Nodal Galerkin Methods for Linear Elasticity

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December 1, 2010

- Consider the following simplified elastic wave equation.

$$\begin{cases} \ddot{u}(\mathbf{x},t) = \nabla \cdot (c^2(\mathbf{x})\nabla u(\mathbf{x},t)) \\ u(\mathbf{x},t=0) = u_0(\mathbf{x}) \\ \dot{u}(\mathbf{x},t=0) = u_1(\mathbf{x}) \end{cases}, \ \mathbf{x} \in \Omega, \ t \geq 0.$$

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- Multiplying by $v(\mathbf{x})$, integrating and applying Green's theorem we obtain

$$\int_\Omega \ddot{u} v \, d\Omega + \int_\Omega c^2 \nabla u \cdot \nabla v \, d\Omega = \oint_\Gamma c^2 \nabla u \cdot \mathbf{n} v \, d\Gamma$$

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- The term on the right hand side is what allows us to "talk" to the boundary Γ.

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- The new problem reads

$$\int_\Omega \ddot{u} v \, d\Omega + \int_\Omega c^2 \nabla u \cdot \nabla v \, d\Omega = \oint_\Gamma c\dot{u} v \, d\Gamma$$

- Choose a set of functions $\{\phi(\mathbf{x})\}_{j=1}^{N}$ for u and v

$$u(\mathbf{x},t) = \sum_{i=1}^{N} \hat{u}_i(t)\phi_i(\mathbf{x})$$

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- The nodal Galerkin method chooses the functions $\phi_j(\mathbf{x})$ from those that act like discrete delta functions on a set of nodes.

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- Appropriate choices are Lagrange polynomials or sinc functions.

