Linear algebra for microseismic sensitivity analysis

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Abstract

Two papers have previously been presented for evaluating the sensitivity of locating a microseismic event: a Monte Carlo method that perturbed the geometry, and a linear algebra method that used singular value decomposition (SVD).

These papers have been combined to be used as a tool to understand the linear algebra behind the SVD method.

Theory

The traveltime equations for raypaths between a source at (x_0, y_0, z_0) and four arbitrarily located receivers at (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), and (x_4, y_4, z_4) are:

$$\begin{aligned}(x_1-x_0)^2+(y_1-y_0)^2+(z_1-z_0)^2&=v^2(t_1-t_0)^2\\(x_2-x_0)^2+(y_2-y_0)^2+(z_2-z_0)^2&=v^2(t_2-t_0)^2\\(x_3-x_0)^2+(y_3-y_0)^2+(z_3-z_0)^2&=v^2(t_3-t_0)^2\\(x_4-x_0)^2+(y_4-y_0)^2+(z_4-z_0)^2&=v^2(t_4-t_0)^2\end{aligned}\qquad(1)$$

If the velocities and traveltimes are known, the location of a microseismic event can be computed. Any noise or error in the estimated arrival times (jitter) at the receivers will produce an error in the source location. If we know the distribution of the noise, then we can define the sensitivity of the location, relative to that noise.

The first method uses a Mote Carlo approach where many tests are conducted when noise is added to the receiver traveltimes. The second method uses Linear Algebra to compute the distribution.

Monte Carlo result

One hundred test were conducted where random noise (jitter) was added to the receiver clock-times. The corresponding source locations were computed and plotted in the displays below. Note the sensitivity in the depth of the estimated locations.

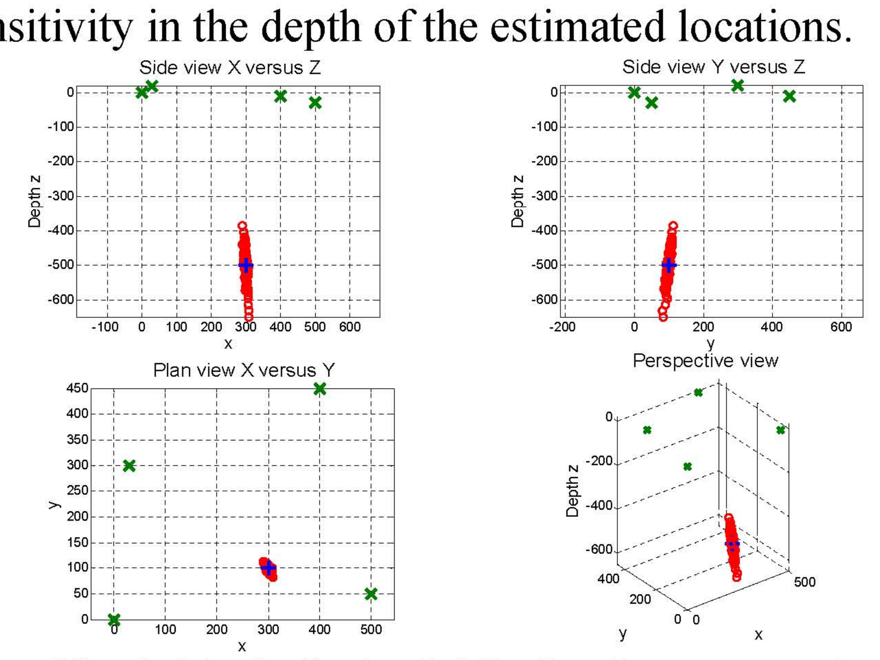

FIG. 1 Monte Carlo distribution for a source location.

Linear algebra

Linear algebra was also used to estimate the error distribution of a microseismic event when jitter is added to the receiver clock-times. The variance of the noise is passed through the covariance matrix to analytically produce the distribution shown below.

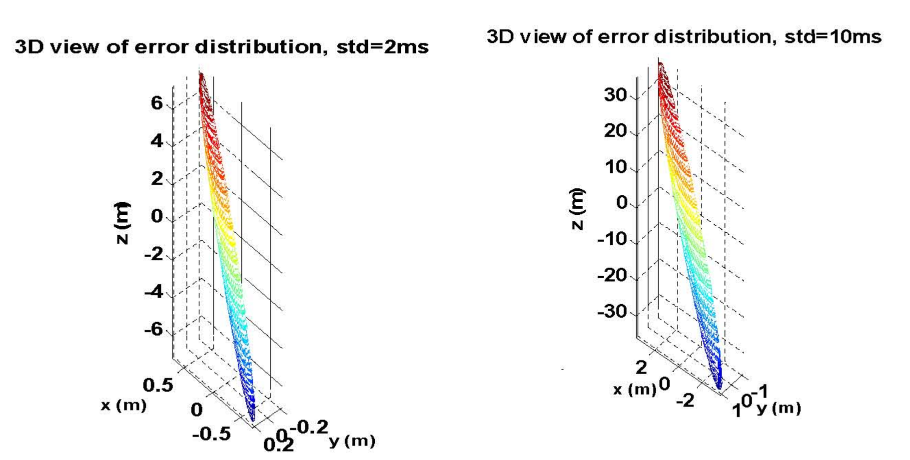

FIG. 1 Linear algebra distribution of a source location.

