuracy of the finite difference process is 2nd order, $O\left(h^2,\delta^2\right)$.

ABSORBING BOUNDARY AT MODEL BOTTOM

If it is assumed that the anisotropic parameters (stiffness coefficients) are spatially independent the Hankel transformed equations take on the simplified forms given previously in equations in equations (21) and (22). With the equivalence of pseudodifferential operators assumed,

$$\frac{\partial}{\partial t} \leftrightarrow (-i\omega) \qquad \frac{\partial}{\partial z} \leftrightarrow (ik_z) \tag{24}$$

Equations (23) and (24) have the form

$$(-i\omega)^2 S = a_{55} (ik_z)^2 S - k_i (a_{13} + a_{55})(ik_z) R - k_i^2 a_{11} S \tag{25}$$

$$(-i\omega)^2 R = a_{33} (ik_z)^2 R + k_i (a_{13} + a_{55})(ik_z) S - k_i^2 a_{55} R \tag{26}$$

with $\tilde{\mathbf{U}} = (S,T)^T$, the transformed equations of motion. Combining results in

$$(-i\omega)^2 \tilde{\mathbf{U}} = \mathbf{D}_1 (ik_z)^2 \tilde{\mathbf{U}} + \mathbf{H} k_i (ik_z) \tilde{\mathbf{U}} - \mathbf{D}_2 k_i^2 \tilde{\mathbf{U}} \tag{27}$$

where the coefficient matrices are defined by

$$\mathbf{D}_1 = \begin{bmatrix} a_{55} & 0 \\ 0 & a_{33} \end{bmatrix} \qquad \mathbf{D}_2 = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{55} \end{bmatrix} \qquad \mathbf{H} = \begin{bmatrix} 0 & -(a_{13} + a_{55}) \\ (a_{13} + a_{55}) & 0 \end{bmatrix}. \tag{28}$$

Rearranging produces

$$\begin{gathered} \left[I - \mathbf{D}_1 \frac{(ik_z)^2}{(-i\omega)^2} - \mathbf{H} \frac{k_i (ik_z)}{(-i\omega)^2} + \mathbf{D}_2 \frac{k_i^2}{(-i\omega)^2} \right] \tilde{\mathbf{U}} = 0 \\ \text{or} \qquad . \\ \left[I - \mathbf{D}_1 \frac{(ik_z)^2}{(-i\omega)^2} - \mathbf{H} \frac{k_i (ik_z)}{(-i\omega)^2} + \mathbf{D}_2 \frac{k_i^2}{(-i\omega)^2} \right] = 0 \quad \text{assuming } \tilde{\mathbf{U}} \neq 0 \end{gathered} \tag{29}$$

Following the method used in the paper of Clayton and Engquist (1977), two paraxial equations of different orders are assumed. The first is of the form

$$\left[(ik_z) \mathbf{I} + \mathbf{B}_1 (-i\omega) \right] \tilde{\mathbf{U}} \qquad \left[\frac{\partial \tilde{\mathbf{U}}}{\partial z} + \mathbf{B}_1 \frac{\partial \tilde{\mathbf{U}}}{\partial t} 0 \right] \tag{30}$$

As it has been assumed that $\tilde{\mathbf{U}} \neq 0$

$$\frac{(ik_z)}{(-i\omega)} \mathbf{I} = -\mathbf{B}_1 \tag{31}$$

In this paraxial approximation substituted into (29) and ignoring terms of order greater than zero in powers of $k_i/(-i\omega)$ has

$$\mathbf{I}-\mathbf{D}_1\left(\mathbf{B}_1\right)^2=0 \quad \rightarrow \quad \mathbf{B}_1=\left(\mathbf{D}_1\right)^{-1/2} \tag{32}$$

from which it follows that

$$\left(\mathbf{D}_1\right)^{-1}=\begin{bmatrix} 1/a_{55} & 0 \\ 0 & 1/a_{33} \end{bmatrix} \quad \text{so that} \quad \mathbf{B}_1=\left(\mathbf{D}_1\right)^{-1/2}=\begin{bmatrix} 1/\sqrt{a_{55}} & 0 \\ 0 & 1/\sqrt{a_{33}} \end{bmatrix} \tag{33}$$