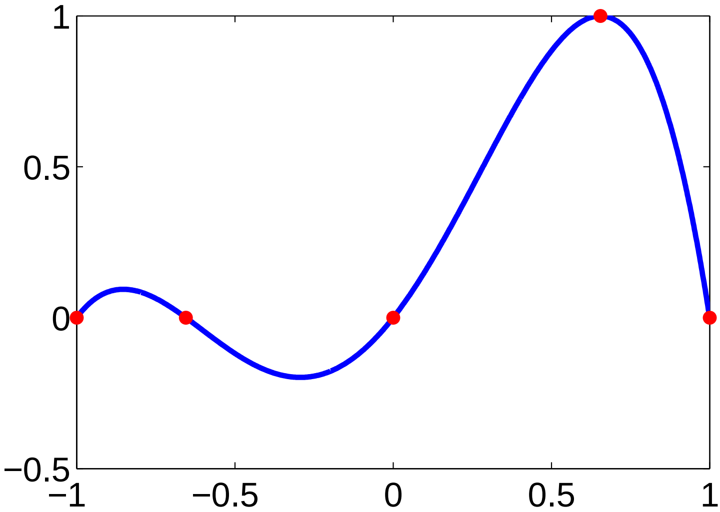


Figure: 1D Lagrange Polynomial

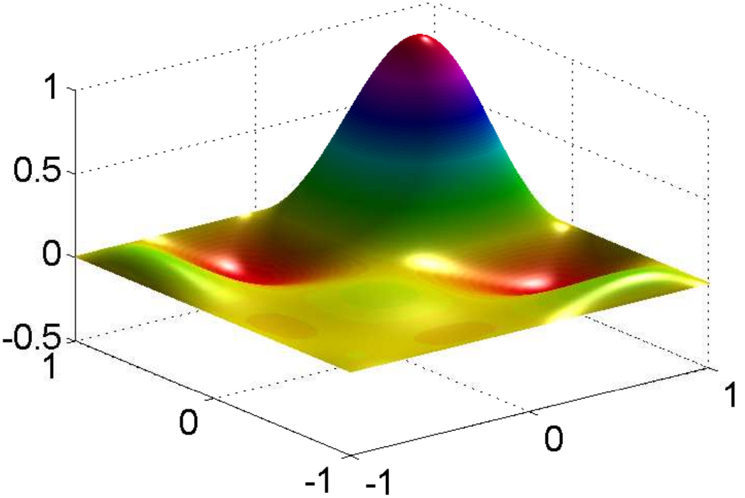


Figure: 2D Lagrange Polynomial

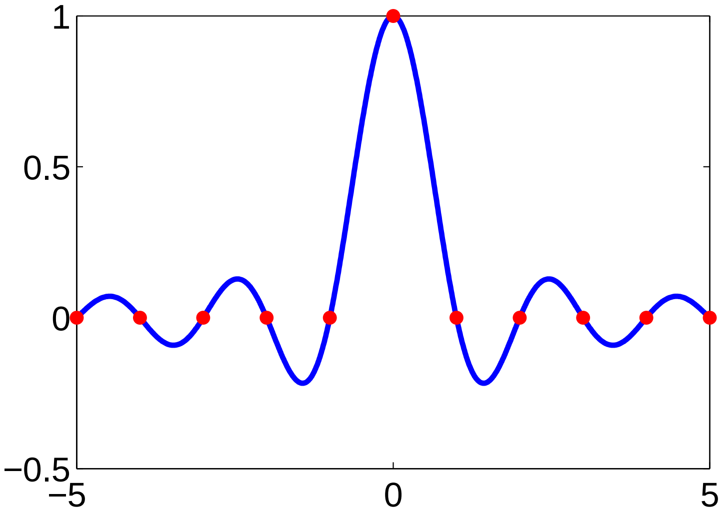
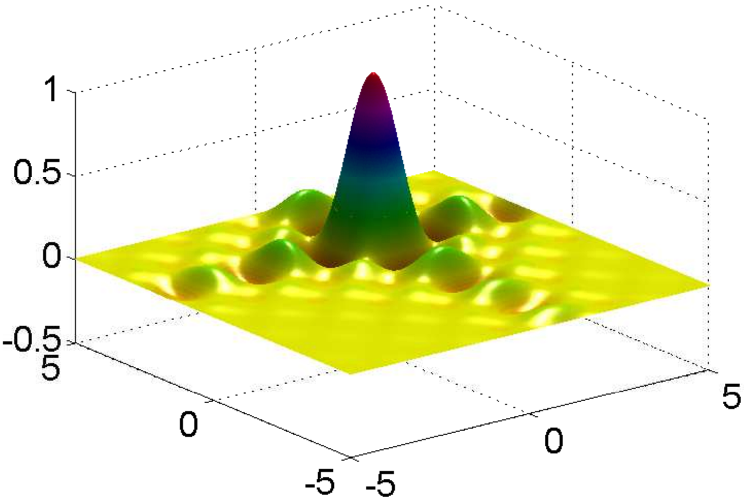


Figure: 1D Sinc Function



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- Associated with a set of nodes is a Differentiation matrix D and a set of *weights* $\{w_j\}_{j=1}^{N}$.
- The d-dimensional version of these nodes, weights, and matrices are defined using the Kronecker-tensor product.

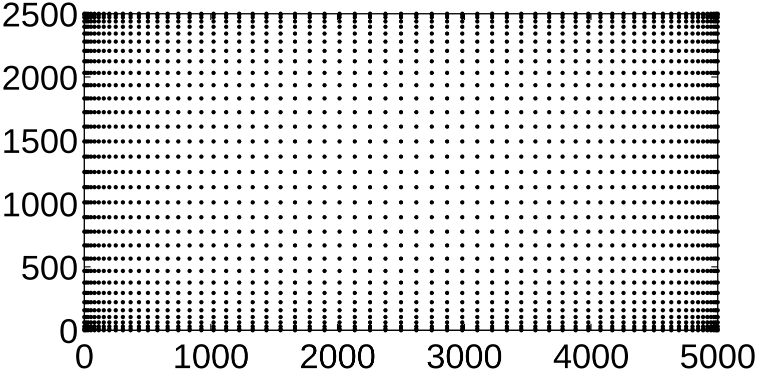

Figure: Legendre-Gauss-Lobatto Nodes mapped to [0,5000]x[0,2500]

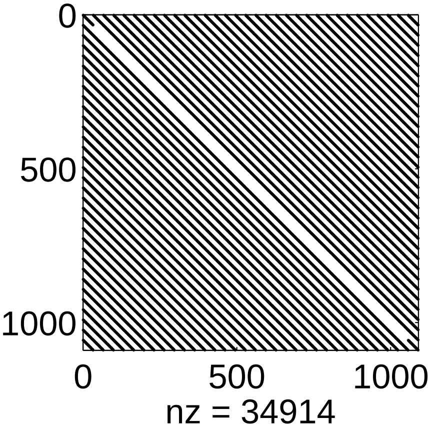

Figure: 2D LGL Differentiation matrix $D_x = D \otimes I_N$ ($\approx 2.94\%$ populated).

