Jean Morlet and the Continuous Wavelet Transform (CWT)

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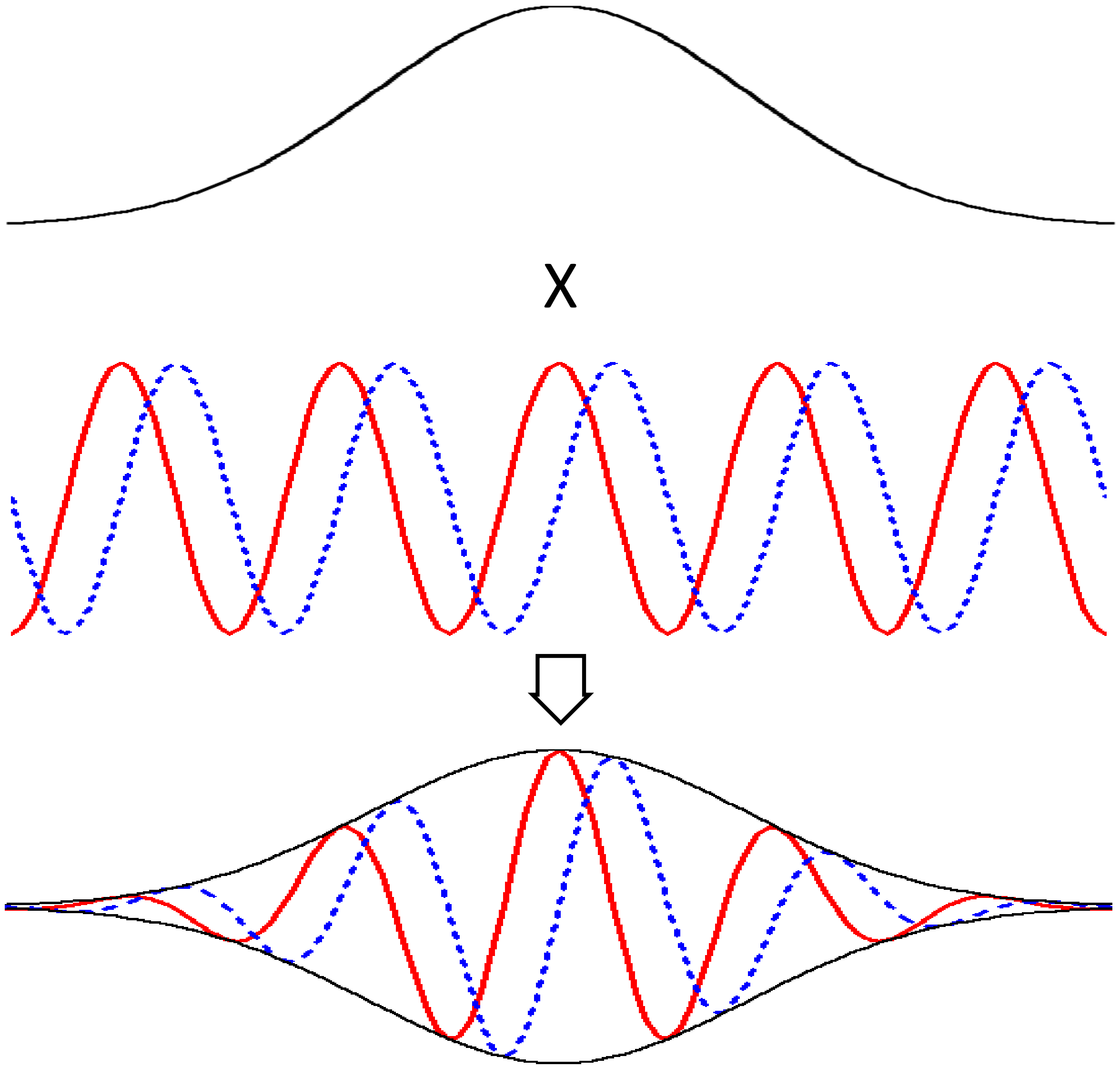
Introduction

- In 1982, Jean Morlet, a geophysicist who worked for Total in Paris, published a new approach to time-frequency analysis in Geophysics (Morlet et al., 1982, I and II).
- Up to that point, seismic frequency analysis in was done with the Fourier Transform, which did not give time-localized estimates of frequency content.
- Morlet took several ideas: the Gabor wavelet, the Heisenberg uncertainty principle, logarithmic frequency increments and cross-correlation, and put them together in such a way that he was able to get a time-localized frequency estimate.
- Geophysicists did not recognize Morlet's originality, but mathematicians did, and his method became the Continuous Wavelet Transform, a new branch of mathematics.
- I will summarize Morlet's approach by analyzing each of the above steps, then show the modern formulation of the CWT and an application to seismic data analysis.

The Gabor wavelet

- The motivation for Morlet’s work was the wavelet shape proposed by Gabor (1941).
- As shown here, the Gabor wavelet is a sine (or cosine) wave modulated by a Gaussian.
- The wavelet shown in the bottom figure has its Gaussian envelope shown, to illustrate the effect of the modulation on the cosine and sine waves.