The second paraxial approximation is given by

$$\frac{\partial^2\mathbf{U}}{\partial t\partial z}+\mathbf{C}_1\frac{\partial^2\mathbf{U}}{\partial t^2}+\mathbf{C}_2 k_i\frac{\partial\mathbf{U}}{\partial t}+\mathbf{C}_3 k_i^2\mathbf{U}=0 \tag{34}$$

After replacing the partial derivatives with pseudodifferential operators the following results

$$\begin{gathered}\left(ik_z\right)\left(-i\omega\right)\tilde{\mathbf{U}}+\mathbf{C}_1\left(-i\omega\right)^2\tilde{\mathbf{U}}+\mathbf{C}_2 k_i\left(-i\omega\right)\tilde{\mathbf{U}}+\mathbf{C}_3 k_i^2\tilde{\mathbf{U}}=0 \\ \text{or} \\ \left[\frac{\left(ik_z\right)}{\left(-i\omega\right)}\mathbf{I}+\mathbf{C}_1+\mathbf{C}_2\frac{k_i}{\left(-i\omega\right)}+\mathbf{C}_3\frac{k_i^2}{\left(-i\omega\right)^2}\right]=0 \quad \text{assuming } \tilde{\mathbf{U}}\neq 0\end{gathered} \tag{35}$$

Without any derivation, it may be seen that $\mathbf{C}_1=\mathbf{B}_1$. Substitute (35) into (29) to obtain.

$$\begin{gathered}\mathbf{I}-\mathbf{D}_1\left(\mathbf{C}_1+\mathbf{C}_2\frac{k_i}{\left(-i\omega\right)}+\mathbf{C}_3\frac{k_i^2}{\left(-i\omega\right)^2}\right)\left(\mathbf{C}_1+\mathbf{C}_2\frac{k_i}{\left(-i\omega\right)}+\mathbf{C}_3\frac{k_i^2}{\left(-i\omega\right)^2}\right)+ \\ \mathbf{H}\frac{k_i}{\left(-i\omega\right)}\left(\mathbf{C}_1+\mathbf{C}_2\frac{k_i}{\left(-i\omega\right)}+\mathbf{C}_3\frac{k_i^2}{\left(-i\omega\right)^2}\right)+\mathbf{D}_2\frac{k_i^2}{\left(-i\omega\right)^2}=0\end{gathered} \tag{36}$$

Retaining only terms in $k_i/\left(-i\omega\right)$ up to order one produces

$$-\mathbf{D}_1\mathbf{C}_1\mathbf{C}_2 - \mathbf{D}_1\mathbf{C}_2\mathbf{C}_1 + \mathbf{H}\mathbf{C}_1 = 0 \tag{37}$$

$$\mathbf{C}_2 = \begin{bmatrix} 0 & c_2^{(12)} \\ c_2^{(21)} & 0 \end{bmatrix} \tag{38}$$

Retaining only terms in $\left(ik_x/-i\omega\right)$ results after a number of matrix multiplications, results in

$$\begin{gathered}\left(\sqrt{a_{55}} + a_{55}/\sqrt{a_{33}}\right)c_2^{(12)} + \left(a_{13} + a_{55}\right)/\sqrt{a_{33}} = 0 \\ \left(\sqrt{a_{33}a_{55}} + a_{55}\right)c_2^{(12)} = -\left(a_{13} + a_{55}\right) \\ c_2^{(12)} = -\frac{\left(a_{13} + a_{55}\right)}{\sqrt{a_{55}}\left(\sqrt{a_{33}} + \sqrt{a_{55}}\right)}\end{gathered} \tag{39}$$