Intersection of planes

Consider the equation of a plane in 3D space of x, y, and z,

$$ax+by+cz=d,\qquad(1)$$

as shown in Figure 2a. Two planes intersect to produce a line in (b) and three planes intersect at a point in (c).

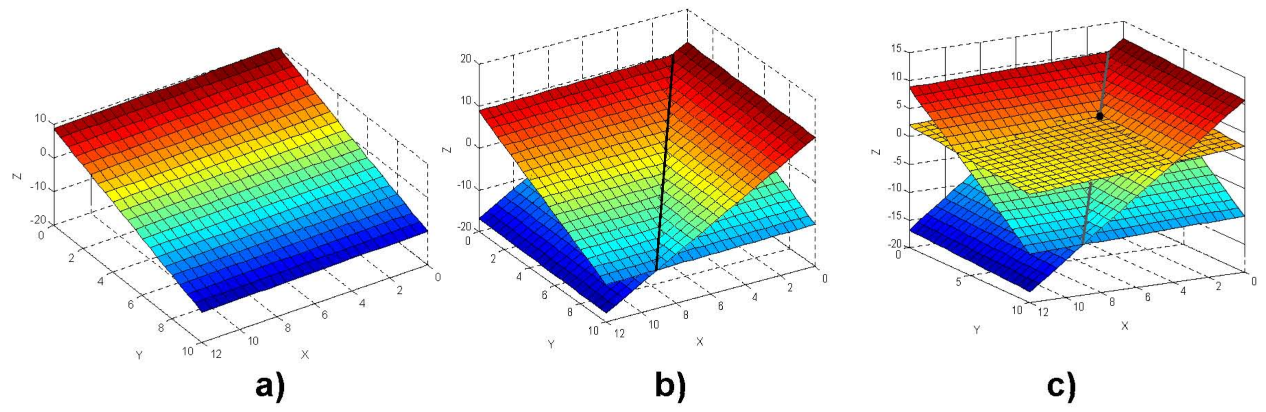

FIG. 2 The intersection of planes.

The equations for the three planes are

$$\begin{aligned}a_1x+b_1y+c_1z&=d_1\\a_2x+b_2y+c_2z&=d_2,\\a_3x+b_3y+c_3z&=d_3\end{aligned}\qquad(2)$$

The location of the intersection point is found by solving these equation for x, y, and z. Written in linear equation form

$$\begin{bmatrix}a_1&b_1&c_1\\a_2&b_2&c_2\\a_3&b_3&c_3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}d_1\\d_2\\d_3\end{bmatrix}.\qquad(3)$$

$$\mathbf{G}\qquad\mathbf{x}\ =\ \mathbf{d}$$

and the solution is found from

$$\mathbf{x}=\mathbf{G}^{-1}\mathbf{d}.\qquad(4)$$

Gauss elimination

Assuming the three equations are linearly independent, i.e. they are not parallel, we can then combine the equations by scaling and addition to solve for **x**.

Jacobi

The **G** matrix is broken into diagonal, upper and lower matrix

$$[\mathbf{G}]=\begin{bmatrix}a_1&b_1&c_1\\a_2&b_2&c_2\\a_3&b_3&c_3\end{bmatrix}=\begin{bmatrix}a_1&0&0\\0&b_2&0\\0&0&c_3\end{bmatrix}+\begin{bmatrix}0&b_1&c_1\\0&0&c_2\\0&0&0\end{bmatrix}+\begin{bmatrix}0&0&0\\a_2&0&0\\a_3&b_3&0\end{bmatrix}.\qquad(5)$$

$$[\mathbf{G}]\quad=\quad[\mathbf{D}]\quad+\quad[\mathbf{U}]\quad+\quad[\mathbf{L}]$$

Then written as

$$(\mathbf{D}+\mathbf{U}+\mathbf{L})\mathbf{G}\mathbf{x}=\mathbf{d},\qquad(6)$$

$$\mathbf{D}\mathbf{x}=\mathbf{d}-(\mathbf{U}+\mathbf{L})\mathbf{x},\qquad(7)$$

$$\mathbf{x}_i=\mathbf{D}^{-1}\left(\mathbf{d}-(\mathbf{U}+\mathbf{L})\mathbf{x}_{i-1}\right).\qquad(8)$$

The last equation is written in an iterative form.

Gauss-Seidel

Equation (6) can be re-written in a form to update the parameters as soon as they are computed,

$$\mathbf{x}_i=(\mathbf{D}+L)^{-1}\left(\mathbf{d}-\mathbf{U}\mathbf{x}_{i-1}\right).\qquad(9)$$

Least squares and the Covariance matrix

Consider six points on a straight line, where there is noise on the y coordinate. If we know the standard deviation (SD) of the noise, we can plot the data in Figure 3, where the vertical axis displays the distribution of the noise for a SD of 0.9 and 0.3. The original line is in gray, and "x" marks the location of the observed value.