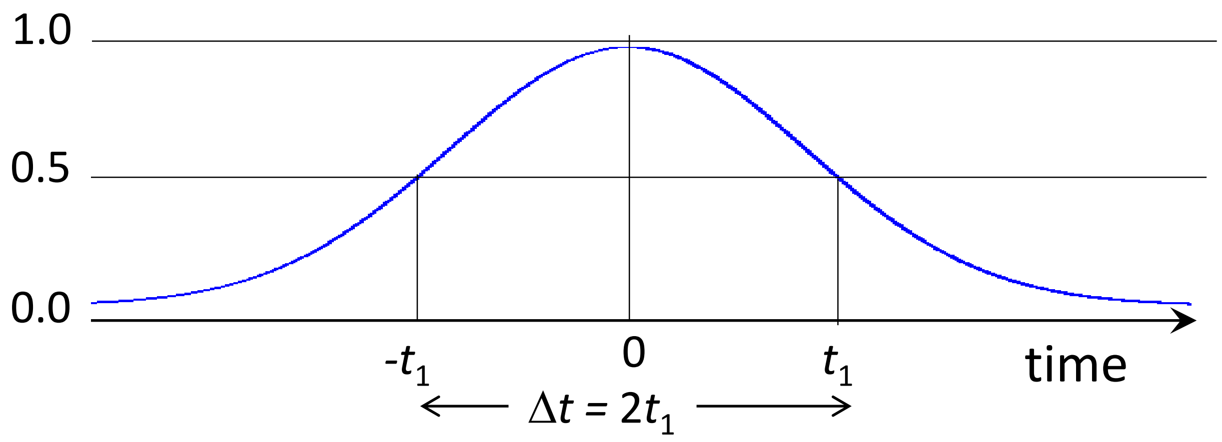
The mathematical Gabor wavelet

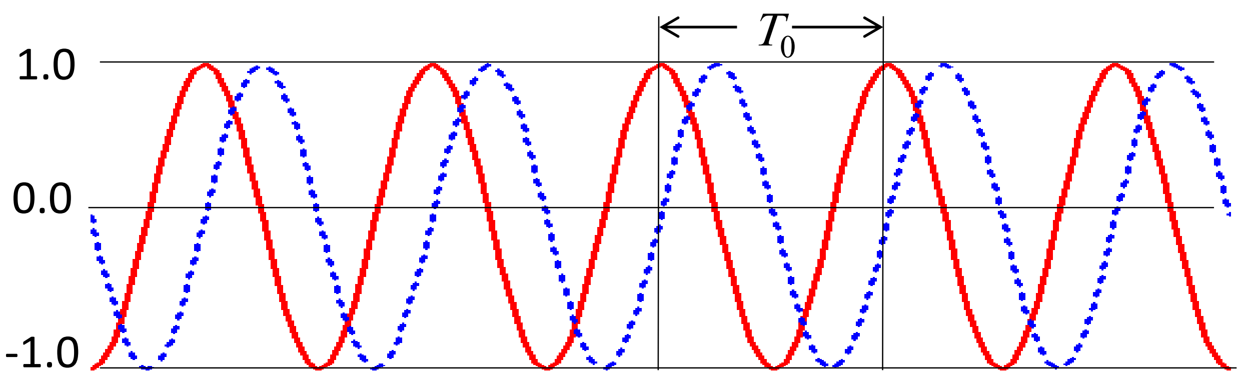
- Morlet writes the Gabor wavelet mathematically as follows, where he defines Δt as the time width between the points where the modulus drops to 1/2:

$$\psi(t) = g(t)s(t), \text{ where:}$$

$$g(t) = \exp\left[-\left(\frac{2t\sqrt{\ln 2}}{\Delta t}\right)^2\right], s(t) = \exp(i\omega_0 t) = \cos\omega_0 t + i\sin\omega_0 t, \omega_0 = \text{mean frequency} = \frac{2\pi}{T_0},$$

$$\Delta t = \text{width of Gaussian between 1/2 amplitude values since } \psi(\pm t_1) = \exp[-\ln 2] = 0.5$$



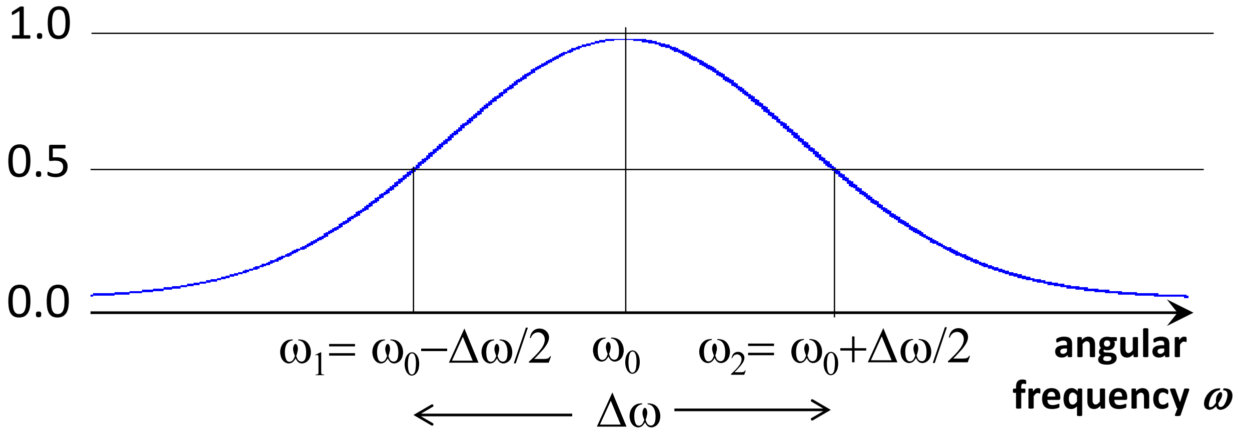


The Fourier Transform of the Gabor Wavelet

- Morlet then computed the modulus of the Fourier transform of the Gabor wavelet and showed that it was equal to:

$$\Psi(\omega) = \frac{1}{2}\sqrt{\frac{\pi}{\ln 2}}\Delta t \exp\left(-\left(\frac{\Delta t(\omega-\omega_0)}{4\sqrt{\ln 2}}\right)^2\right),$$

$$\text{where } \omega-\omega_0 = \frac{\Delta\omega}{2}.$$



- Setting the amplitudes at angular frequencies ω_1 and ω_2 to ½ gives the constant product of $\Delta\omega$ and Δt in Heisenberg's time-frequency uncertainty principle :

$$\frac{\Psi(\omega_1)}{\Psi(\omega_0)} = \frac{\exp\left(-\frac{(\Delta t\Delta\omega)^2}{64\ln 2}\right)}{\exp(0)} = \frac{1}{2} \Rightarrow \ln 2 = \frac{(\Delta t\Delta\omega)^2}{64\ln 2} \Rightarrow \Delta\omega\Delta t = 8\ln 2 \Rightarrow \Delta f\Delta t = \frac{8\ln 2}{2\pi} \approx 0.883$$