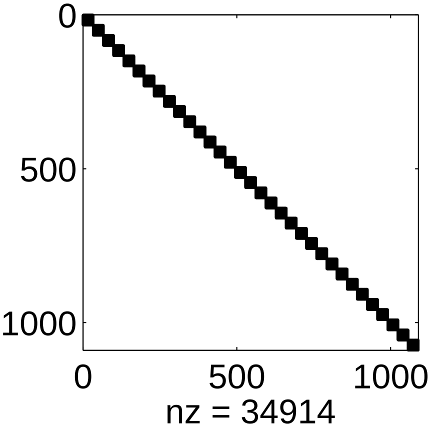

Figure: 2D LGL Differentiation matrix $D_y = I_N \otimes D$ ($\approx 2.94\%$ populated).

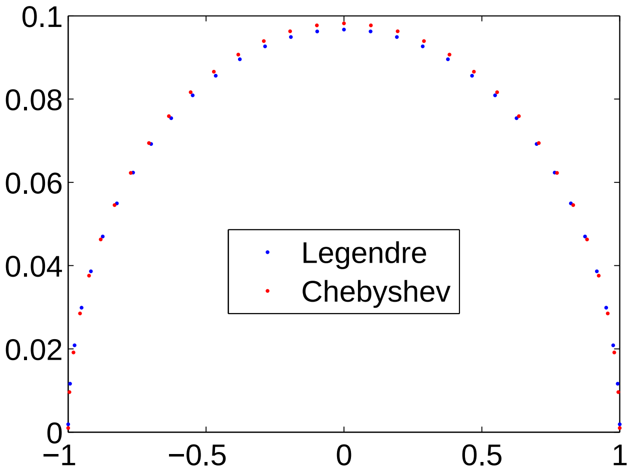

Figure: 1D Legendre and Chebyshev Gauss-Lobatto weights

- Replace integration and differentiation by their nodal counterparts in

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- System of ordinary differential equations for evolution in time

$$\begin{cases} M\ddot{\mathbf{U}}(t) + A\dot{\mathbf{U}}(t) + K\mathbf{U}(t) = 0 \\ \mathbf{U}(0) = \mathbf{U}_0 \\ \dot{\mathbf{U}}(0) = \mathbf{U}_1 \end{cases}$$

- Discretize using centered finite difference in time

$$\left[M + \frac{dt}{2}A\right]\mathbf{U}(t_{j+1}) + \left[dt^2K - 2M\right]\mathbf{U}(t_j)$$

$$+\left[M - \frac{dt}{2}A\right]\mathbf{U}(t_{j-1}) = 0$$

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- Or let $\mathbf{V} = \dot{\mathbf{U}}$ and write as first order system

$$\begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}\begin{bmatrix} \dot{\mathbf{U}} \\ \dot{\mathbf{V}} \end{bmatrix}(t) + \begin{bmatrix} 0 & I \\ K & A \end{bmatrix}\begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix}(t) = 0$$

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- Initial conditions

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- In square tripartite medium with speeds $c = 2, 3, 4$.

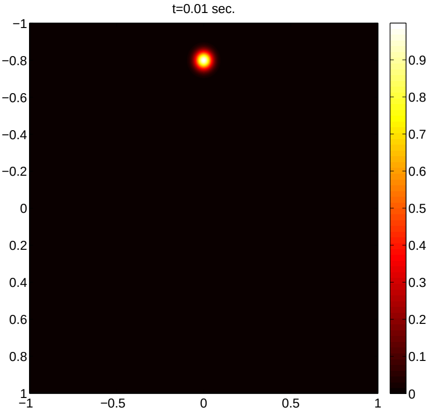

Figure: Numerical simulation of P-wave propagation with first-order ABC's.

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- To test this, consider a forcing term with Ricker wavelet time-component

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- In square bipartite medium with properties.

Region	ρ	V_p	V_s
1	2.064	2305	997
2	2.14	4500	2600

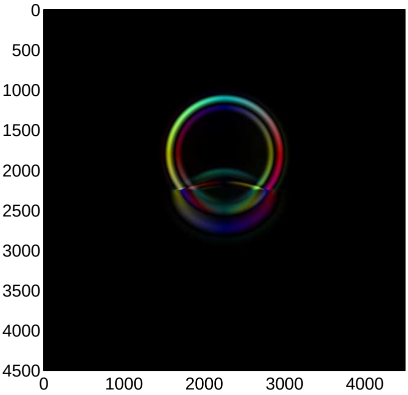

Figure: Numerical simulation of elastic wave propagation with periodic boundary conditions.

- Computation times for comparison with 2,4,6,8 order finite differences on 401 by 401 node grid.

