Efficient Pseudo Gauss-Newton FWI in the Time-Ray Parameter [$\tau - \mathbf{p}$] Domain

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Outline

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General Principle of FWI

$$\phi\left(s_0^{(n)}(\mathbf{r})\right) = \frac{1}{2}\int d\omega \left(\sum_{\mathbf{r}_s}\sum_{\mathbf{r}_g} \left\|\delta P\left(\mathbf{r}_g, \mathbf{r}_s, \omega | s_0^{(n)}(\mathbf{r})\right)\right\|_2\right)$$

Least-squares misfit function for full waveform inversion

General Principle of FWI

$$s^{(n)}(\mathbf{r}) = s_0^{(n)}(\mathbf{r}) + \mu^{(n)} \delta s_0^{(n)}(\mathbf{r})$$

The model can updated iteratively

$$\delta s_0^{(n)} = -\int d\mathbf{r}' H^{(n)-}(\mathbf{r}, \mathbf{r}') g^{(n)}(\mathbf{r})$$

The model perturbation can be constructed by gradient and inverse Hessian

Why FWI fails in industry practice?

- ❑ Extensively computational burden
- ❑ Slow convergence rate
- ❑ Cycle skipping problem

Strategies

- ❑ Source encoding method
- ❑ Hessian approximations
- ❑ Multiscale approach

Gradient

$$g^{(n)}(\mathbf{r}) = -\sum_{\mathbf{r}_s}\sum_{\mathbf{r}_g}\int d\omega \Re\left(\frac{\delta G\left(\mathbf{r}_g,\mathbf{r}_s,\omega|s_0^{(n)}\right)}{\delta s_0^{(n)}(\mathbf{r})}\delta P^*\left(\mathbf{r}_g,\mathbf{r}_s,\omega|s_0^{(n)}\right)\right)$$

Gradient

$$g^{(n)}(\mathbf{r}) = \sum_{\mathbf{r}_s}\sum_{\mathbf{r}_g}\int d\omega \Re\left(\omega^2 \mathcal{F}_s(\omega) G(\mathbf{r},\mathbf{r}_s,\omega) G(\mathbf{r}_g,\mathbf{r},\omega)\delta P^*\left(\mathbf{r}_g,\mathbf{r}_s,\omega|s_0^{(n)}\right)\right)$$

Gradient can be constructed using adjoint state method

Gradient

$$g^{(n)}(\mathbf{r}) = -\sum_{\mathbf{r}_s}\sum_{\mathbf{r}_g}\int d\omega \Re\left(\frac{\delta G\left(\mathbf{r}_g, \mathbf{r}_s, \omega | s_0^{(n)}\right)}{\delta s_0^{(n)}(\mathbf{r})}\delta P^*\left(\mathbf{r}_g, \mathbf{r}_s, \omega | s_0^{(n)}\right)\right)$$

Gradient

$$g^{(n)}(\mathbf{r}) = \sum_{\mathbf{r}_s}\sum_{\mathbf{r}_g}\int d\omega \Re\left(\omega^2 \mathcal{F}_s(\omega) G(\mathbf{r}, \mathbf{r}_s, \omega) G(\mathbf{r}_g, \mathbf{r}, \omega)\delta P^*\left(\mathbf{r}_g, \mathbf{r}_s, \omega | s_0^{(n)}\right)\right)$$

RTM image based on cross-correlation imaging condition

Phase Encoded Gradient

$$\tilde{\mathbf{d}}\left(\mathbf{r}_g, \mathbf{r}_s, p, \omega\right) = \int \mathbf{d}\left(\mathbf{r}_g, \mathbf{r}_s, \omega\right) e^{i\omega p(x_s - x_0)} d\mathbf{r}_s$$

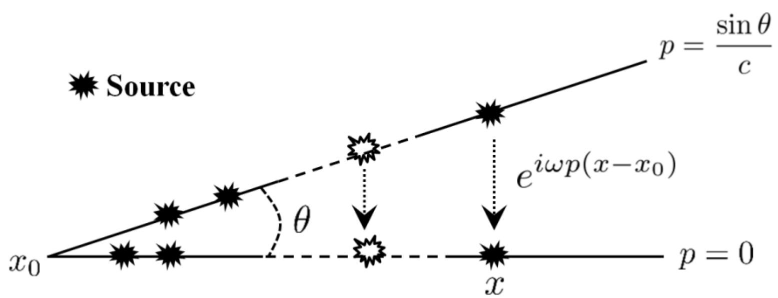
The construction of the encoded wavefields can be interpreted $\tau - p$ **as transform.**

Phase Encoded Gradient

$$\tilde{g}\left(\mathbf{r},\mathbf{p}^g,\omega\right)=\sum_{\omega}\sum_{\mathbf{r}_s}\sum_{\mathbf{r}_g}\sum_{i=1}^{N_p^g}\Re\left\{\omega^2\mathcal{F}(\omega)G\left(\mathbf{r},\mathbf{r}_s,\omega\right)G^*\left(\mathbf{r},\mathbf{r}_g,\omega\right)e^{i\omega p_i^g(x_s-x_g)}\right\}$$

Linear phase encoded gradient

where p_s is the ray-parameter, x_s and x_g are the sources and receivers coordinates.



Linear phase encoding strategy

- ❑ **Slant stacking over a set of ray parameters can disperse the crosstalk noise.**
- ❑ **Different ray parameters can balance the gradient and update.**

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The Role of Hessian in Least-Squares Inverse Problem

Hessian Matrix ?

- ❑ **Improve the convergence rate**

The Role of Hessian in Least-Squares Inverse Problem

Hessian Matrix = Nonstationary Deconvolution Operator

- ❑ **Improve the convergence rate**
- ❑ **Compensate the geometrical spreading effects and balance the amplitude**
- ❑ **Suppress the multiple scattering effects and improve the resolution**

Approximate Hessian

$$H^{(n)}(\mathbf{r}', \mathbf{r}) = H_1^{(n)} + H_2^{(n)}$$

Full Hessian

$$H_a^{(n)} = H_2^{(n)} \simeq \sum_{\mathbf{r}_s, \mathbf{r}_g} \int d\omega \Re \left\{ \omega^4 G\left(\mathbf{r}_g, \mathbf{r}', \omega | s_0^{(n)}\right) G\left(\mathbf{r}', \mathbf{r}_s, \omega | s_0^{(n)}\right) G^*\left(\mathbf{r}_g, \mathbf{r}'', \omega | s_0^{(n)}\right) G^*\left(\mathbf{r}'', \mathbf{r}_s, \omega | s_0^{(n)}\right) \right\}$$

Approximate Hessian in Gauss-Newton Method by Gary and Kris (2011)

$$H_a^{(n)} \simeq \mathbf{Diag}\left(H_a^{(n)}\right)$$

Diagonal part of the approximate Hessian

Pseudo-Hessian

$$f_{virtual} = -\omega^2 \mathcal{F}_s(\omega) G\left(\mathbf{r}, \mathbf{r}_s, \omega | s_0^{(n)}(\mathbf{r})\right)$$

Virtual Source

$$H_{pseudo}^{(n)} = f_{virtual} f_{virtual}^* = \sum_{\mathbf{r}_s} \int d\omega \Re\{\omega^4 |\mathcal{F}_s(\omega)|^2 G(\mathbf{r}', \mathbf{r}'_s, \omega) G^*(\mathbf{r}'', \mathbf{r}_s, \omega)\}$$

Pseudo-Hessian

$$I_{dec} = \frac{\sum_{\mathbf{r}_s, \mathbf{r}_g} \int d\omega \Re\left\{\omega^2 \mathcal{F}_s(\omega) G(\mathbf{r}, \mathbf{r}_s, \omega) G(\mathbf{r}_g, \mathbf{r}, \omega) \delta P^*\left(\mathbf{r}_g, \mathbf{r}_s, \omega\right)\right\}}{\sum_{\mathbf{r}_s} \int d\omega \omega^4 \Re\left\{|\mathcal{F}_s(\omega)|^2 G\left(\mathbf{r}', \mathbf{r}_s, \omega\right) G^*\left(\mathbf{r}', \mathbf{r}_s, \omega\right)\right\} + \lambda I}$$

Deconvolution imaging condition

Phase Encoded Hessian

$$H_{encoded} = \sum_{\mathbf{r}_s} \int d\omega \Re \left\{ \omega^4 G(\mathbf{r}', \mathbf{r}_s, \omega) G^*(\mathbf{r}'', \mathbf{r}_s, \omega) \right\}$$

$$\times \sum_{\mathbf{p}_g} \int d\omega \Re \left\{ G(\mathbf{r}'', \mathbf{r}'_g, \omega) e^{i\omega p_g (x'_g - x_{initial})} G^*(\mathbf{r}'', \mathbf{r}_g, \omega) e^{-i\omega p_g (x_g - x_{initial})} \right\}$$

Receiver-side linear phase encoded Hessian

$$H_{encoded} = H_{exact} + H_{crosstalk}$$

By Tang (2009)

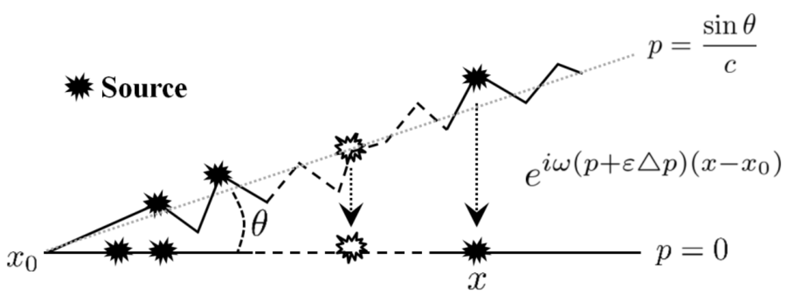
$$H_{encoded} = H_{exact}, \mathbf{p}_g \in (-\infty, +\infty)$$

By Tao and Sen (2013)