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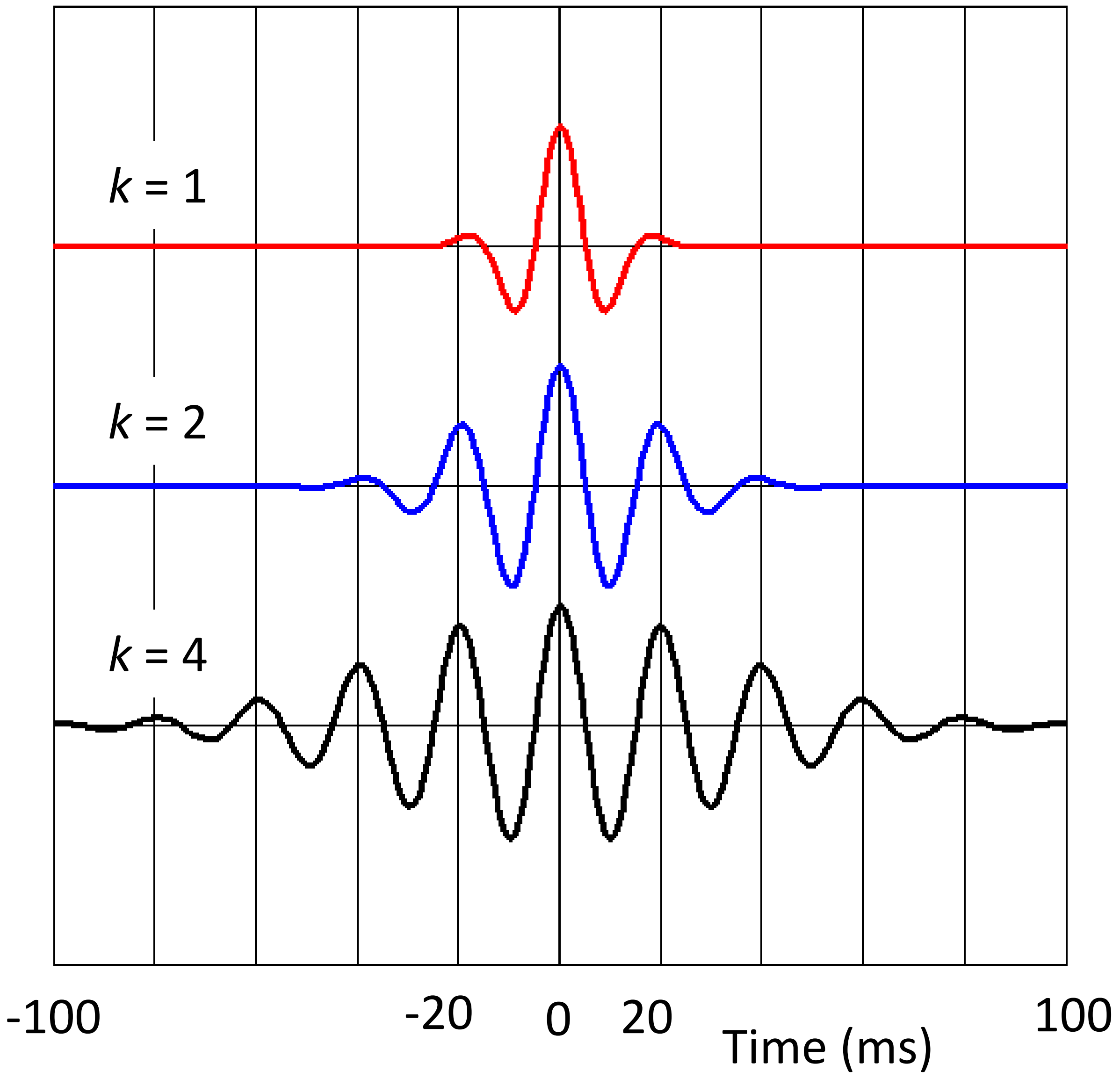
Varying shape ratio with constant period

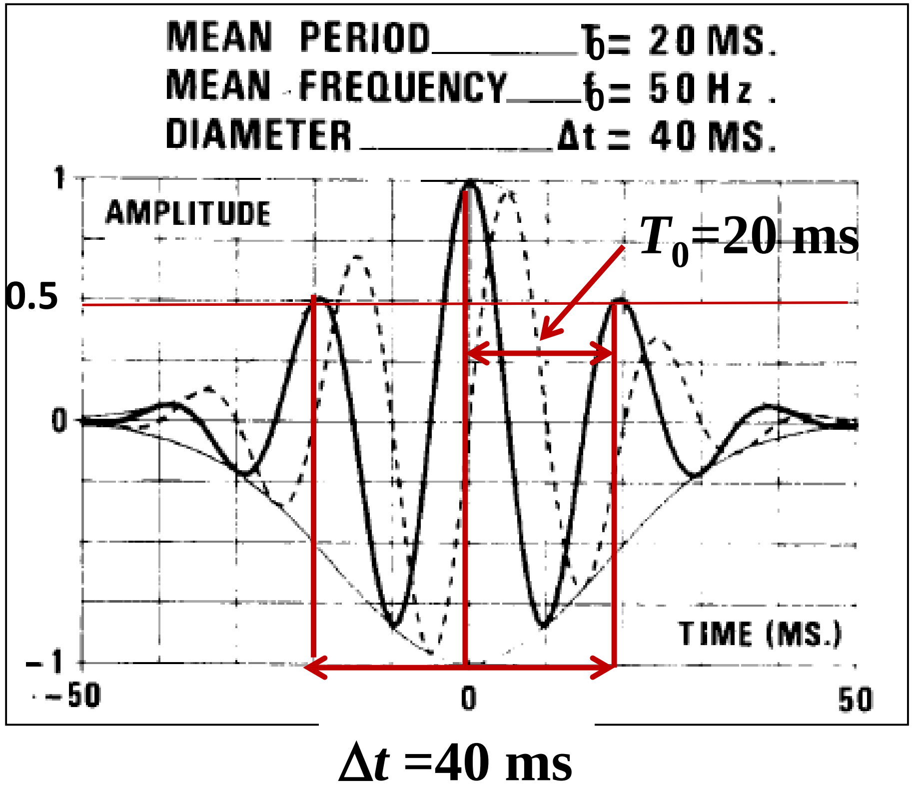


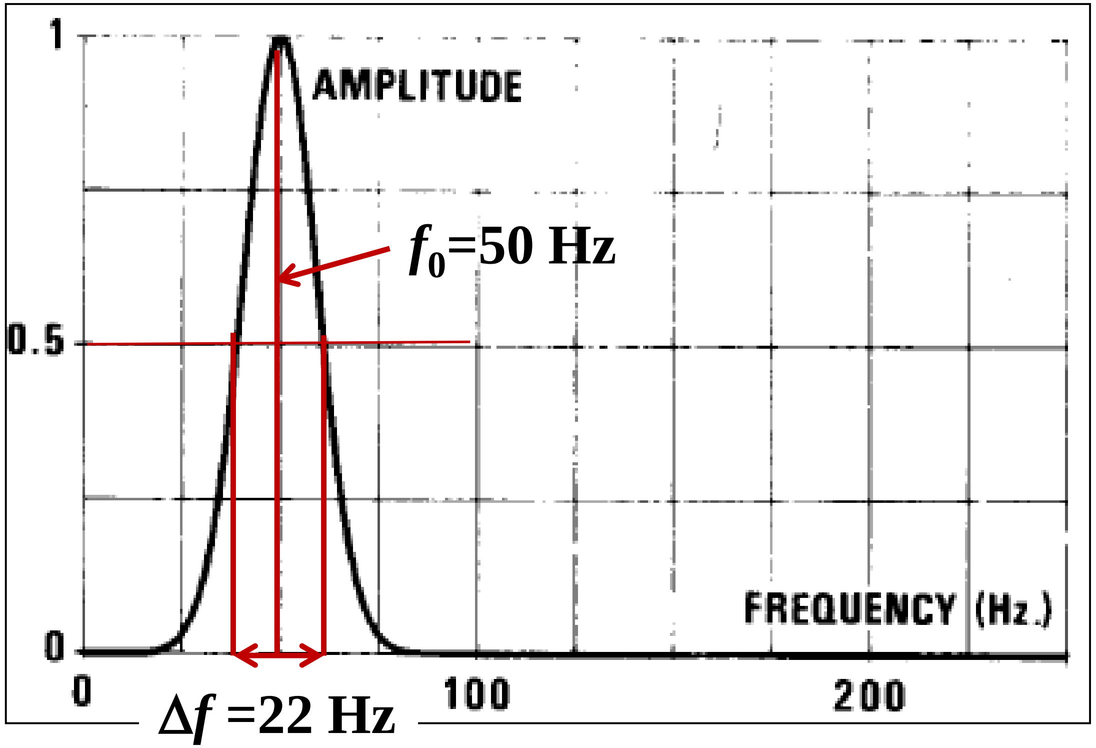
- Morlet also defines the **shape ratio** k, which relates the time width at half-amplitude to the mean period, or:

$$\Delta t = kT_0 \Rightarrow \Delta f = \frac{0.883}{kT_0}$$

- The figure on the left shows three different shape ratios (1, 2 and 4) for a constant $T_0 = 20\text{ ms}$, or $f_0 = 50\text{ Hz}$.
- Higher shape ratios have more side lobes.

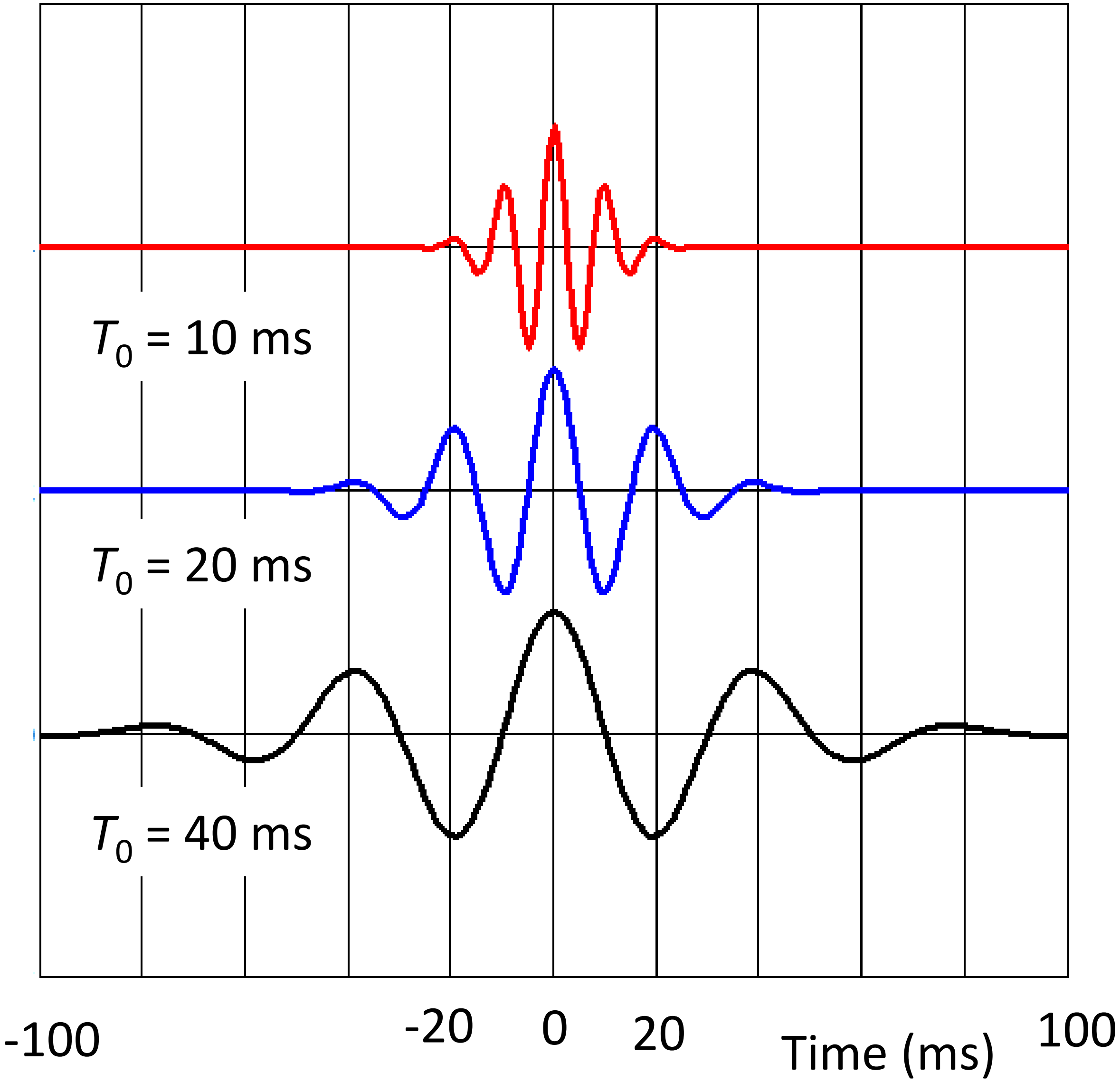
Figure 1 from Morlet et al.





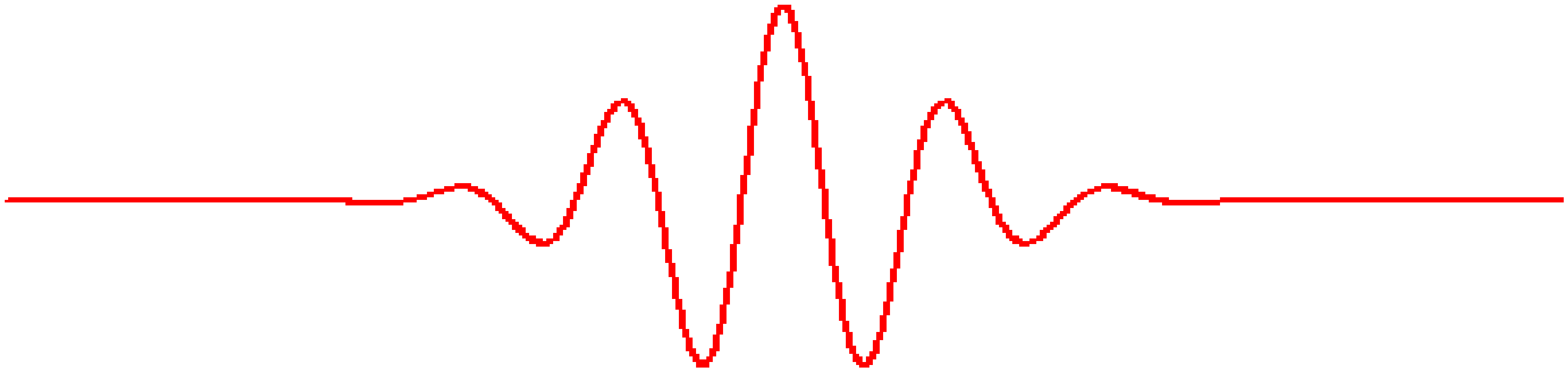
- Here is Morlet's illustration of a wavelet and its Fourier transform (right), with k = 2.
- Note that f_0 = 50 Hz, T_0 = 1/50 Hz = 20 ms, $\Delta t = 2T_0$ = 40 ms, Δf = 0.883/0.04 sec = 22 Hz.