The second term in $\mathbf{C}_2$ follows as

$$\begin{gathered}\sqrt{a_{33}}c_2^{(21)} + a_{33}c_2^{(21)}/\sqrt{a_{55}} - \left(a_{13} + a_{55}\right)/\sqrt{a_{55}} = 0 \\ \left(\frac{\sqrt{a_{33}}\sqrt{a_{55}} + a_{33}}{\sqrt{a_{55}}}\right)c_2^{(21)} - \left(a_{13} + a_{55}\right)/\sqrt{a_{55}} = 0 \\ c_2^{(21)} = \frac{\left(a_{13} + a_{55}\right)}{\left(\sqrt{a_{33}}\sqrt{a_{55}} + a_{33}\right)} = \frac{\left(a_{13} + a_{55}\right)}{\sqrt{a_{33}}\left(\sqrt{a_{55}} + \sqrt{a_{33}}\right)}\end{gathered} \tag{40}$$

The following quantity is required later in the derivation of the terms in C_3.

$$c_2^{(21)}c_2^{(12)} = -\frac{\left(a_{13} + a_{55}\right)}{\sqrt{a_{33}}\left(\sqrt{a_{55}} + \sqrt{a_{33}}\right)}\frac{\left(a_{13} + a_{55}\right)}{\sqrt{a_{55}}\left(\sqrt{a_{33}} + \sqrt{a_{55}}\right)} = -\frac{\left(a_{13} + a_{55}\right)^2}{\sqrt{a_{33}a_{55}}\left(\sqrt{a_{55}} + \sqrt{a_{33}}\right)^2} \tag{41}$$

In the isotropic limit, $c_3^{(22)}$

$$c_2^{(21)} = \frac{\left(a_{13} + a_{55}\right)}{\sqrt{a_{33}}\left(\sqrt{a_{55}} + \sqrt{a_{33}}\right)} = \frac{\left(\alpha^2 - \beta^2\right)}{\beta\left(\alpha + \beta\right)} = \frac{\left(\alpha - \beta\right)}{\beta} \tag{42}$$

$$c_2^{(12)} = -\frac{\left(a_{13} + a_{55}\right)}{\sqrt{a_{55}}\left(\sqrt{a_{33}} + \sqrt{a_{55}}\right)} = -\frac{\left(\alpha^2 - \beta^2\right)}{\beta\left(\alpha + \beta\right)} = -\frac{\left(\alpha - \beta\right)}{\beta} \tag{43}$$

Finally it is required determine $\mathbf{C}_3$. As with $\mathbf{C}_2$ the starting equation is (35). Expand this equation and retaining only those terms in powers of $(ik_x)^2/(-i\omega)^2$ and equating this quantity to zero has

$$-\mathbf{D}_1\mathbf{C}_3\mathbf{C}_1 - \mathbf{D}_1\mathbf{C}_2\mathbf{C}_2 - \mathbf{D}_1\mathbf{C}_1\mathbf{C}_3 + \mathbf{H}\mathbf{C}_2 + \mathbf{D}_2 = 0 \tag{44}$$

$$-\sqrt{a_{33}}c_3^{(22)} - c_2^{(12)}c_2^{(21)}a_{33} - \sqrt{a_{33}}c_3^{(22)} - \frac{\left(a_{13}+a_{55}\right)^2}{\sqrt{a_{55}}\left(\sqrt{a_{33}}+\sqrt{a_{55}}\right)} + a_{55} = 0 \tag{45}$$

$$-\left(\sqrt{a_{33}}+\sqrt{a_{33}}\right)c_3^{(22)} + \frac{a_{33}\left(a_{13}+a_{55}\right)^2}{\sqrt{a_{33}a_{55}}\left(\sqrt{a_{55}}+\sqrt{a_{33}}\right)^2} - \frac{\left(a_{13}+a_{55}\right)^2}{\sqrt{a_{55}}\left(\sqrt{a_{33}}+\sqrt{a_{55}}\right)} + a_{55} = 0 \tag{46}$$

$$-2\sqrt{a_{33}}c_3^{(22)} - \frac{\left(a_{13}+a_{55}\right)^2}{\sqrt{a_{55}}\left(\sqrt{a_{33}}+\sqrt{a_{55}}\right)}\left(1 - \frac{a_{33}}{\sqrt{a_{33}}\left(\sqrt{a_{55}}+\sqrt{a_{33}}\right)}\right) + a_{55} = 0 \tag{47}$$