Method	CFD2	CFD4	CFD6	CFD8	Sinc
Time(sec)	17	21	24	27	52

Model Properties

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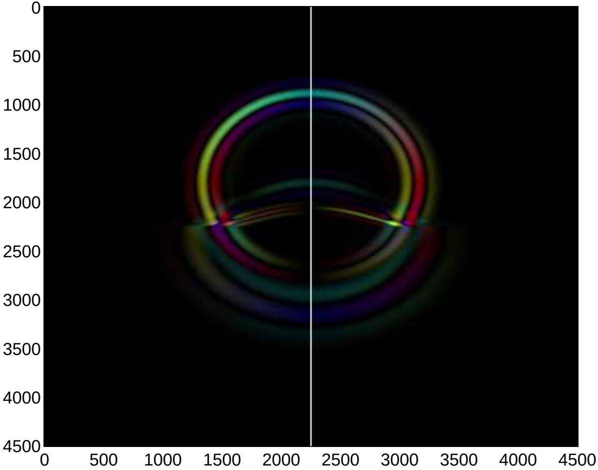


Figure: Elastic wave propogated to $t = 1$ sec. White line indicates receivers.

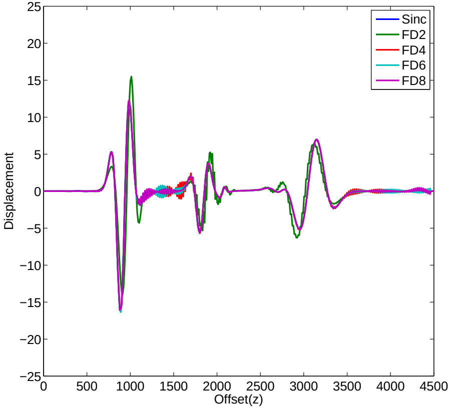
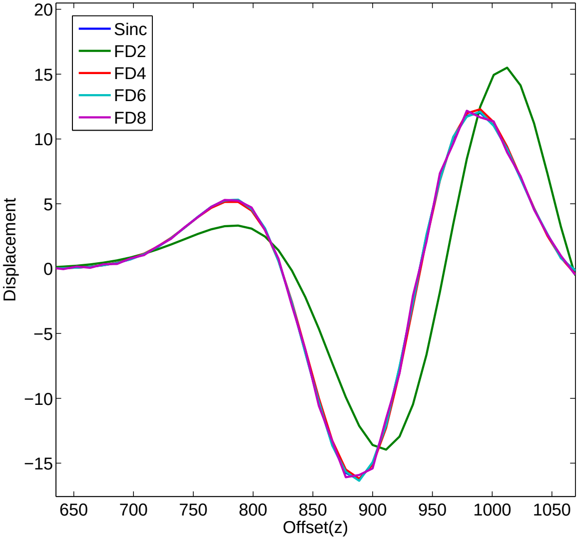
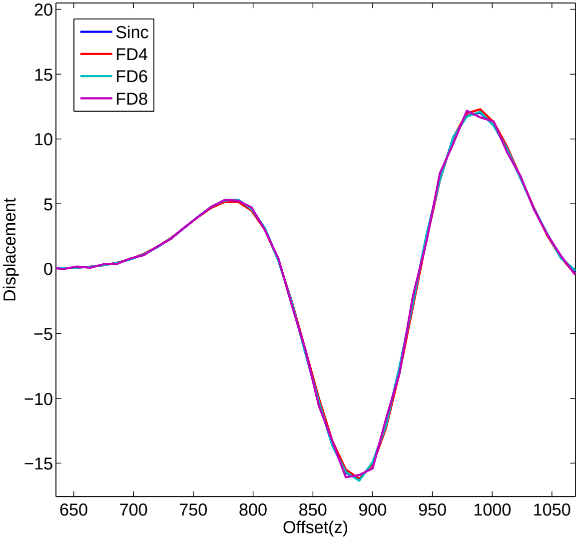


Figure: Centerline of the vertical component in presence of a jump in the velocity model.