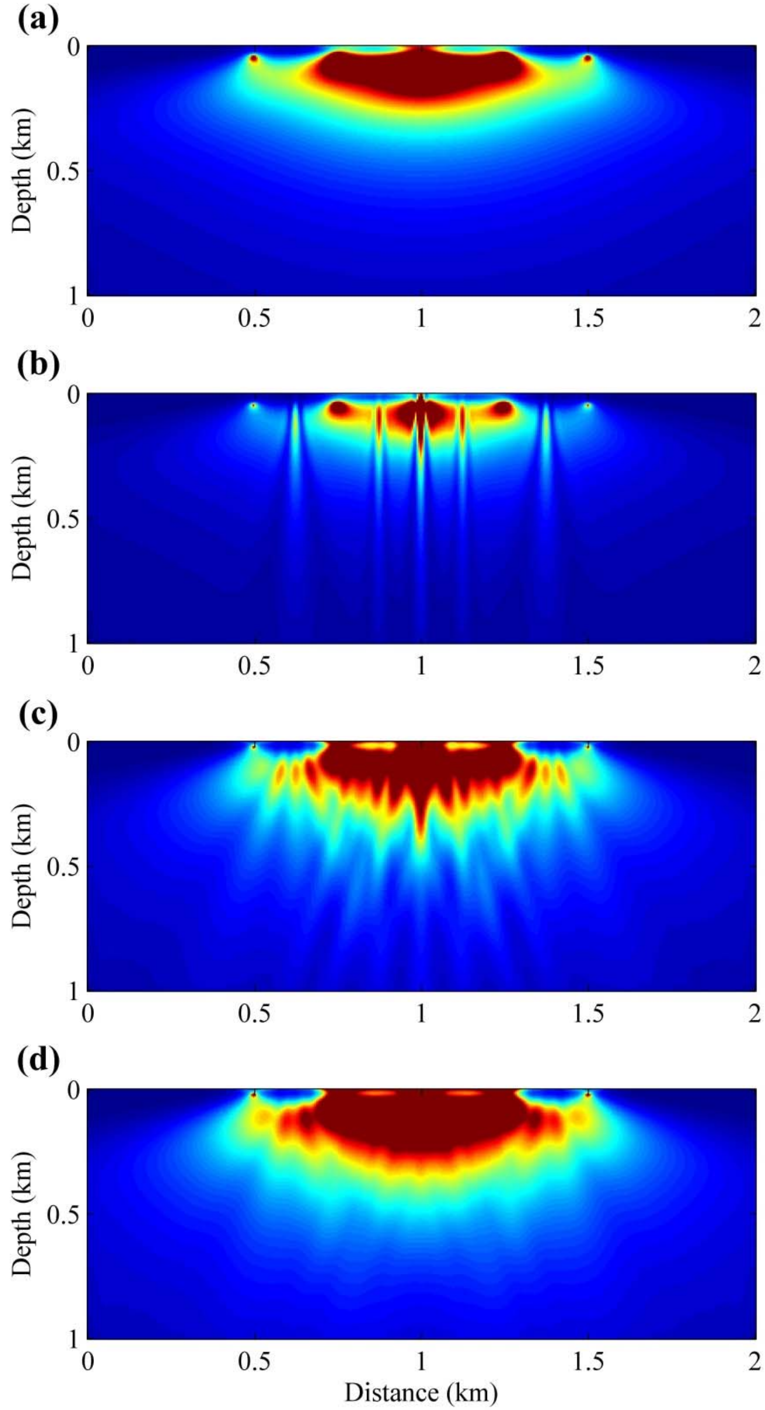
Phase Encoded Hessian

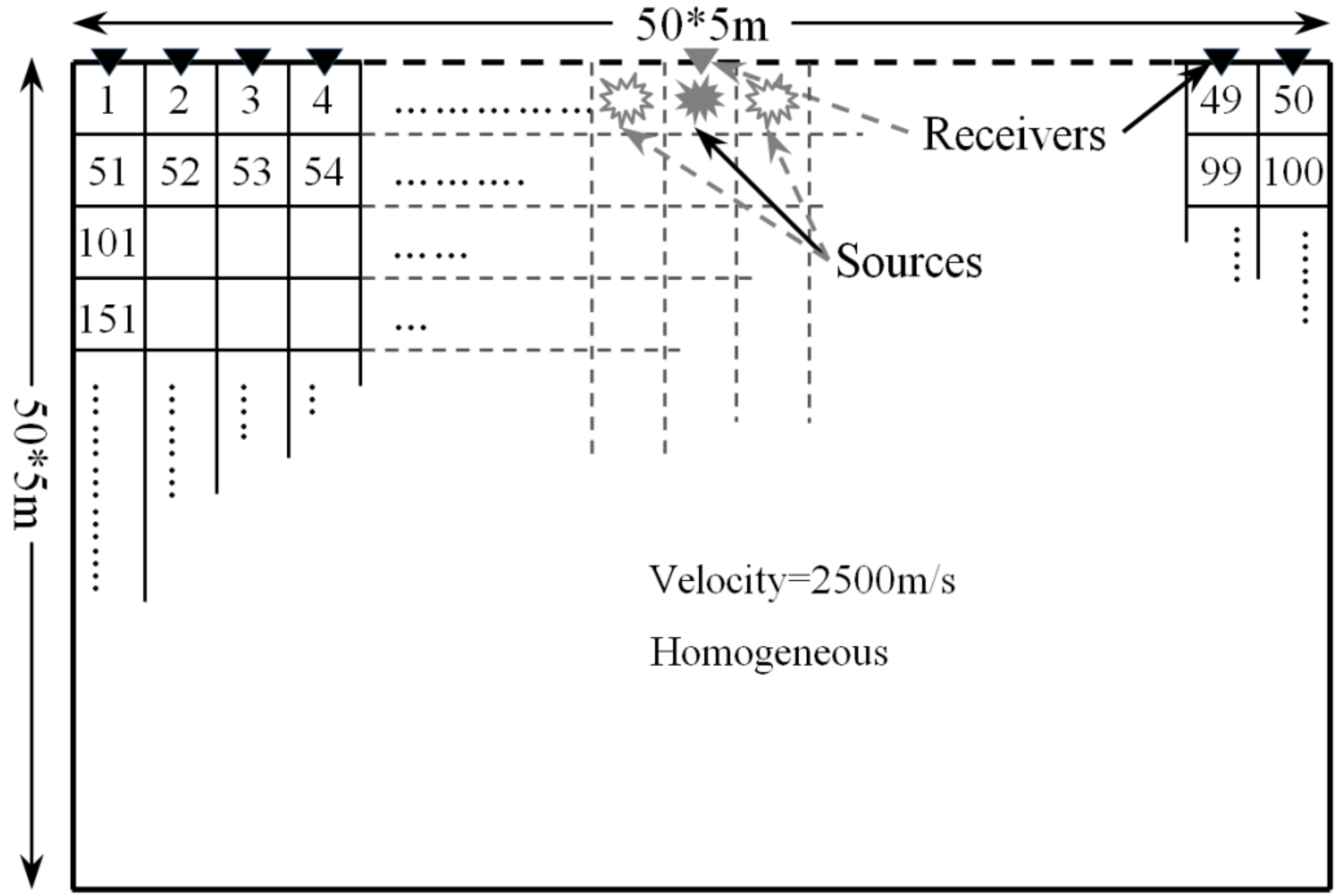
$$H_{chirp} = \sum_{\mathbf{r}_s} \int d\omega \Re \left\{ \omega^4 G(\mathbf{r}', \mathbf{r}_s, \omega) G^*(\mathbf{r}', \mathbf{r}_s, \omega) \right\}$$
$$\times \sum_{\mathbf{p}_g} \int d\omega \Re \left\{ G(\mathbf{r}'', \mathbf{r}'_g, \omega) G^*(\mathbf{r}'', \mathbf{r}_g, \omega) e^{i\omega(p_g + \varepsilon \triangle p)(x'_g - x_g)} \right\}$$