$$-2\sqrt{a_{33}}c_3^{(22)} - \frac{\left(a_{13}+a_{55}\right)^2}{\left(\sqrt{a_{33}}+\sqrt{a_{55}}\right)^2} + a_{55} = 0 \tag{48}$$

$$c_3^{(22)} = \frac{a_{55}}{2\sqrt{a_{33}}} - \frac{\left(a_{13}+a_{55}\right)^2}{2\sqrt{a_{33}}\left(\sqrt{a_{33}}+\sqrt{a_{55}}\right)^2} \tag{49}$$

$$c_3^{(22)} = \frac{1}{2\sqrt{a_{33}}}\left(a_{55} - \frac{\left(a_{13}+a_{55}\right)^2}{\left(\sqrt{a_{33}}+\sqrt{a_{55}}\right)^2}\right) \tag{50}$$

In the isotropic limit, $c_3^{(22)}$

$$\begin{aligned} c_3^{(22)} &= \frac{1}{2\alpha}\left(\beta^2 - \frac{\left(\alpha^2-\beta^2\right)^2}{\left(\alpha+\beta\right)^2}\right) = \frac{1}{2\alpha}\left(\beta^2 - \left(\alpha-\beta\right)^2\right) \\ &= \frac{1}{2\alpha}\left(\beta^2 - \alpha^2 + 2\alpha\beta - \beta^2\right) = -\frac{\left(\alpha-2\beta\right)}{2} \end{aligned} \tag{51}$$

The term $c_3^{(11)}$ is obtained in a manner similar to that used for $c_3^{(22)}$

$$a_{55}c_2^{(12)}c_2^{(21)} + 2\sqrt{a_{55}}c_3^{(11)} + \frac{\left(a_{13}+a_{55}\right)^2}{\sqrt{a_{33}}\left(\sqrt{a_{55}}+\sqrt{a_{33}}\right)} - a_{11} = 0 \tag{52}$$

so that

$$2\sqrt{a_{55}}c_3^{(11)} = a_{11} - a_{55}c_2^{(12)}c_2^{(21)} - \frac{\left(a_{13}+a_{55}\right)^2}{\sqrt{a_{33}}\left(\sqrt{a_{55}}+\sqrt{a_{33}}\right)}. \tag{53}$$

With the relation

$$\begin{aligned} c_2^{(21)}c_2^{(12)} &= -\frac{\left(a_{13}+a_{55}\right)}{\sqrt{a_{33}}\left(\sqrt{a_{55}}+\sqrt{a_{33}}\right)}\frac{\left(a_{13}+a_{55}\right)}{\sqrt{a_{55}}\left(\sqrt{a_{33}}+\sqrt{a_{55}}\right)} \\ &= -\frac{\left(a_{13}+a_{55}\right)^2}{\sqrt{a_{33}a_{55}}\left(\sqrt{a_{55}}+\sqrt{a_{33}}\right)^2} \end{aligned} \tag{54}$$

and using the following sequence of steps

$$2\sqrt{a_{55}}c_3^{(11)} = a_{11} + a_{55}\frac{\left(a_{13}+a_{55}\right)^2}{\sqrt{a_{33}a_{55}}\left(\sqrt{a_{55}}+\sqrt{a_{33}}\right)^2} - \frac{\left(a_{13}+a_{55}\right)^2}{\sqrt{a_{33}}\left(\sqrt{a_{55}}+\sqrt{a_{33}}\right)} \tag{55}$$

$$2\sqrt{a_{55}}c_3^{(11)} = a_{11} - \frac{\left(a_{13}+a_{55}\right)^2}{\sqrt{a_{33}}\left(\sqrt{a_{55}}+\sqrt{a_{33}}\right)}\left(1 - \frac{\sqrt{a_{55}}}{\left(\sqrt{a_{55}}+\sqrt{a_{33}}\right)}\right) \tag{56}$$

$$2\sqrt{a_{55}}c_3^{(11)} = a_{11} - \frac{\left(a_{13}+a_{55}\right)^2}{\sqrt{a_{33}}\left(\sqrt{a_{55}}+\sqrt{a_{33}}\right)}\left(\frac{\sqrt{a_{55}}+\sqrt{a_{33}}-\sqrt{a_{55}}}{\left(\sqrt{a_{55}}+\sqrt{a_{33}}\right)}\right) \tag{57}$$

the term $c_3^{(11)}$ has the final form

$$c_3^{(11)} = \frac{1}{2\sqrt{a_{55}}}\left(a_{11} - \frac{\left(a_{13}+a_{55}\right)^2}{\left(\sqrt{a_{55}}+\sqrt{a_{33}}\right)^2}\right) \tag{58}$$