Varying mean periods with constant shape ratio



- The second key concept in the Morlet wavelet is to keep the **shape ratio** constant for varying mean periods (or frequencies).
- This figure shows three different mean periods (10, 20 and 40 ms) for a constant shape ratio of 2.
- Note that all the wavelets now have the same number of side lobes.

Varying mean periods with constant shape ratio



- Here is a movie of the Morlet wavelet as the frequency gets lower.

The extended Gabor expansion (Figure 2 from Morlet et al. part II)

- Since complex signals have a discontinuity at zero, Morlet next introduced the **extended Gabor expansion**, which involved using a logarithmic frequency scale in octaves.
- Morlet did not write down the math for doing this, instead only supplying the figure shown here.
- You have to look carefully to notice that the scale is logarithmic!

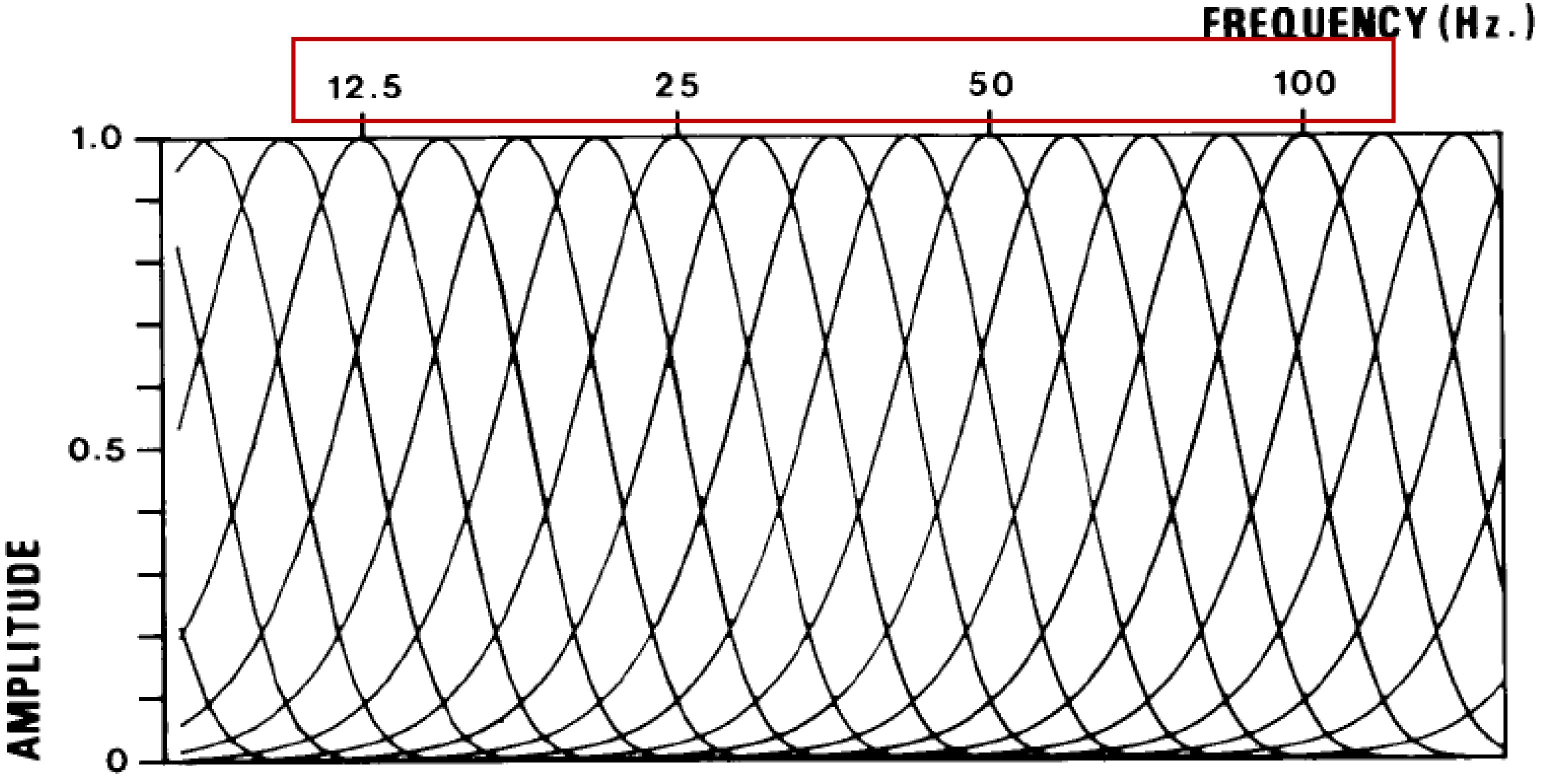


FIG. 2. Set of Gabor basic wavelets used in the Gabor expansion (practical example).

The extended Gabor expansion (theory)

- To understand the mathematics, I have shown one octave of Morlet's wavelets from f_0 to $f_0/2$.
- Morlet uses four wavelets per octave, so we can express Δ as:

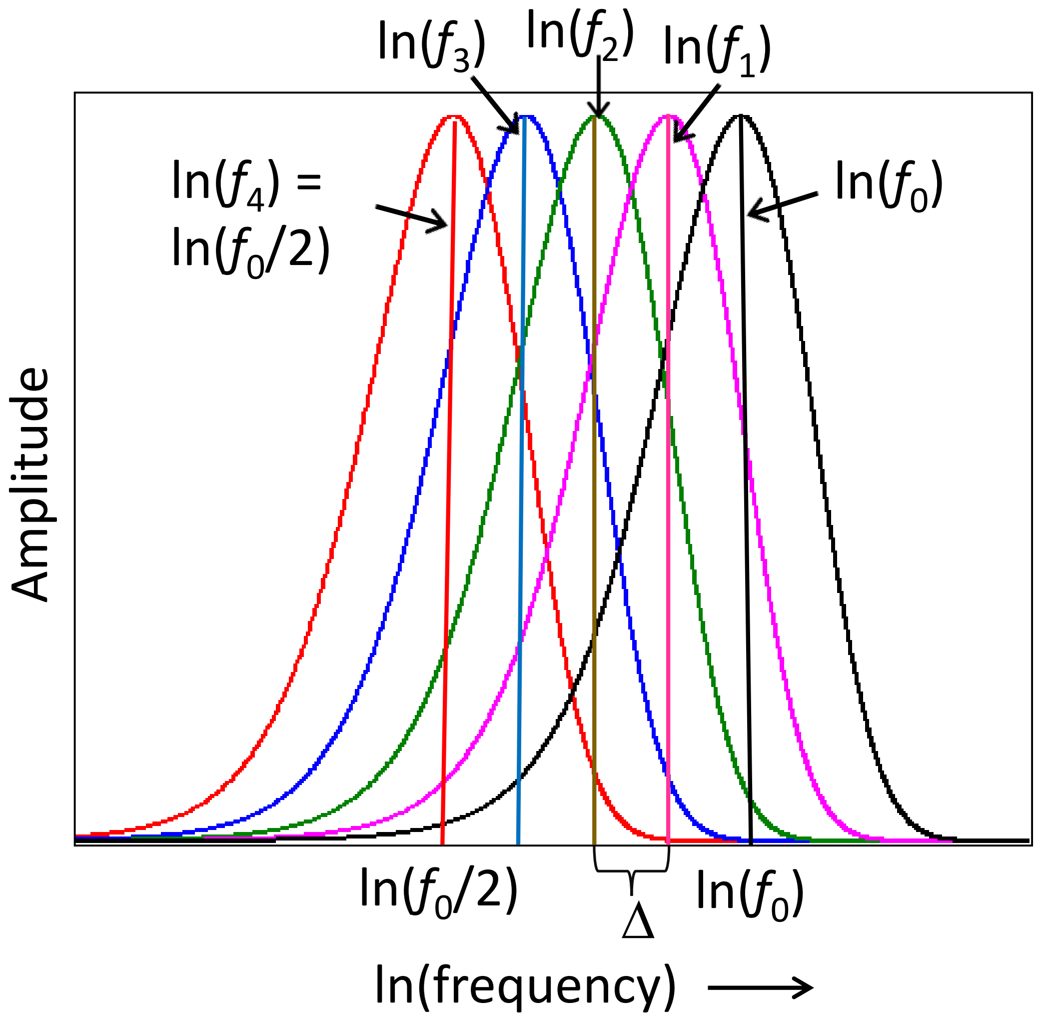
$$\Delta = \frac{\ln(f_0) - \ln(f_0/2)}{4} = \frac{\ln 2}{4}$$

- Thus, each frequency increment is:

$$\ln(f_n) = \ln(f_0) - (n/4)\ln 2$$

- This expression can also be expressed as:

$$f_n = \exp\left(\ln(f_0)\right)\cdot\exp\left((n/4)\ln 2\right) = \frac{f_0}{2^{n/4}}$$



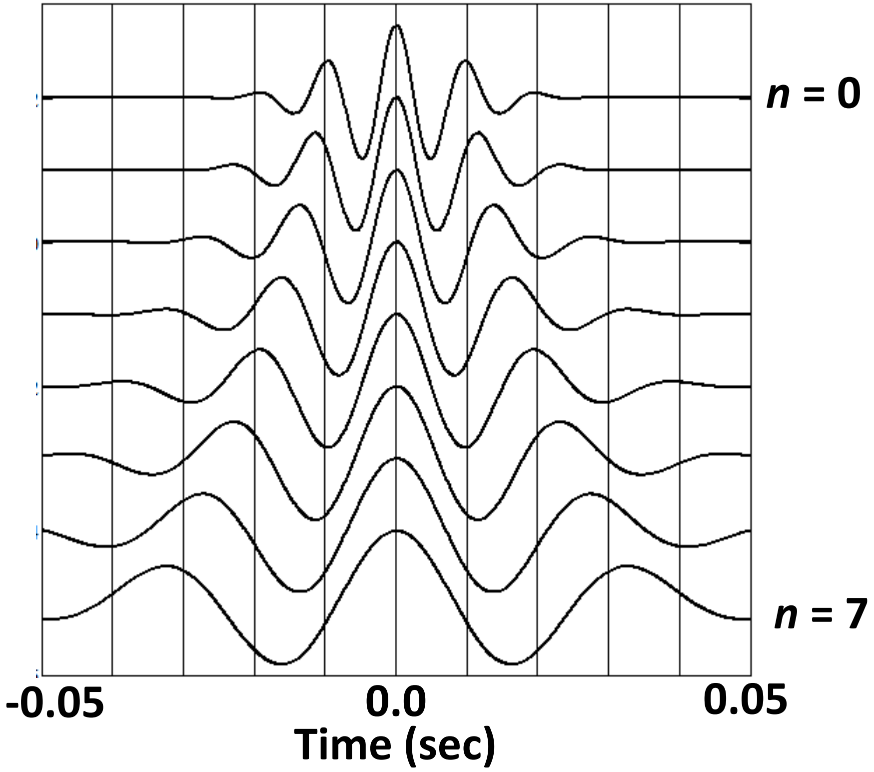
Wavelet scale

- We can thus introduce the scale parameter $s = 2^{n/m}$, where n is the number of steps below the starting frequency and m is the number of wavelets per octave.

- Using the scale parameter s and the shape parameter k allows us to re-write the Gabor wavelet using only one other parameter, the dominant frequency ω_0:

$$\psi = \exp\left[-\ln 2\left(\frac{\omega_0 t}{sk\pi}\right)^2\right]\exp\left(i\frac{\omega_0}{s}t\right)$$

- The figure on the right shows the real components of eight Gabor wavelets from $n = 0$ to 7, where $m = 4$, $k = 2$ and $f_0 = 100$ Hz.



Correlation with the wavelets

- Now we get to Morlet's final step, which involved cross-correlating each wavelet ψ_t with the seismic trace s_t to obtain its wavelet transform.
- Recall that the discrete correlation formula is given as follows where, since we are dealing with a complex wavelet, we must take the complex conjugate of the wavelet first (indicated by the asterisk on ψ_k^*):

$$\phi_{\psi s}(\tau) = \sum_{k=0}^{N-1} \psi_k^* s_{k-\tau}, \qquad \tau = -(M-1), \ldots, (N-1),$$

$$\text{where: } \psi_t = \left(\psi_{-N/2}, \ldots, \psi_0, \ldots, \psi_{N/2}\right), \text{ and } s_t = \left(s_0, s_1, \ldots, s_{M-1}\right).$$

- However, a faster approach is to apply frequency domain correlation, which is used in our real data example.