Chirp phase encoded Hessian



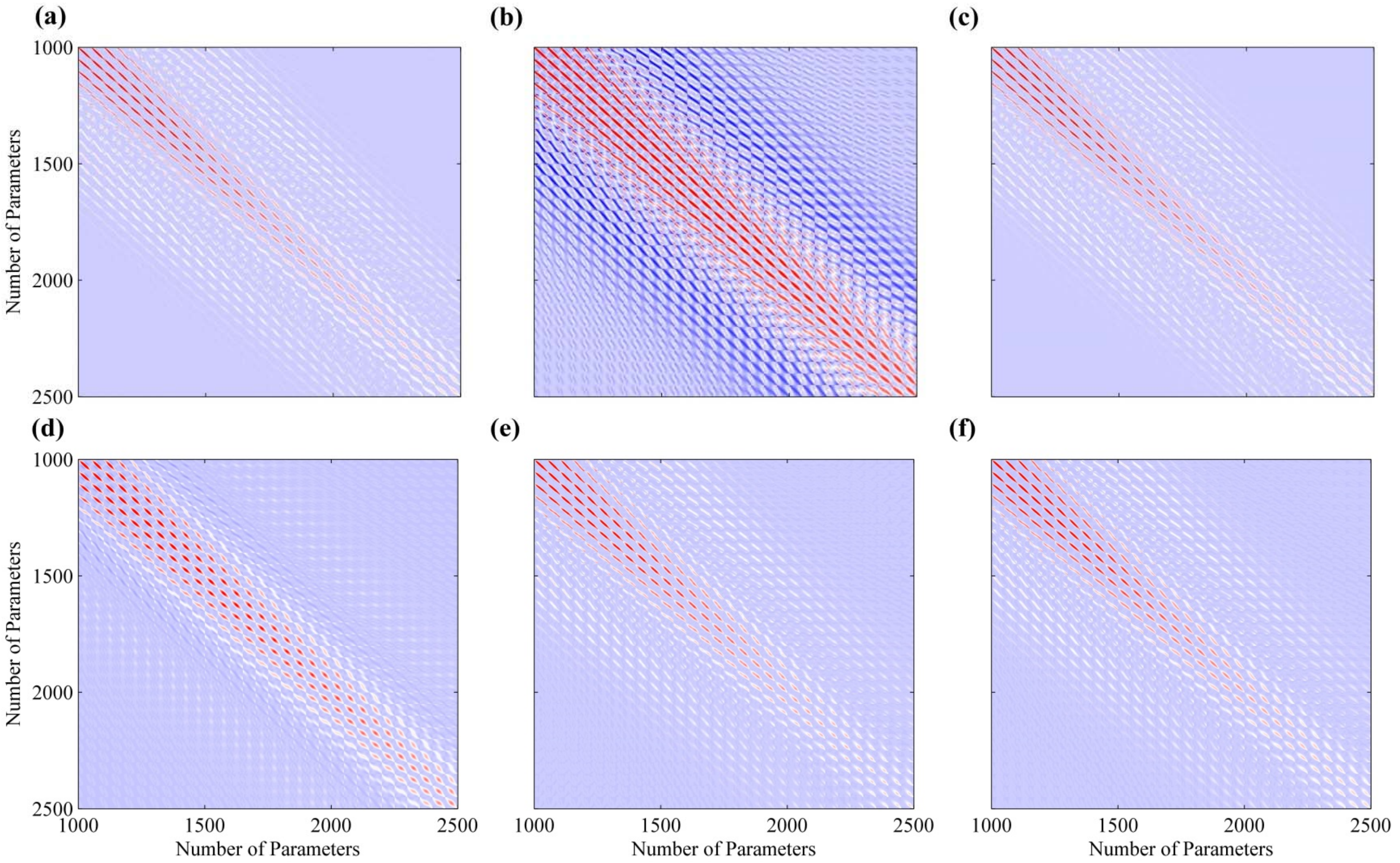
Chirp phase encoding strategy


(a)
Depth (km)
0
0.5
1
0
0.5
1
1.5
2
(b)
Depth (km)
0
0.5
1
0
0.5
1
1.5
2
(c)
Depth (km)
0
0.5
1
0
0.5
1
1.5
2
(d)
Depth (km)
0
0.5
1
0
0.5
1
1.5
2
Distance (km)

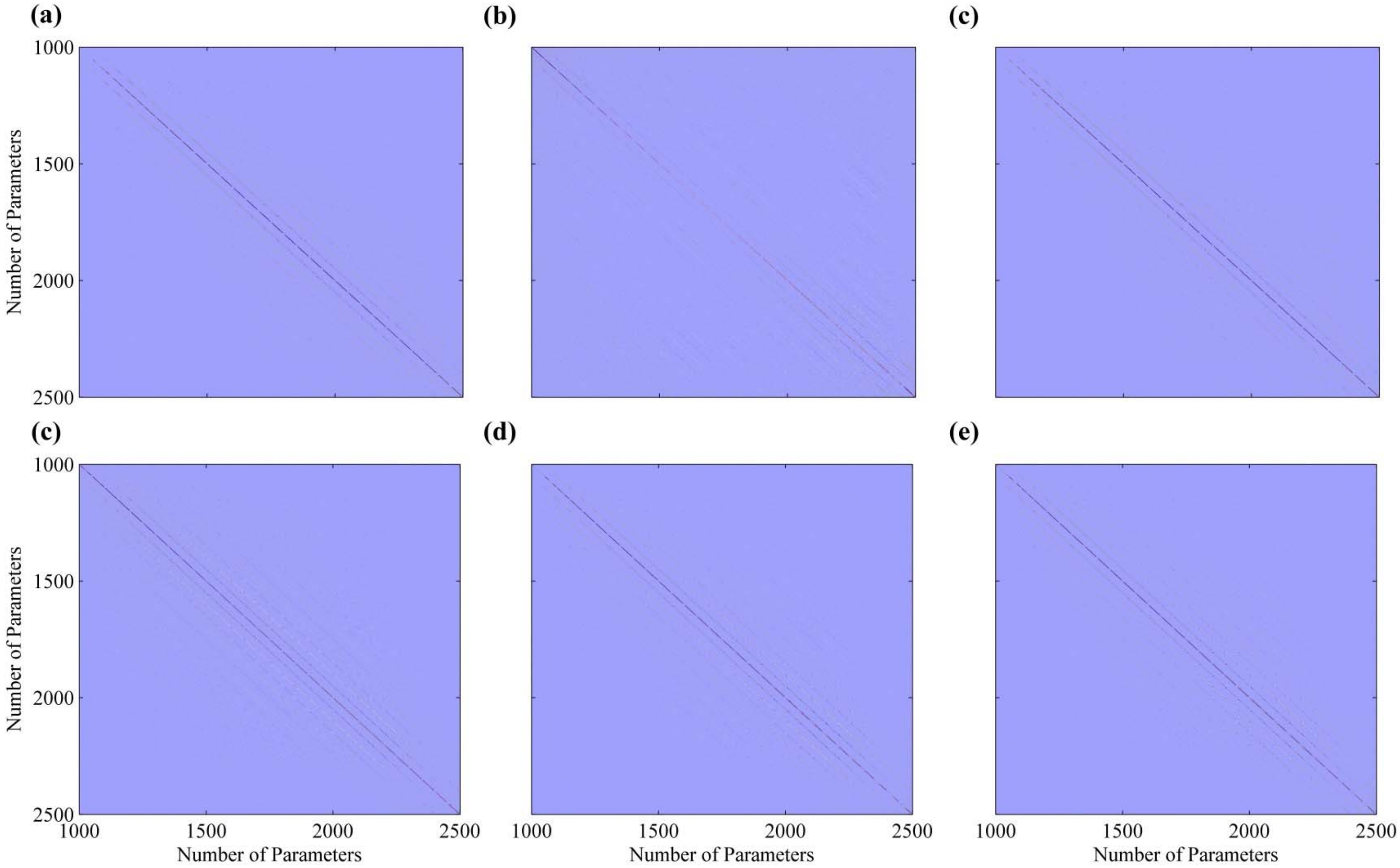


50*5m
50*5m
1
2
3
4
49
50
51
52
53
54
99
100
101
151
Receivers
Sources
Velocity=2500m/s
Homogeneous