As previously, in the isotropic limit, $c_3^{(11)}$ becomes

$$c_3^{(11)} = \frac{1}{2\beta}\left(\alpha^2 - \frac{\left(\alpha+\beta\right)^2\left(\alpha-\beta\right)^2}{\left(\alpha+\beta\right)^2}\right) = \frac{1}{2\beta}\left(\alpha^2 - \left(\alpha-\beta\right)^2\right) \tag{59}$$

$$\begin{aligned} c_3^{(11)} &= \frac{1}{2\beta}\left(\alpha^2 - \alpha^2 + 2\alpha\beta - \beta^2\right) \\ &= -\frac{1}{2\beta}\left(\beta^2 - 2\alpha\beta\right) = -\frac{1}{2}\left(\beta - 2\alpha\right) \end{aligned} \tag{60}$$

NUMERICAL RESULTS

A simple layered model is used here with the vertical velocities given in Fig. 1. The vertical component of the VSP particle displacement is shown in Fig. 2 and Fig. 3. The reason for this choice is that the signal at depth is of the same magnitude as are the spurious reflected arrivals from the model bottom. It is clear that the formulae derived remove unwanted model edge reflections.

SUMMARY AND CONCLUSIONS

The theory and development of finite difference analogues for $qP-qS_V$ wave propagation in a plane parallel layered transversely isotropic model has been presented. The radial coordinate was removed using a finite Hankel transform prior to implementation of finite difference process. What results are a coupled system of finite difference equations in only depth and time. Both components of particle displacement are recovered by applying inverse Hankel transform summations, which although infinite, may be truncated if a band limited source wavelet is used. The synthetic traces produced using this method have 3D spreading and the amount of computer resources is reduced considerably as the vertical and horizontal components of particle displacement as well as all required elastic parameters need only to be specified at a sequence of depth points – one spatial dimension.

The finite difference analogues given are accurate to second order in both time and space (depth). The analogues for a surface point as well as general points within the medium are given. Provisions for either a vertical or explosive point source of P – waves are included in the derivations. A number of points regarding this seismic modeling process, especially where some mathematical rigor is required are dealt with in the Appendix.

Using the formulae presented here it should be possible write a hybrid finite difference – finite integral transform programs for a transversely isotropic medium for a variety of source – receiver configurations including AVO and VSP. The Appendix in Clayton and Engquist (1977) is quite useful in accomplishing this.

APPENDIX A: FINITE HANKEL TRANSFORM

Although the two following finite Hankel transform methods may be found in the literature (Sneddon, 1972, for example), it was felt that for completeness they should be included here, at least in an abbreviated theorem formulation. The finite Hankel transform of the first kind is a direct application of the following theorem.

Theorem I: If $f(x)$ satisfies Dirichlet's conditions in the interval $(0,a)$ and if its Hankel transform in that range is defined to be

$$H_\mu^{(1)}\left[f(x)\right] \equiv f_{\mathrm{J}}\left(\xi_j\right) = \int_0^a x\, f(x)\, J_\mu\left(\xi_j x\right) dx \tag{A.1}$$

where ξ_j is a root of the transcendental equation

$$J_{\mu}\left(\xi_j a\right)=0 \tag{A.2}$$

then, at any point in the interval $\left(0,a\right)$ at which the function $f\left(x\right)$ is continuous ,

$$f\left(x\right)=\frac{2}{a^2}\sum_{j=1}^{\infty} f_{\mathrm{J}}\left(\xi_j\right)\frac{J_{\mu}\left(\xi_j x\right)}{\left[J_{\mu+1}\left(\xi_j x\right)\right]^2} \tag{A.3}$$

where the sum is taken over all the positive roots of equation (A.2).