Sinc
FD2
FD4
FD6
FD8
Displacement
Offset(z)
20
15
10
5
0
−5
−10
−15
650
700
750
800
850
900
950
1000
1050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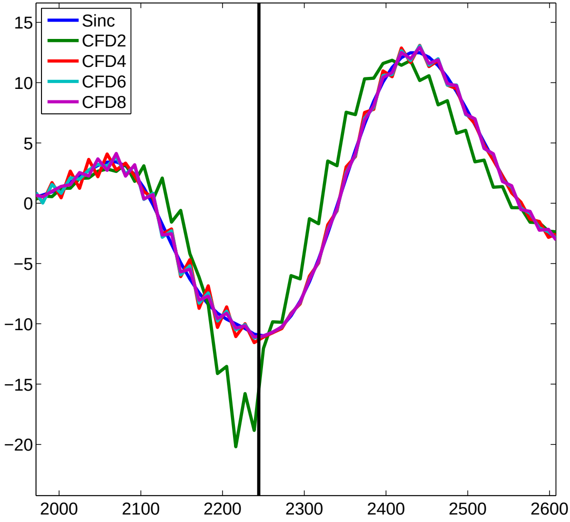
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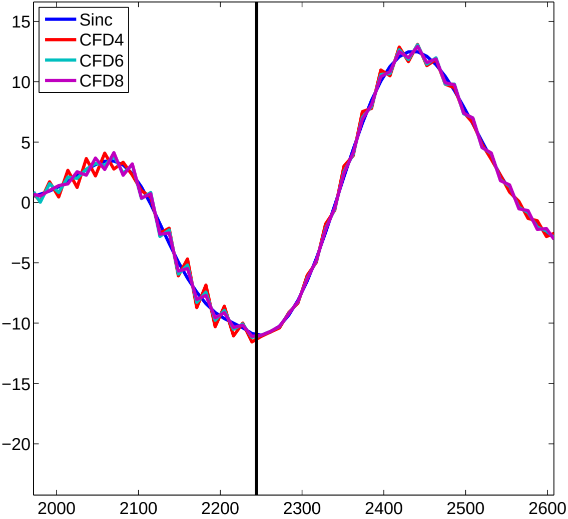
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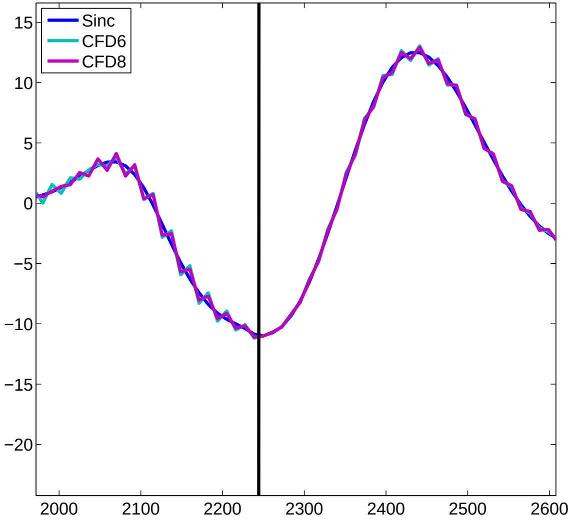
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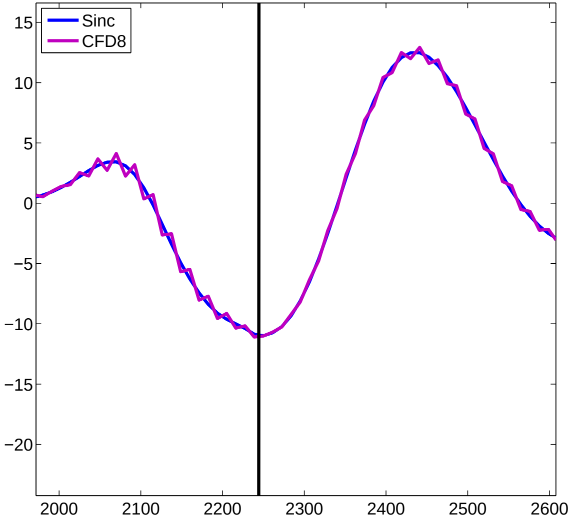
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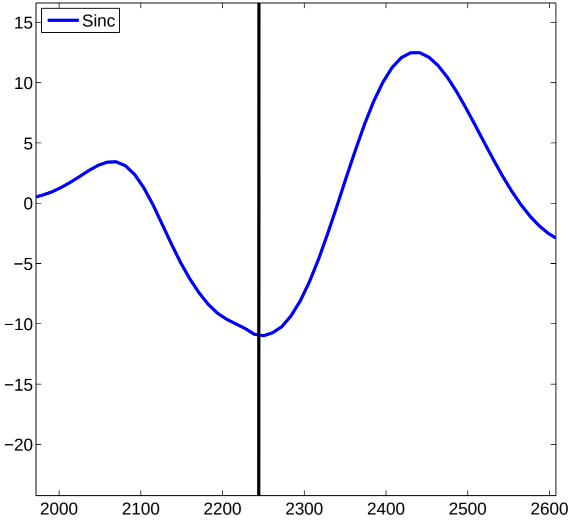
Sinc
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- # Thank you!
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  - Gary Margrave
  - Laura Baird
  - Kevin Hall
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  - U of C, POTSI, MITACS, NSERC