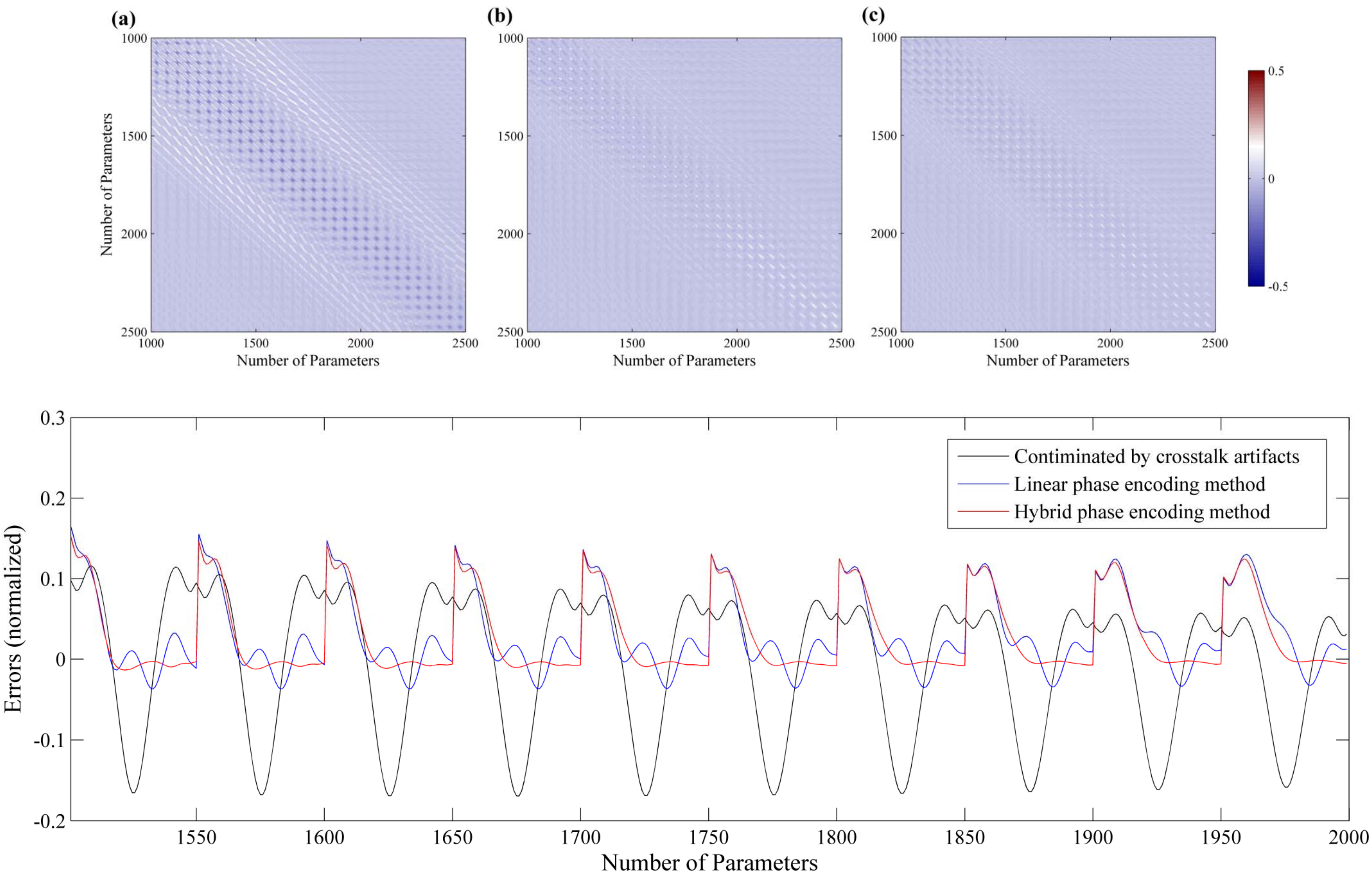
Hessian Approximations Comparison



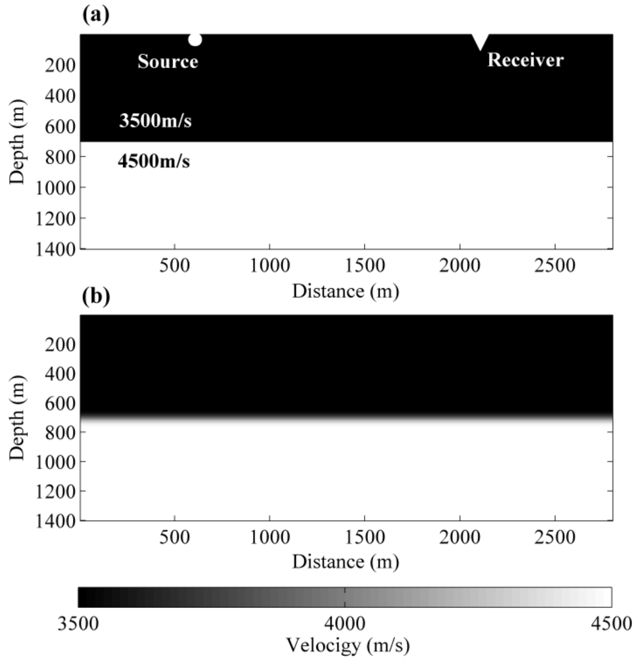
Inverse Hessian Comparison

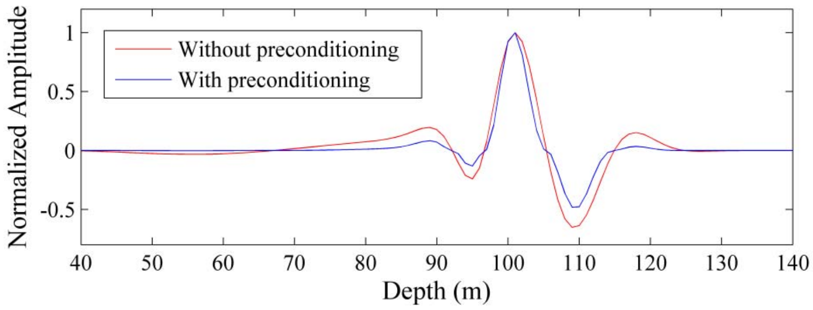


Error Comparison



Gradient Contribution Analysis





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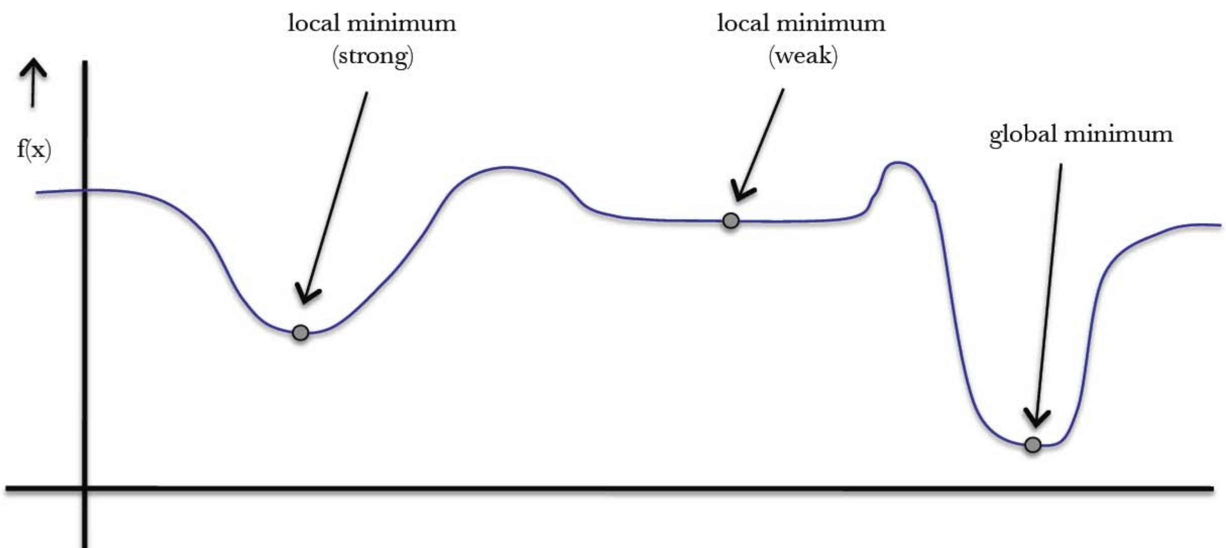
Multiscale Approach



By Kris (2013)

Multiscale Approach

- Low frequency is responsible to catch the low wavenumber component
- High frequency is responsible to add detailed information

$$s(\mathbf{r}) = s(\mathbf{r})^{low} + s(\mathbf{r})^{high}$$

Pseudo Gauss-Newton Step

To reduce the computational cost further, we proposed to use one ray parameter in one FWI iteration but change the ray parameter regularly for different iterations.

$$\delta s\left(\mathbf{r}\right)=\frac{\int d\omega\Re\left\{\omega^2\mathcal{F}_s(\omega)G(\mathbf{r},\mathbf{r}_s,\omega)G(\mathbf{r}_g,\mathbf{r},\omega)e^{i\omega p_s(x_s'-x_s)}\delta P^*\right\}}{\mathbf{Diag}\left(H_{phase\_encoded}\right)+\lambda I}$$

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Pseudo Code for PGN method

BEGIN $\leftarrow s_0$, initial model;

WHILE $\varepsilon \leq \varepsilon_{min}$ or $n \leq n_{max}$

Identify the ray parameter $p_s^{(n)}$

Identify the frequency band $f^{(n)} = f_0 \rightarrow f_{max}, f_{interval}$, every k iterations

Generate the data residual δP and apply low-pass filtering $\delta \tilde{P} = \textbf{low\_pass}\left(\delta P, f^{(n)}\right)$

Generate the linear phase encoded gradient $g^{(n)}\left(p_s^{(n)}\right)$

FOR $i = 1$ to $\mathbf{p}_s^H, \mathbf{p}_r^H$, every 1 or m iterations

Construct the diagonal part of the hybrid phase encoded Hessian $diag\left(H_{en\_a}^{(n)}\right)$

END FOR

Calculate the step length $\mu^{(n)}$ using the line search method

update the velocity model:

$$s^{(n+1)}(\mathbf{r}) = s^{(n)}(\mathbf{r}) - \mu^{(n)} \left\{ diag\left(H_{en\_a}^{(n)}\right) + \lambda I \right\}^{-1} g^{(n)}\left(p_s^{(n)}\right)$$

Calculate the relative least-squares error:

$$\varepsilon = \frac{\|s^{(n)}(\mathbf{r}) - s^{true}(\mathbf{r})\|_2}{\|s^{true}(\mathbf{r})\|_2}$$

END WHILE

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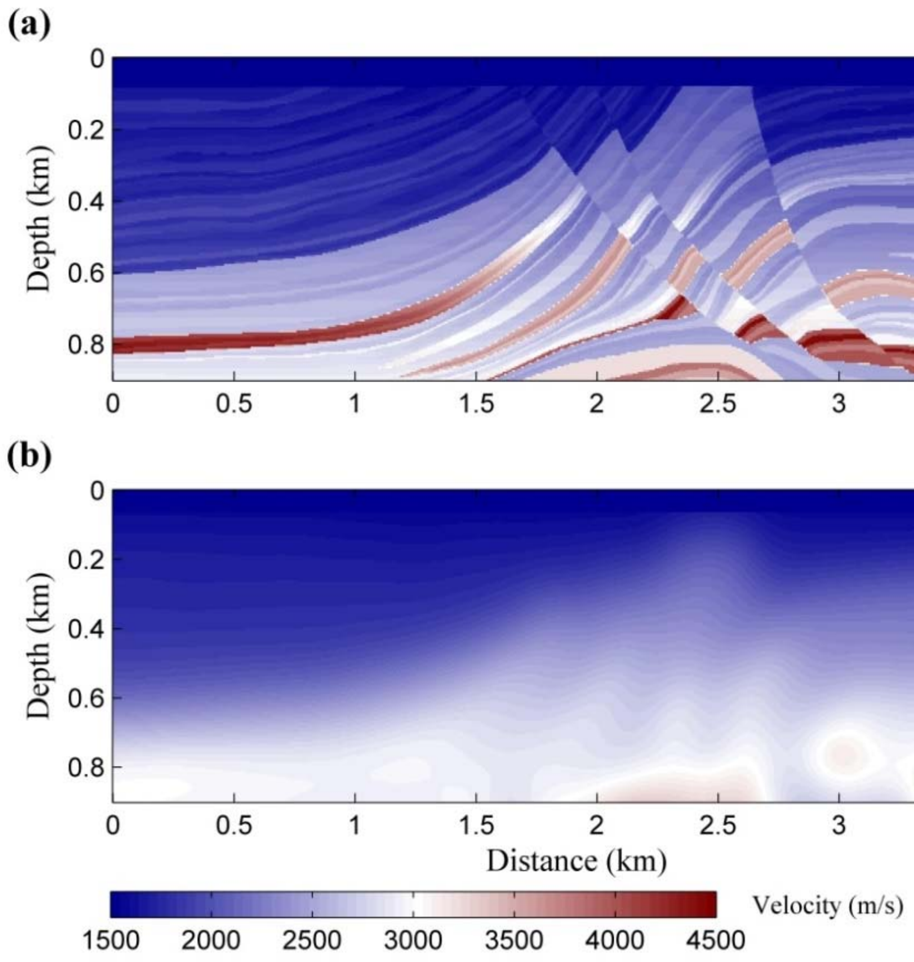
Computational Cost Comparison

Table2. Computational cost comparison for different strategies

Methods	Gradient	H_a	$diag\,(H_{en\_a})$	Step length	Cost for one iteration
TGN Method	$2N_s$	$N_s \times N_r$	\	1	$2N_s + N_s \times N_r$
SEGN Method	$2N_p^g$	\	$N_{ps}^H + N_{pr}^H$	1	$2N_p^g + N_{ps}^H + N_{pr}^H$
PGN Method	2	\	$N_{ps}^H + N_{pr}^H$	1	$N_{ps}^H + N_{pr}^H + 2$