The finite Hankel transform and inverse of the second kind used in the text are given as follows:

Theorem II: If $f\left(x\right)$ satisfies Dirichlet's conditions in the interval $\left(0,a\right)$ and if its Hankel transform in that range is defined to be

$$H_{\mu}^{(1)}\left[f\left(x\right)\right]\equiv f_{\mathrm{J}}\left(\xi_j\right)=\int_0^a x\,f\left(x\right)J_{\mu}\left(\xi_j x\right)dx \tag{A.4}$$

in which ξ_j is a root of the transcendental equation

$$\xi_j J'_{\mu}\left(\xi_j a\right)+h\,J_{\mu}\left(\xi_j a\right)=0 \tag{A.5}$$

then, at each point in the interval $\left(0,a\right)$ at which the function $f\left(x\right)$ is continuous ,

$$f\left(x\right)=\frac{2}{a^2}\sum_{j=1}^{\infty}\frac{\xi_j^2 f_{\mathrm{J}}\left(\xi_j\right)}{h^2+\left(\xi_j^2-\mu^2/a^2\right)}\frac{J_{\mu}\left(\xi_j x\right)}{\left[J_{\mu}\left(\xi_j x\right)\right]^2} \tag{A.6}$$

where the sum is taken over all the positive roots of (A.5) and h is determined from a boundary operator $\mathbf{N}$ at $x=a$ defined as

$$\mathbf{N}\left[f\right]=\frac{df\left(a\right)}{dx}+h\,f\left(a\right)=0\,. \tag{A.7}$$

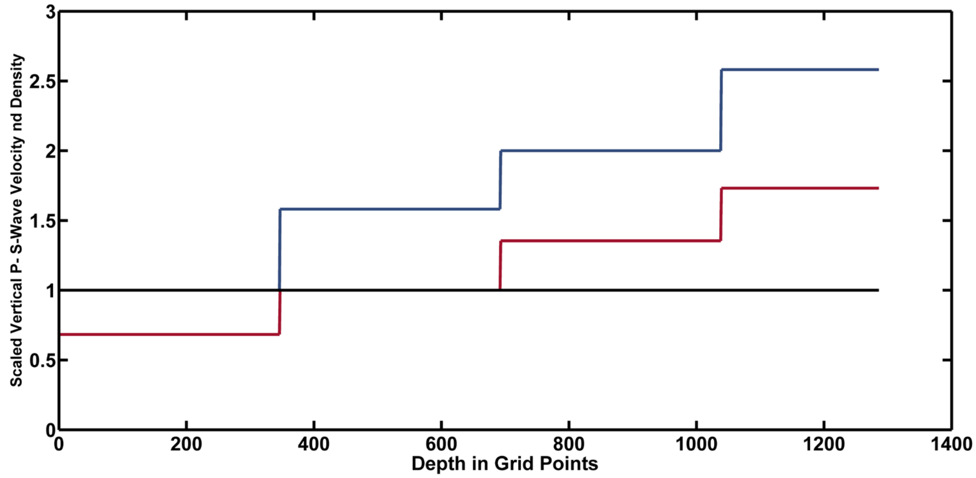

Fig. 1. Scaled velocity/density depth model used in the computations in this report.