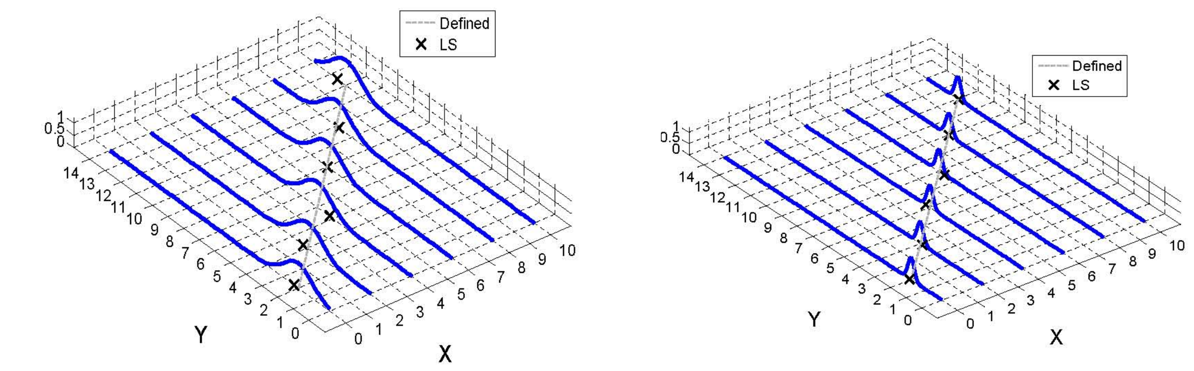

FIG. 3 Six point and their noise distributions of 0.9 and 0.3.

We use these points to estimate the equation of the line $y_1=\hat{m}x_1+\hat{c}$ where we have more points (observations) than unknowns (m, c). The solution is typically found using least squares, where the equations are written in matrix form

$$\mathbf{G}\mathbf{x}=\mathbf{y},\qquad(10)$$

and the least squares solution

$$\mathbf{x}=\left(\mathbf{G}^{\mathrm{T}}\mathbf{G}\right)^{-1}\mathbf{G}^{\mathrm{T}}\mathbf{y}.\qquad(11)$$

is shown as the red line in Figure 4

Consider Figure 4 again in which we have five points with a SD of 0.9, and one with 0.09. We weight the equations inversely proportional to the SDs with a weight vector **w**

$$\mathbf{w}\mathbf{G}\mathbf{x}=\mathbf{w}\mathbf{y},\qquad(12)$$

or the matrix **W** giving a least squares solution

$$\mathbf{G}^{\mathrm{T}}\mathbf{W}^{\mathrm{T}}\mathbf{W}\mathbf{G}\mathbf{x}=\mathbf{G}^{\mathrm{T}}\mathbf{W}^{\mathrm{T}}\mathbf{y},\qquad(13)$$

$$\mathbf{x}=\left(\mathbf{G}^{\mathrm{T}}\mathbf{W}^{\mathrm{T}}\mathbf{W}\mathbf{G}\right)^{-1}\mathbf{G}^{\mathrm{T}}\mathbf{W}^{\mathrm{T}}\mathbf{y}.\qquad(14)$$

The weighted least squares (WLS) solution is the cyan line in Figure 3. It is closer to the gray line and passes very close to the fifth solution point.

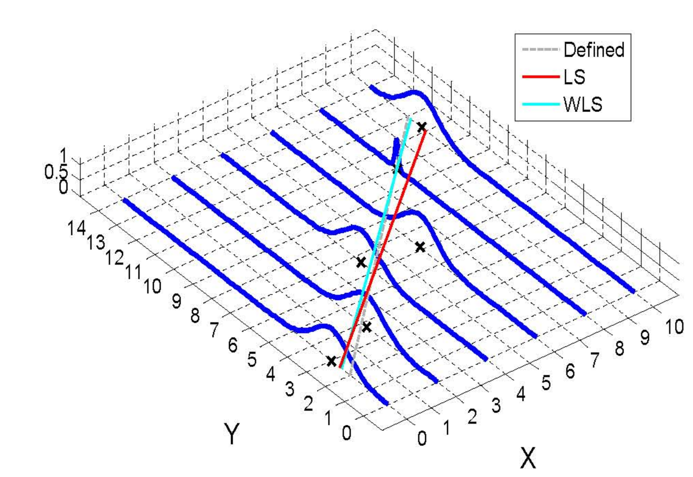

FIG. 4 LS and WLS solution.

We call the $\mathbf{C}=\mathbf{W}^{\mathrm{T}}\mathbf{W}$ matrix the Covariance matrix at the diagonal elements contain the variance of the observations.

The next steps are to proceed with eigenvectors and singular value decomposition (SVD) to solve our stability problem.