Numerical Experiment



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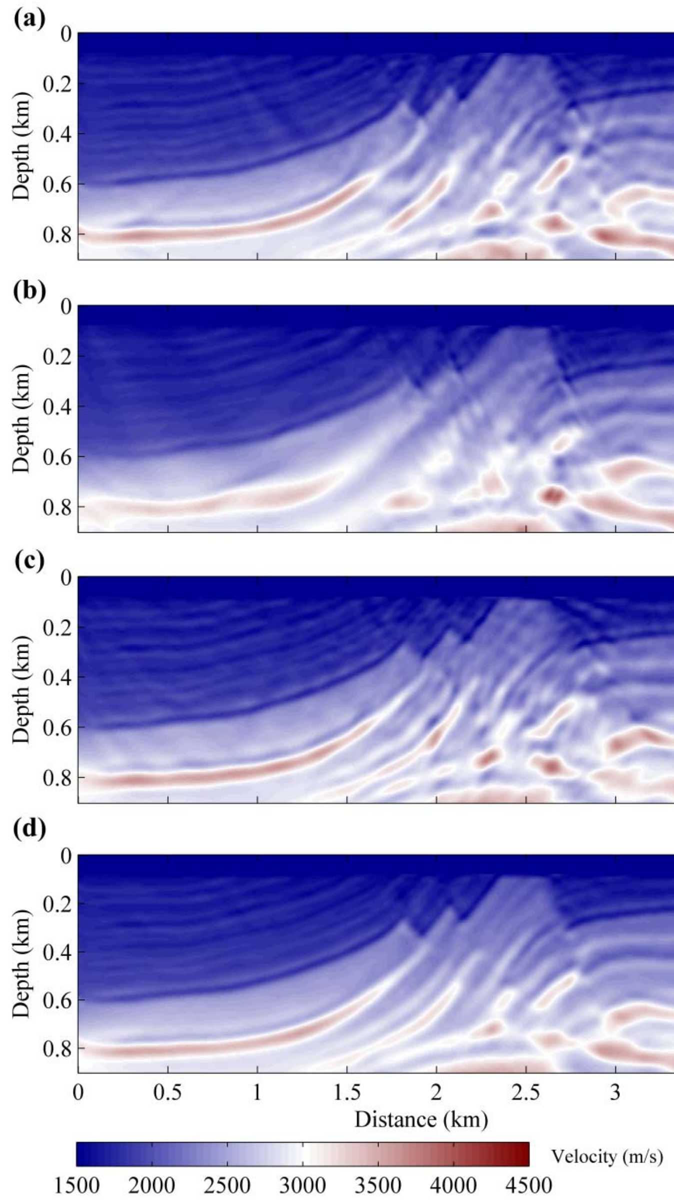