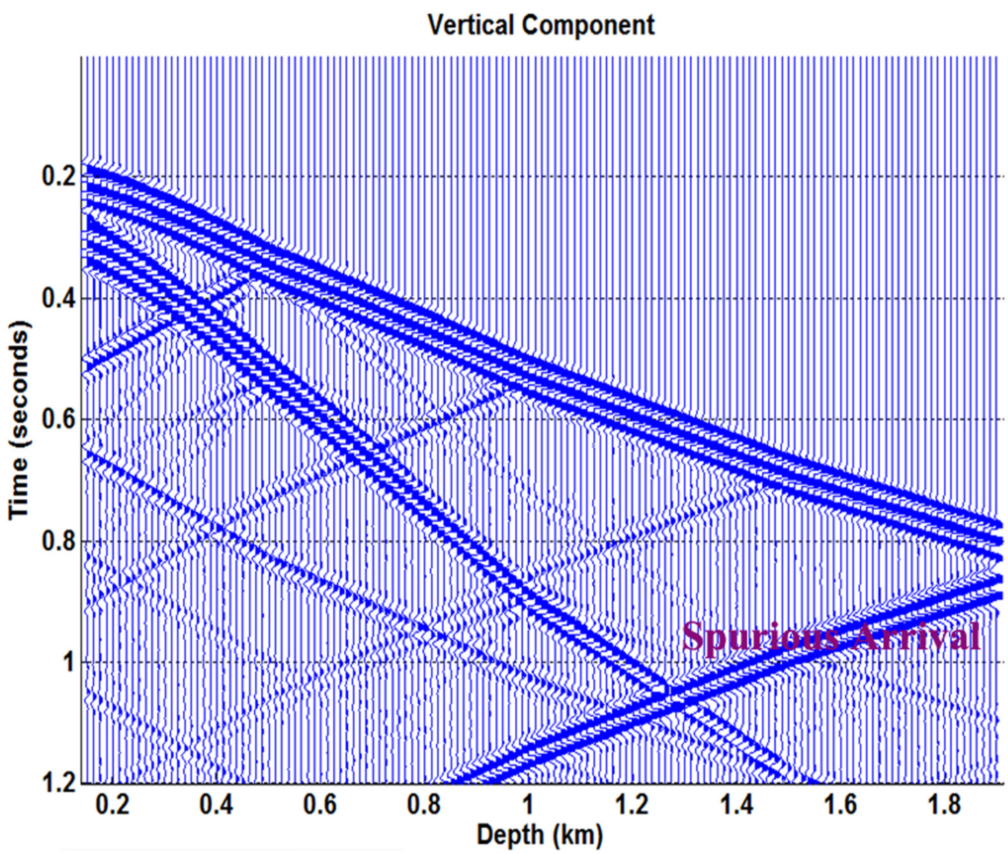

Fig. 2: The vertical component of the VSP synthetic of the model described in the text. In this panel the computations continued to the proper time indicating that *spurious reflections* from the model bottom are included in the traces.