The modern formulation of the CWT

- This leads to the modern mathematical formulation of the continuous wavelet transform (e.g. Daubechies), which is written:

$$S_W(\tau,a)=\frac{1}{\sqrt{a}}\int_{-\infty}^{\infty}s(t)\psi^*\left(\frac{t-\tau}{a}\right)dt,\ \text{where}\ a=a_0^m,\ \text{and}\ a_0=1.03.$$

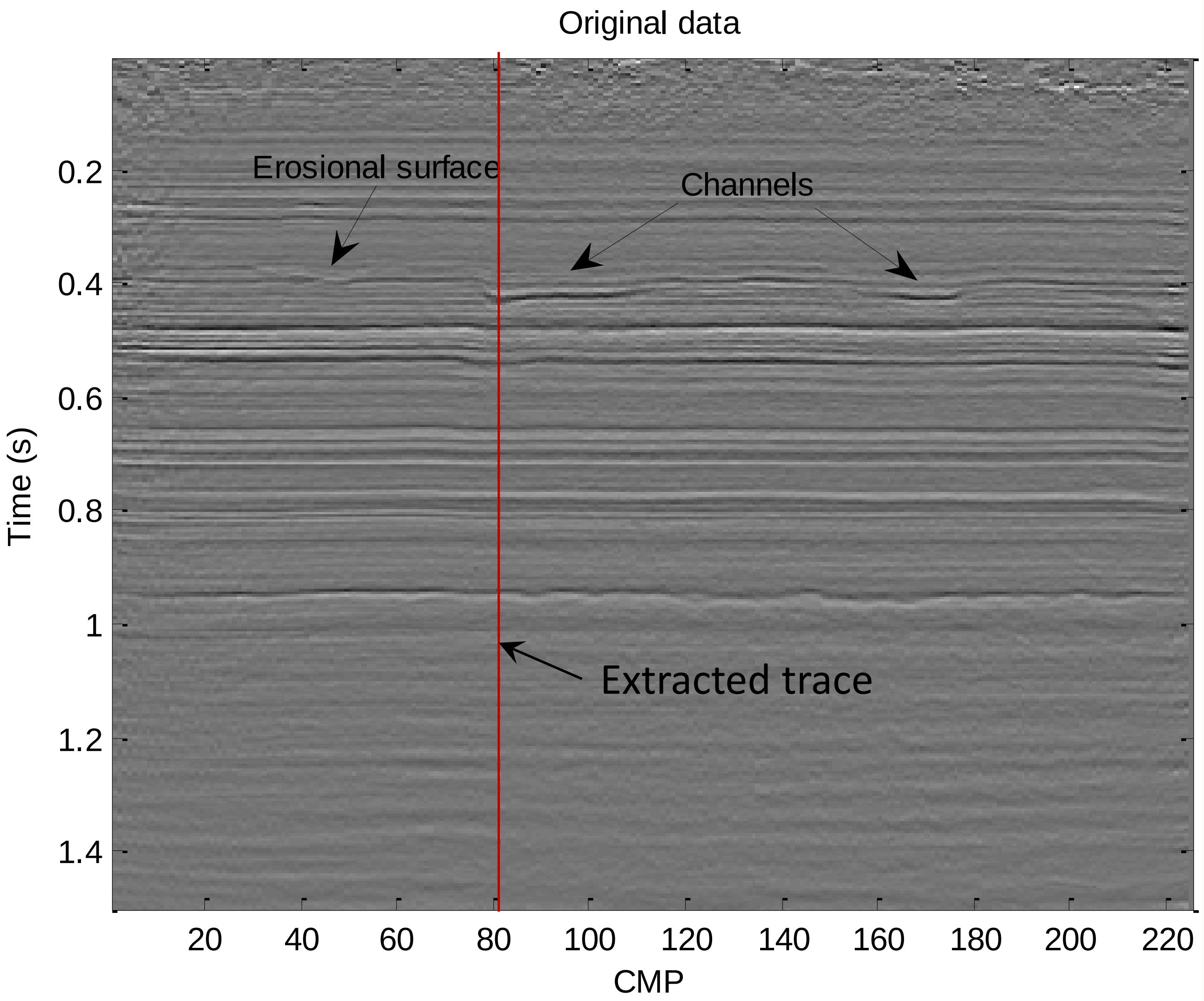
- The term a is the scale, and $\psi(t)$ is called the "mother" wavelet, defined by:

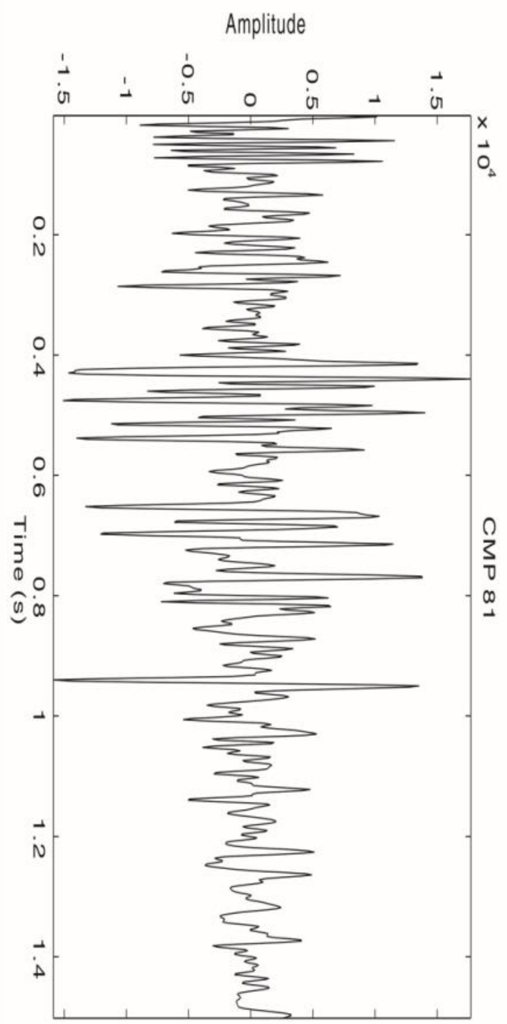
$$\psi(t)=\pi^{-1/4}e^{-t^2/2}\left(e^{-i\omega_0 t}-e^{-\omega_0^2/2}\right),\ \text{where}\ \omega_0=\sqrt{\frac{2}{\ln 2}}.$$

- Although the concepts of scale and shape have been preserved, and the wavelet improved by the subtraction of a scaling term, the mathematical formulation has lost all the physical intuition that Morlet used.

Field data example with CMP 81 extracted

- Finally, we will look at a field data example, where the 2D section is shown to the immediate right and one extracted trace, at CMP 81, on the far right.
- Note that the extracted trace crosses one of the channels at 0.42 sec.