$p = 0 s/km$

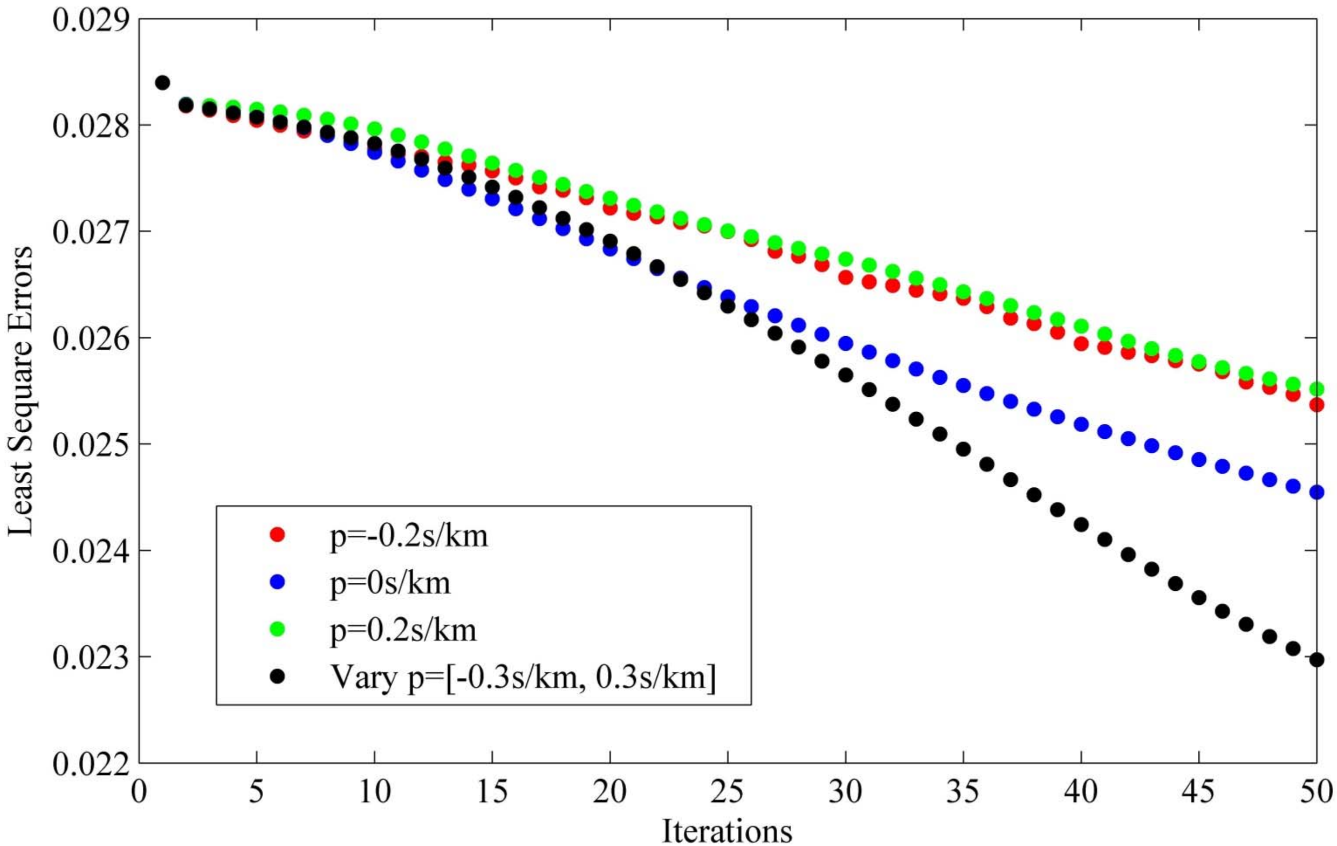
$p = -0.2 s/km$

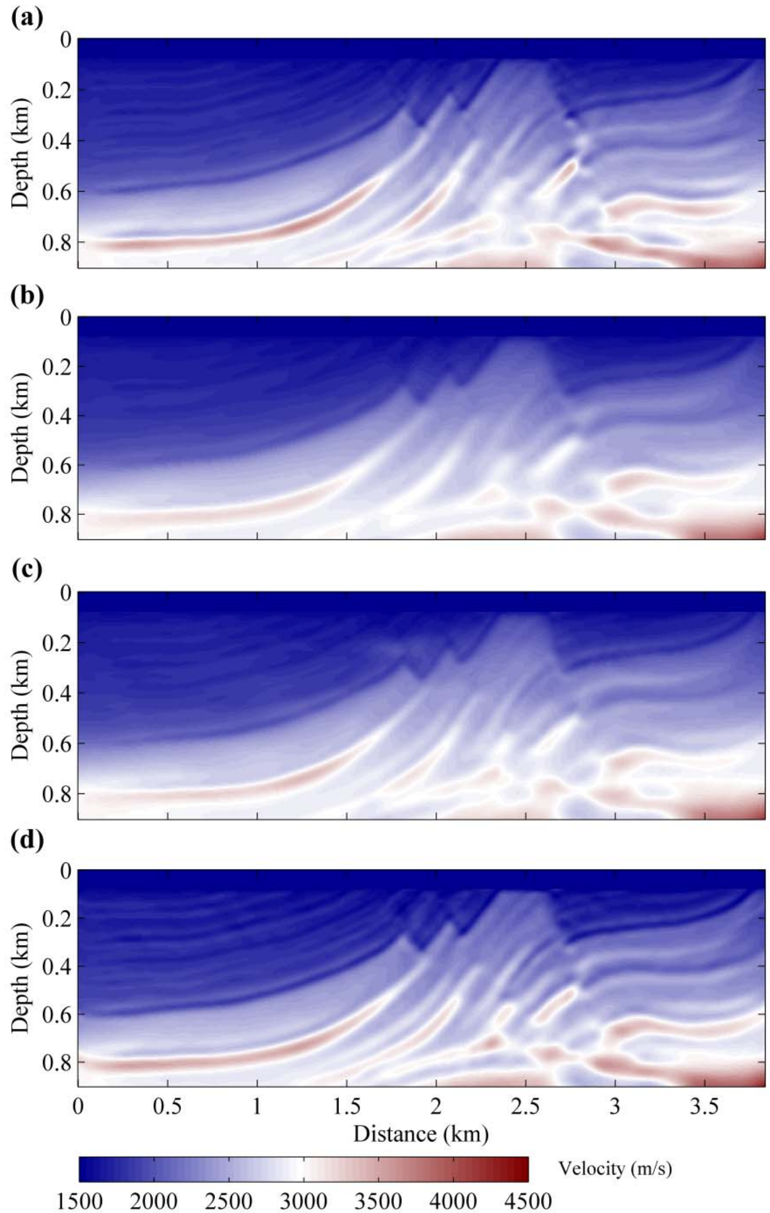
$p = 0.2 s/km$

$p = [-0.3 s/km, 0.3 s/km]$

Effects for Fixed and Varying Ray Parameter

Inversion Results Quality Evaluation





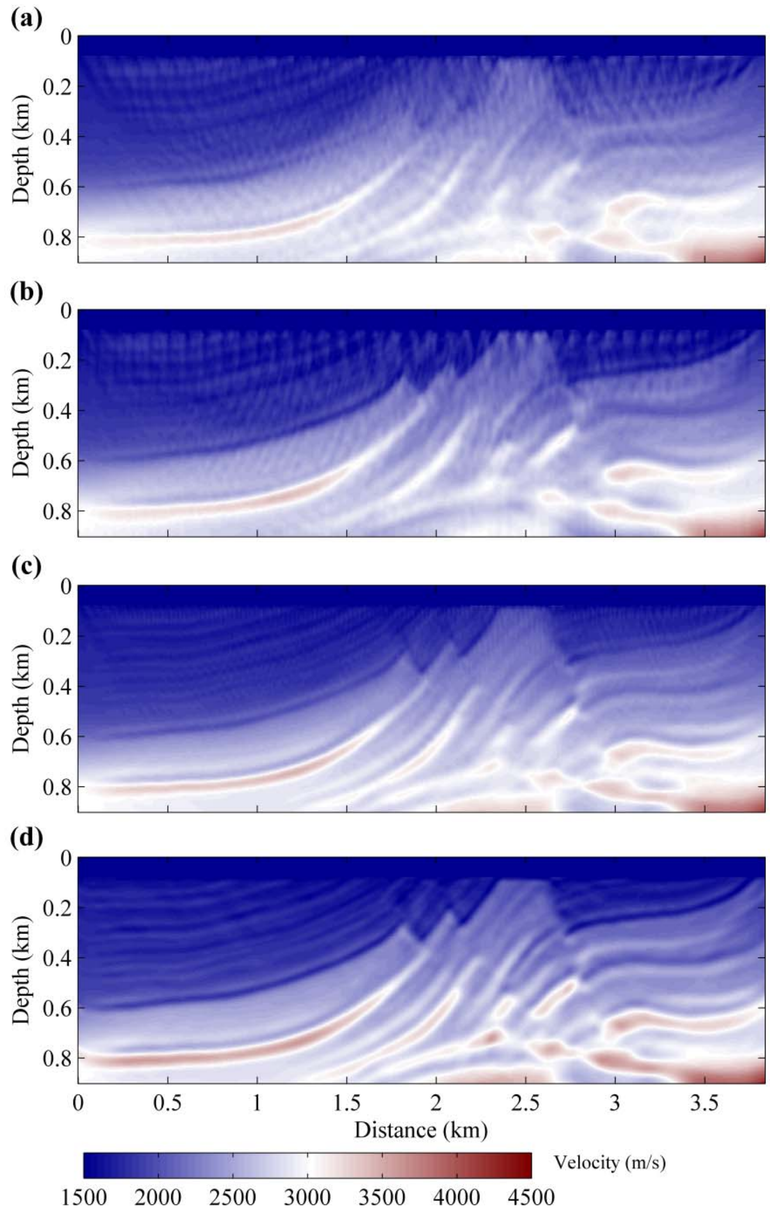
$$p = [-0.1s/km, 0.1s/km]$$
$$step = 0.02s/km$$