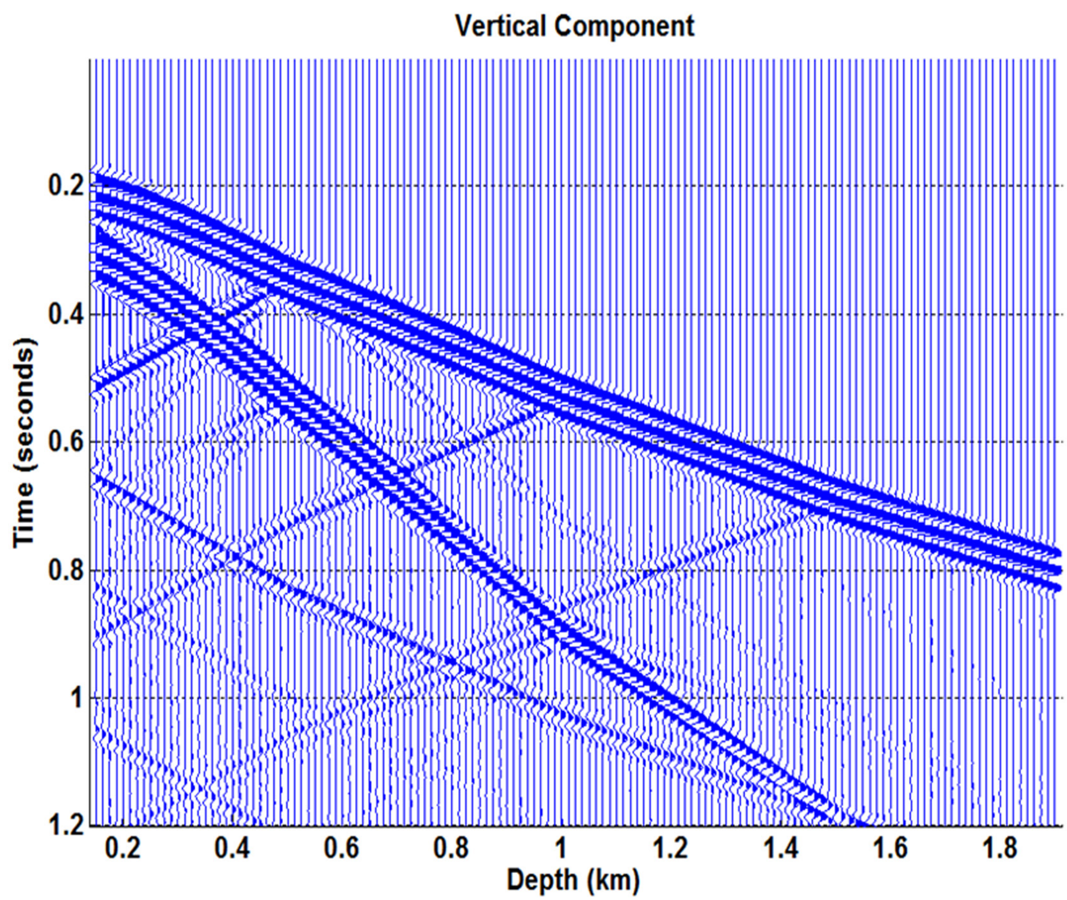

Fig. 3: The vertical component of the VSP synthetic of the model described in the text. In this panel the computations continued to the proper time indicating that *no spurious reflections* from the model are included in the traces.

REFERENCES

Alekseev, A.S. and Mikhailenko, B.G., 1980, Solution of dynamic problems of elastic wave propagation in inhomogeneous media by a combination of partial separation of variables and finite difference methods, Journal of Geophysics, 48, 161-172.

Ames, W.F., 1969, Numerical Methods for Partial Differential Equations: Barnes & Noble, New York.

Cerjan, C., Kosloff, D., Kosloff, R. and Reshef, M., 1985, A nonreflecting boundary condition for discrete acoustic and elastic wave equations, Geophysics, 50, 705-708.

Clayton, R. and Enquist, B., 1977, Absorbing boundary conditions for acoustic and elastic wave equations, Bulletin of the Seismological Society of America, 67, 1529-1540.

Gazdag, J., 1973, Numerical convective schemes based on the accurate computation of space derivatives, Journal of Computational Physics, 13, 100-113.

Gazdag, J., 1981, Modeling of the acoustic wave equation with transform methods, Geophysics, 46, 854-859.

Kosloff, D. and Baysal, E., 1982, Forward modeling by a Fourier method, Geophysics, 47, 1402-1412.

Martynov, V.N. and Mikhailenko, B.G., 1984, Numerical modelling of propagation of elastic waves in anisotropic inhomogeneous media for the half-space and the sphere, Geophysical Journal of the Royal Astronomical Society, 76, 53-63.

Mitchell, A.R., 1969, Computation Methods in Partial Differential Equations: John Wiley & Sons, Inc.

Mikhailenko, B.G., 1980, Personal communication.

Mikhailenko, B.G. and Korneev, V.I., 1984,Calculation of synthetic seismograms for complex subsurface geometries by a combination of finite integral Fourier transforms and finite difference techniques, Journal of Geophysics, 54, 195-206.

Mikhailenko, B.G., 1985, Numerical experiment in seismic investigations, Journal of Geophysics, 58, 101-124.

Reynolds, A.C., 1978, Boundary conditions for the numerical solution of wave propagation problems, Geophysics, 43, 1099-1110.

Richtmeyer, R.D. and Morton, K.W., 1967, Difference Methods for Initial-value Problems, Krieger Publishing Company.

Smarskii, A. A., 1971, Introduction to the Theory of Difference Schemes, Nauka, Moscow (in Russian).

Sneddon, I.A, 1995, Fourier Transforms, Dover Publications, New York.

Zahradnik, J., P. Moczo, and F. Hron, 1993, Testing four elastic finite difference schemes for behavior at discontinuities, Bulletin of the Seismological Society of America, 83, 107-129.