Extracted trace

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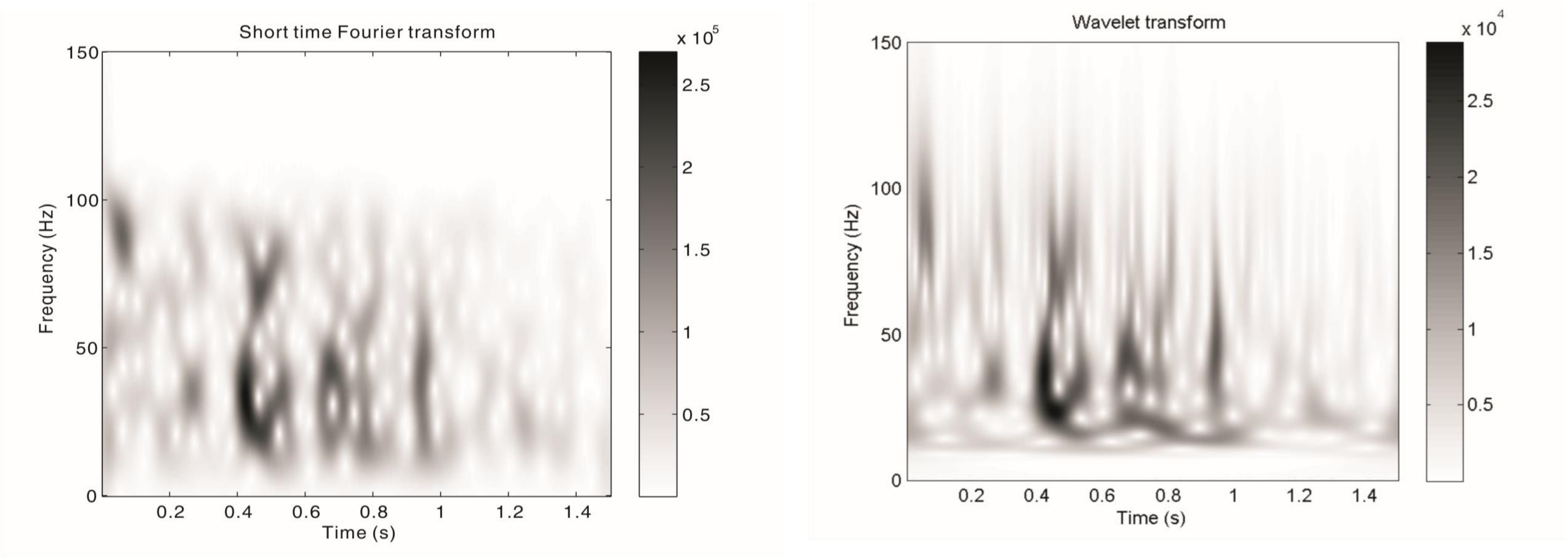
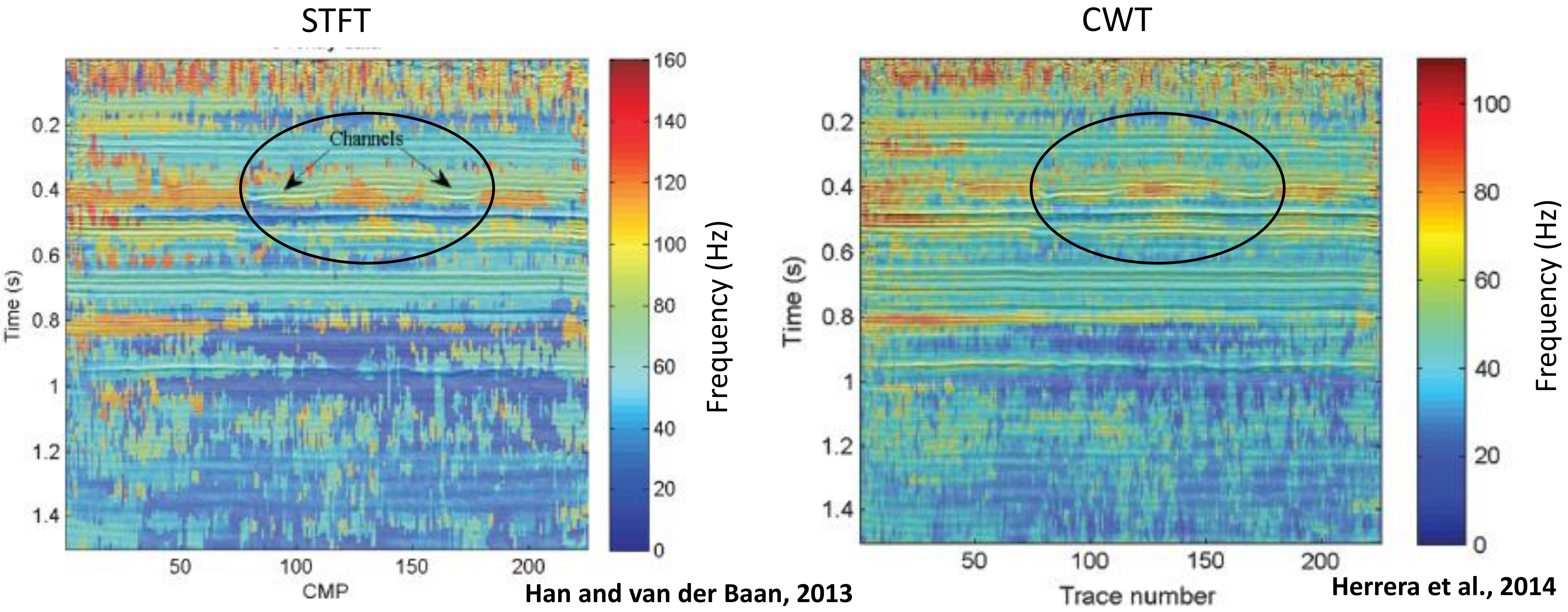
Short time Fourier transform (STFT) with 50 ms window and CWT



Han and van der Baan, 2013

- The STFT of CMP 81 is shown on the left and the CWT of CMP 81 on the right.
- Note the improvement in resolution on the CWT.

STFT with 50 ms window and CWT



Han and van der Baan, 2013

Herrera et al., 2014

- The STFT of the complete seismic section is shown on the left and the CWT of the full section on the right, using 350 scales starting at 150 Hz.
- Note the improvement in resolution of the channels on the CWT.

Conclusions

- In this talk, I have shown how Morlet formulated a new approach to seismic frequency analysis in his classic 1982 papers.
- Morlet took a number of concepts that were familiar to geophysicists at the time, such as wavelets, the Fourier transform, logarithmic bandwidth and cross-correlation, and put them together in a totally new way.
- His conceptual approach was then formalized by mathematicians into a comprehensive new theory called the Continuous Wavelet Transform, or CWT.
- Although the new theory was much more complete than Morlet's original theory, it was also much less rooted in physical intuition.
- Thus, there is a need for both geophysicists and mathematicians in this world!
- A real data example showed the improvement in resolution obtained using the CWT over the Short Time Fourier Transform (STFT).

Acknowledgements

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