$$p = [-0.5s / km, 0.5s/km]$$
$$step = 0.08s / km$$

$$p = [-0.6s/km, 0.6s/km]$$
$$step = 0.12s/km$$

$$p = [-0.3s/km, 0.3s/km]$$
$$step = 0.1s/km$$

Sensitivity to the Ray-Parameter Range



$N_s = 38$

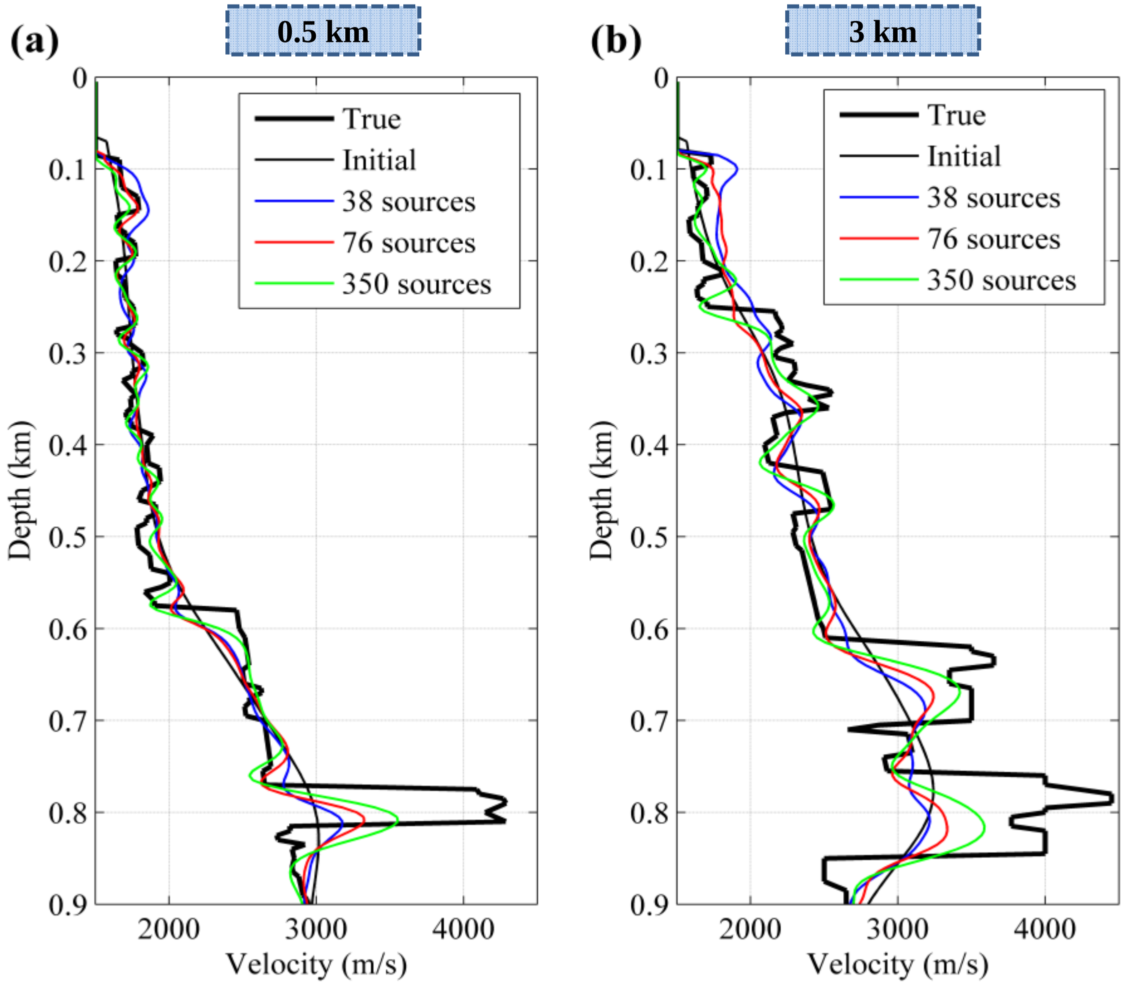
$N_s = 38$

$$\sum_{-0.2s/km}^{0.2s/km} p$$

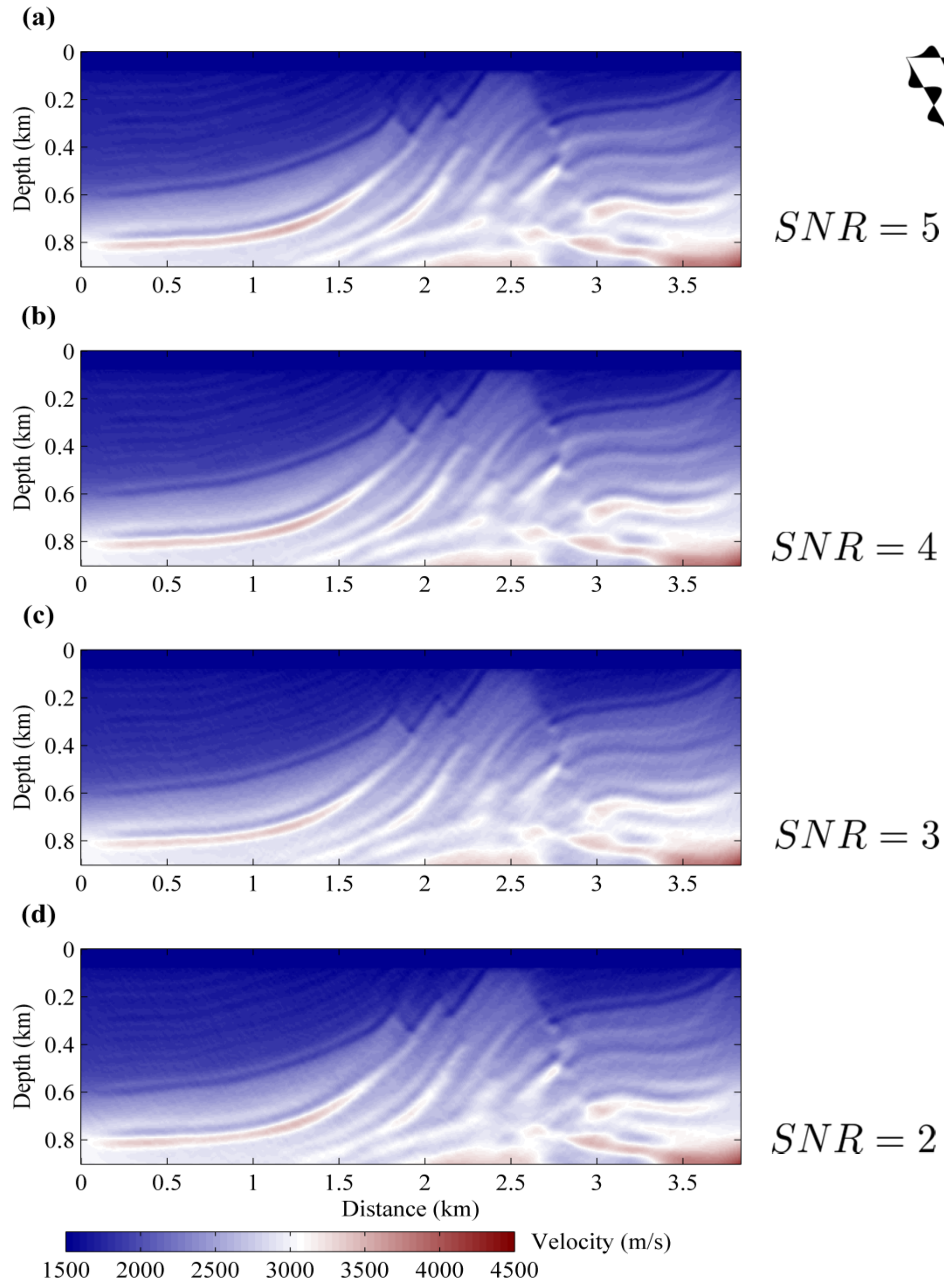
$N_s = 76$

$N_s = 350$

Sensitivity to the Encoded Sources

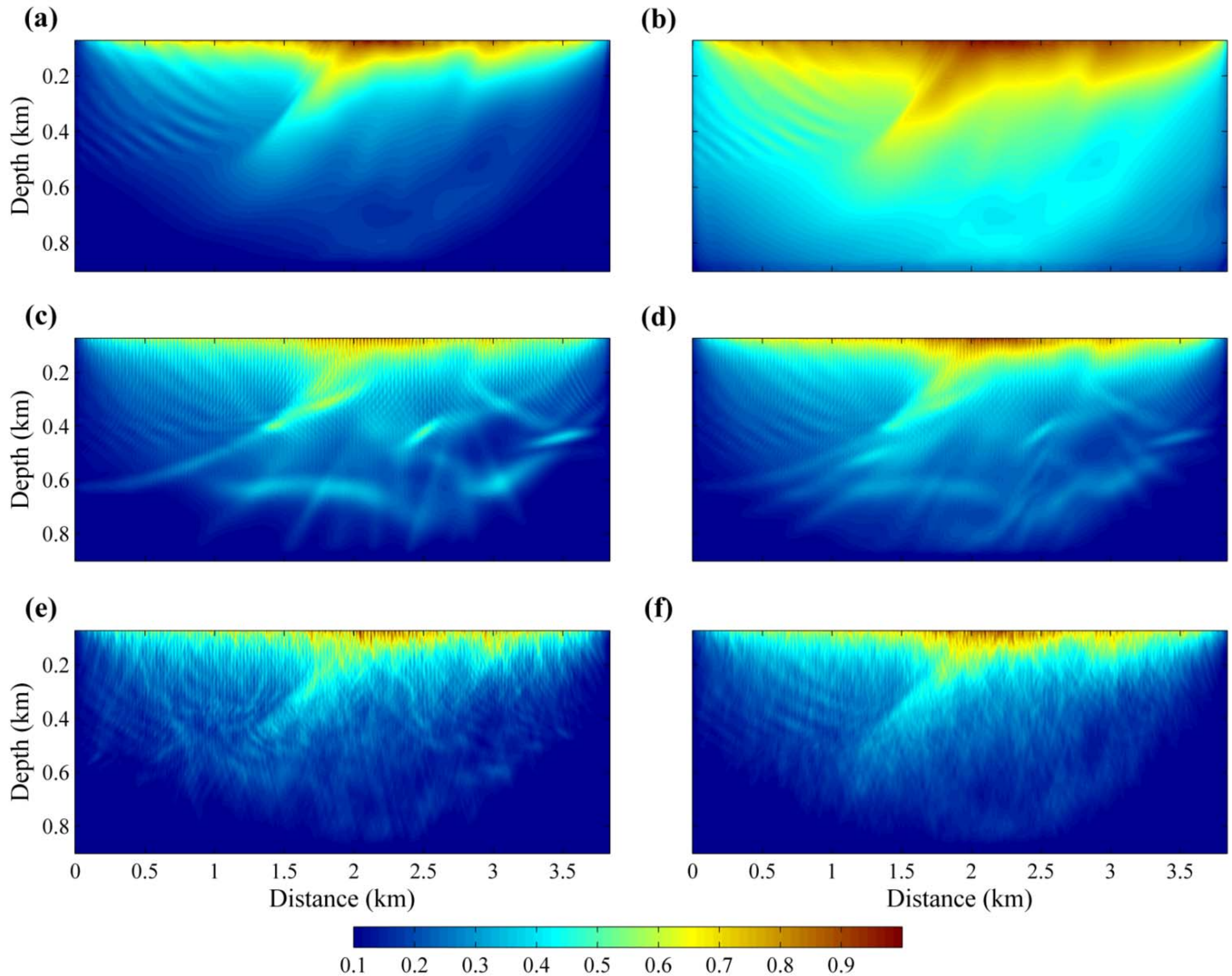


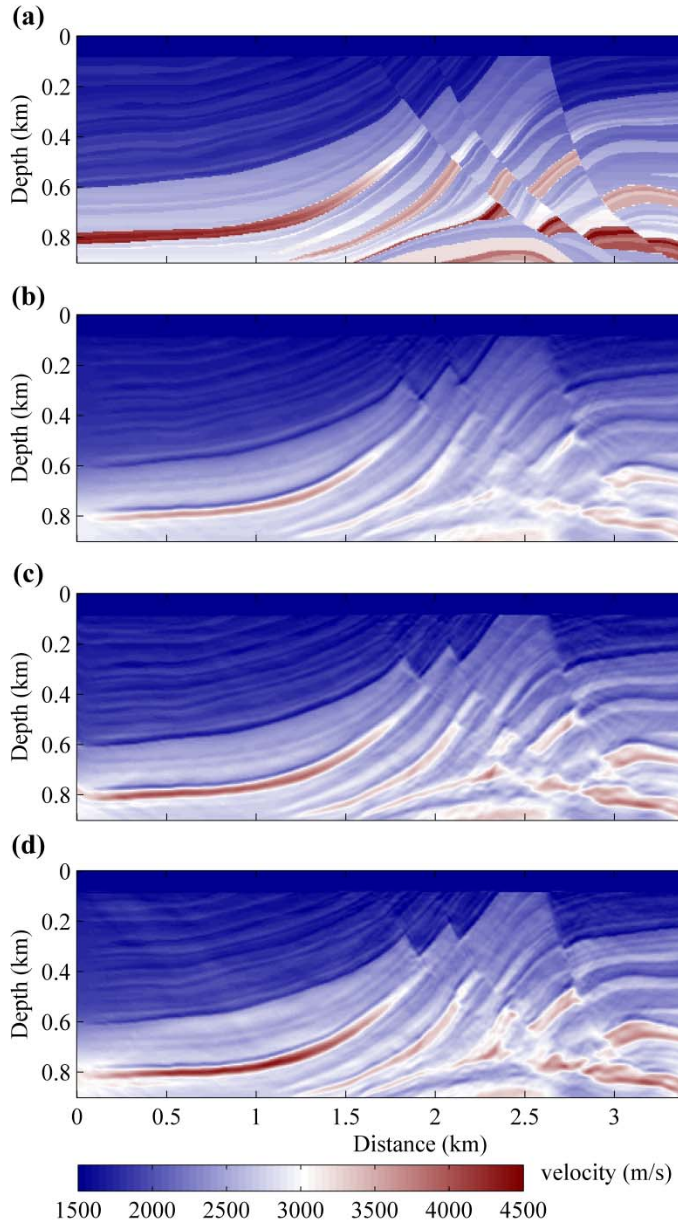
Sensitivity to the Encoded Sources



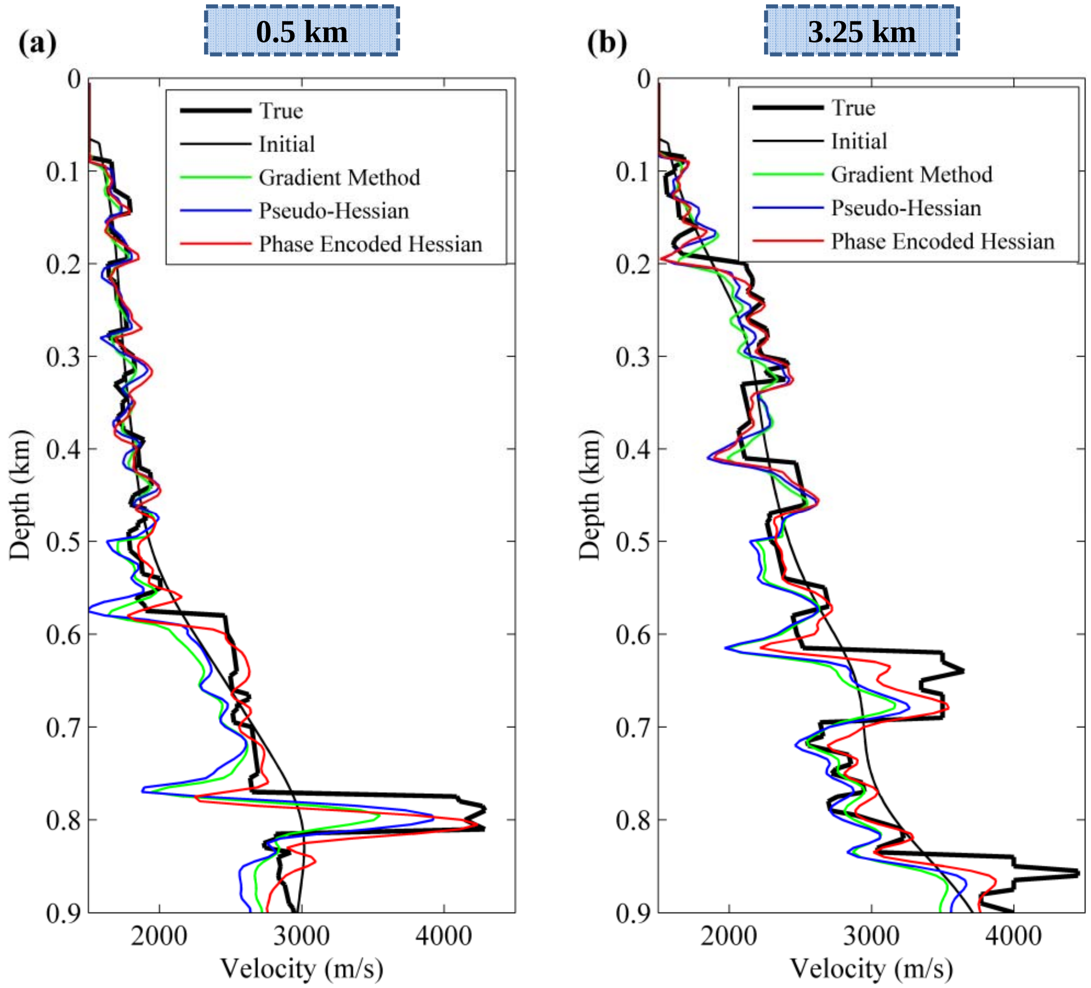
Sensitivity to Gaussian Noise

Initial Velocity Model





Different Scaling Methods for FWI


(a)
0.5 km
(b)
3.25 km
True
Initial
Gradient Method
Pseudo-Hessian
Phase Encoded Hessian
Depth (km)
Velocity (m/s)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
2000
3000
4000

Reverse Time Migration Image Comparison

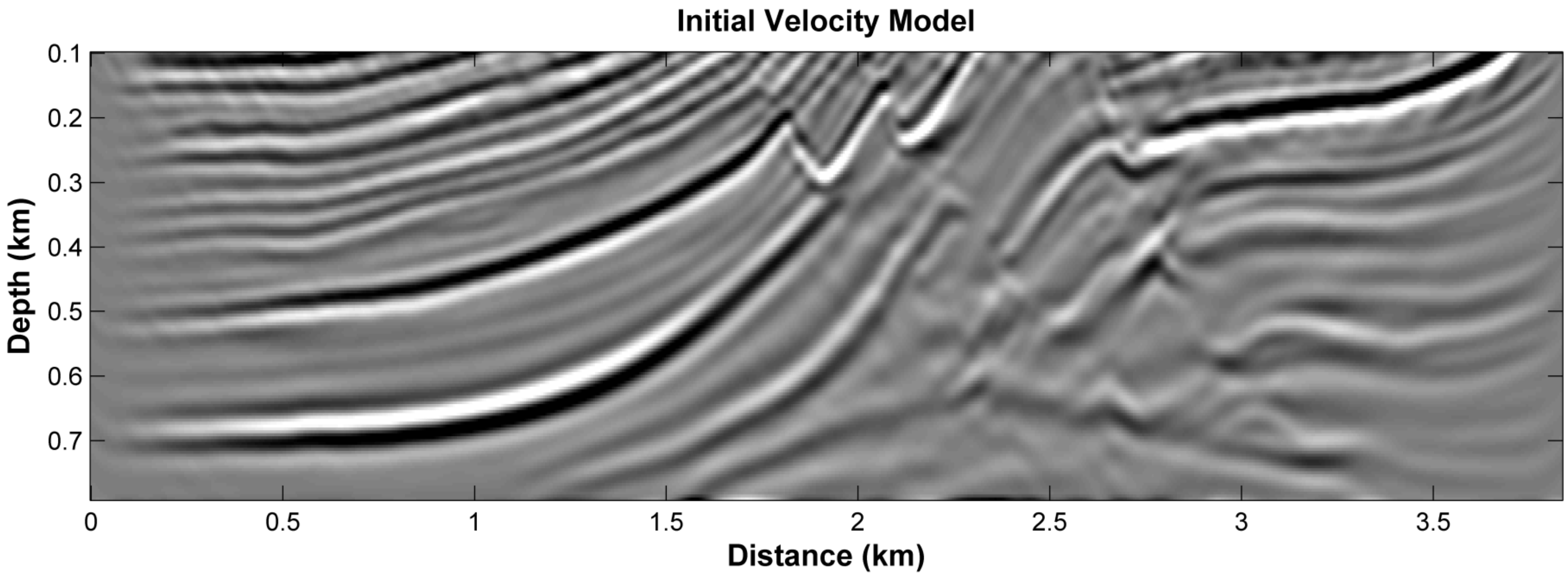


Image By Initial Velocity Model

Reverse Time Migration Image Comparison

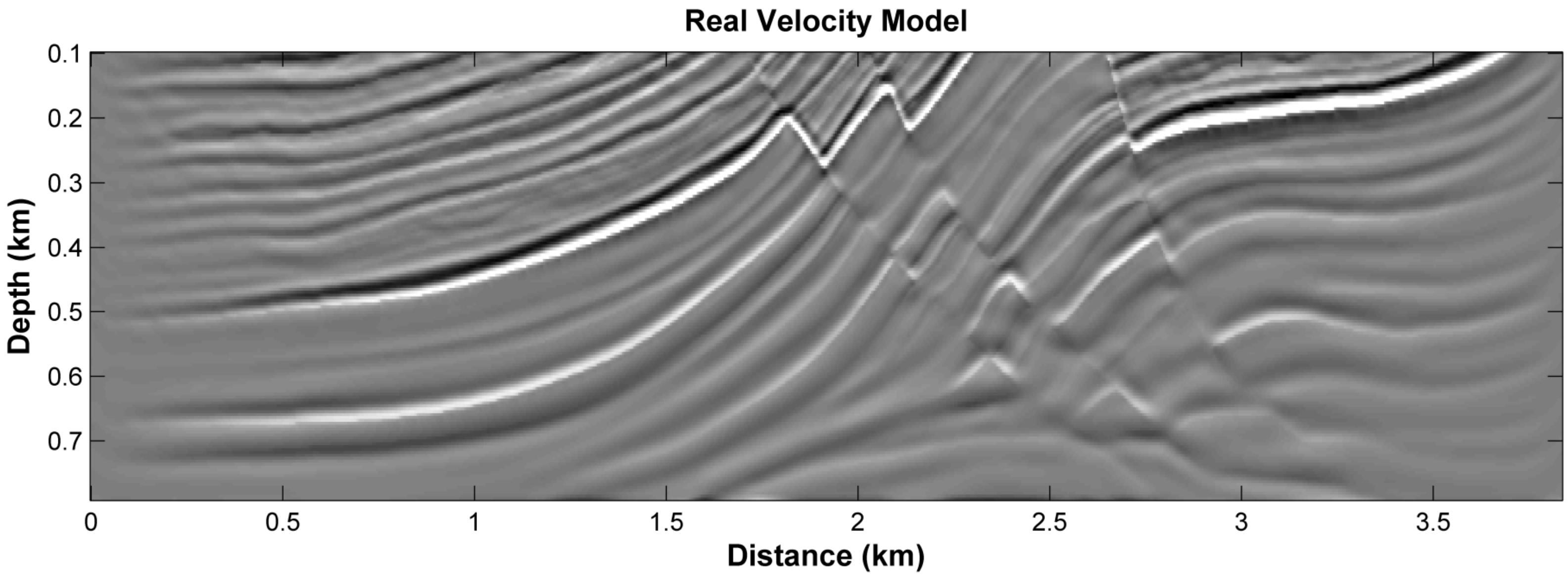


Image By True Velocity Model

Reverse Time Migration Image Comparison

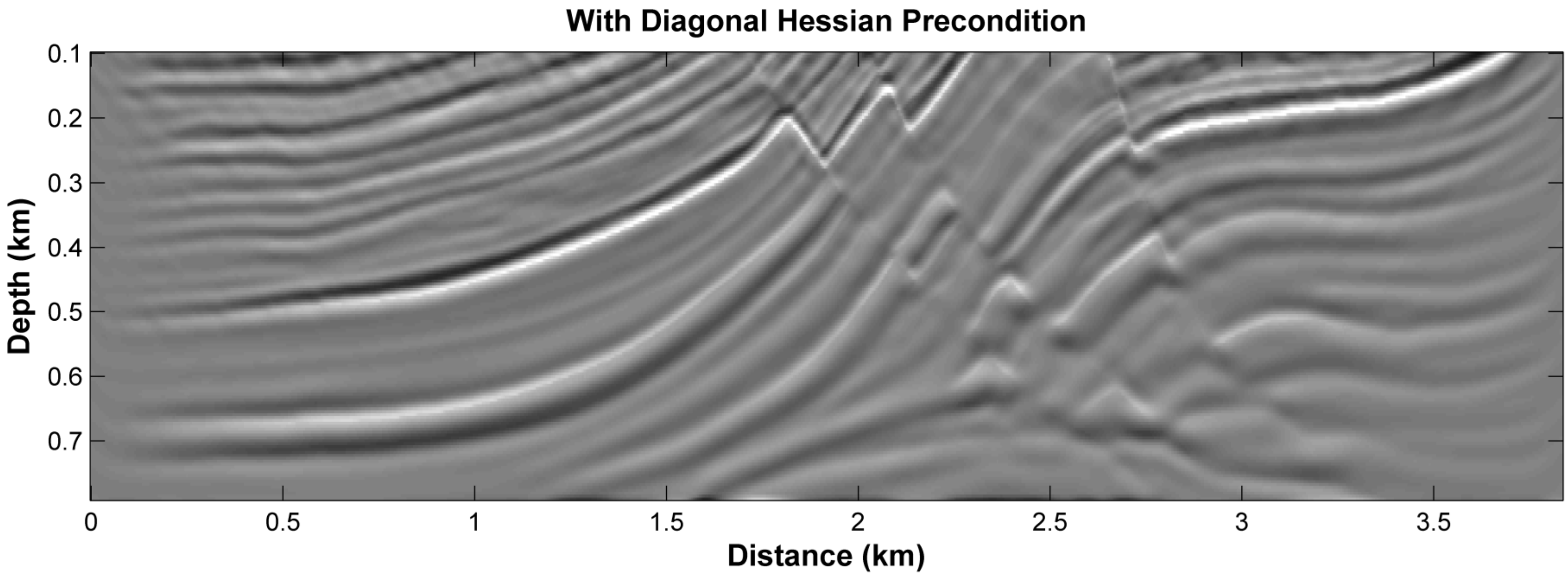


Image By Inverted Velocity Model

Conclusion

- Hessian matrix servers as a nonstationary deconvolution operator to improve the convergence rate of least-squares inverse problem.
- Varying ray-parameter during iterations can reduce the computational cost further and balance the model update.
- If the ray-parameter range is too small, the layers with dip angles cannot be inversed in balance, if the ray-parameter range is too large, the convergence rate will be decreased.
- If the encoded sources are sparsely distributed, the crosstalk noise will be very obvious, especially for shallow layers.
- Hybrid phase encoding strategy can reduce the crosstalk noise better than linear phase encoding strategy with the same number of simulations.
- Diagonal part of the phase encoded Hessian can server as a good approximation of the Hessian to precondition the gradient and increase the convergence rate.

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- Joe Wong, Raul, Babatunde
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Thank You !