Minimum phase and attenuation models in continuous time

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Introduction

A common assumption in seismic applications is that transmitted waves are minimum phase, in the case of horizontally stratified absorptive earth with vertically traveling plane compressional waves. Field experiments indicate the source signature of a dynamite blast is itself minimum phase. This suggests a mathematical inverse problem where we seek to identify the specific linear operator that represents wave propagation through the earth. How do we characterize such an operator? It turns out any minimum phase preserving operator must be in the specific form of a product-composition operator in the Laplace transform domain. How do we determine what the operator is? It turns out that two distinct measurements are required.

We extend our earlier work on minimum phase attenuation models in discrete-time signal processing to the continuous time setting, where real physical processes occur. It is appropriate to ask the analogous questions about such physical linear operators in this continuous time setting, which is what is done in the present work.

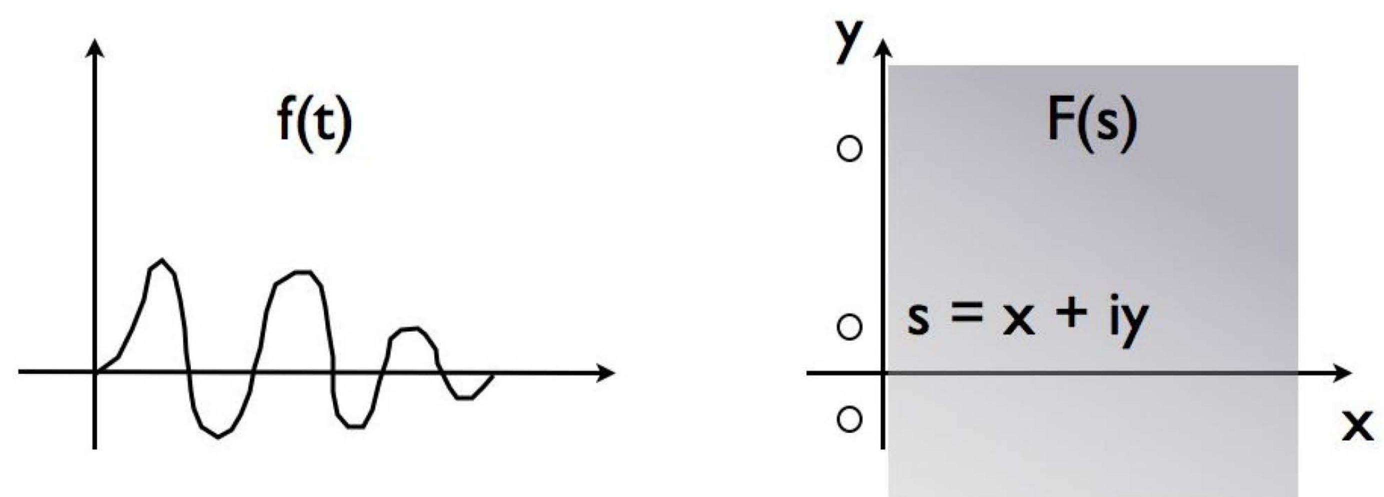
Minimum phase, outer functions

In continuous time, a causal signal $f \in L^2(\mathbb{R}^+)$ is represented by its Laplace transform in the s-domain,

$$F(s) = \int_0^\infty f(t)e^{-ts}\,dt,$$

which defines an analytic function on the right half of the complex plane

$$\{s \in \mathbb{C} : Re(s) > 0\}.$$



A causal signal and domain of s-transform.

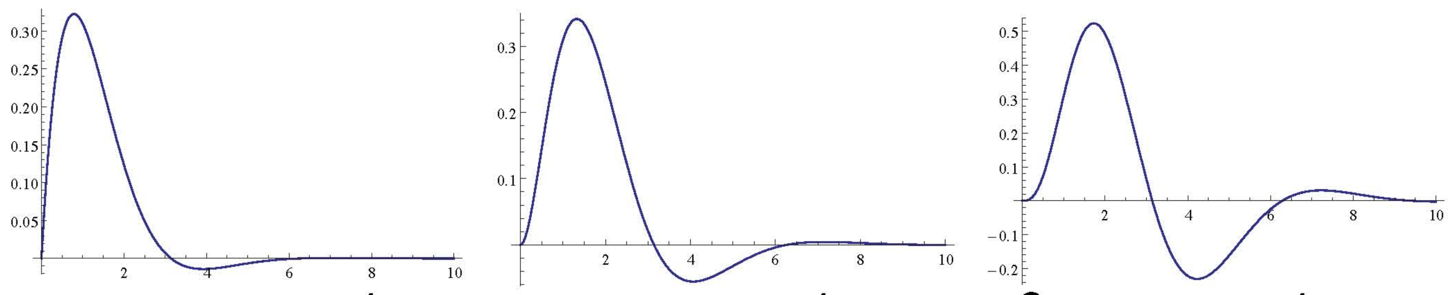
Using the analogy from the discrete time case, a causal signal is considered minimum phase if its Laplace transform (or s-transform) is an outer function on the right half-plane of the complex plane.
An outer function has the property that it maximizes the energy at the start of the signal, of all signals with a specified Fourier power spectrum. More precisely, the analytic function $F(s)$ is outer if its values on the right half plane are determined by its values on the imaginary axis,

$$F(s) = \lambda \exp\left(\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{ys+i}{y+is} \log|F(iy)| \frac{dy}{1+y^2}\right), \text{ for } Re(s) > 0.$$

A signal is minimum phase if its s-transform is completely determined by the log amplitude (Fourier) spectrum of the signal. Equivalently, the amplitude and phase of the Fourier transform are related by the Kramer-Kronig relations.
When the s-transform is a rational function, this is the same are requiring all the roots and poles to lie on the left half plane. Not all s-transforms are rational, however.

Min phase signals

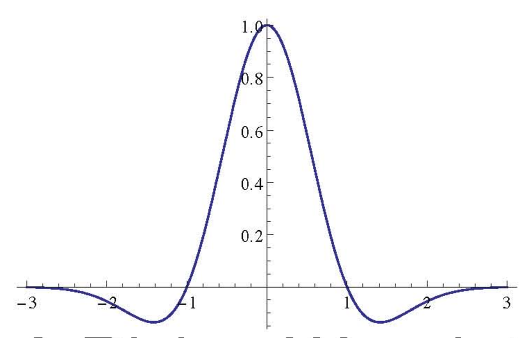
Minimum phase signals in continuous time are easily constructed using polynomials with trig functions and decaying exponentials. Three examples are given here, with the corresponding transforms:



$$\sin(t)e^{-t}, \qquad t\sin(t)e^{-t}, \qquad t^2\sin(t)e^{-t}.$$

$$\frac{1}{s^2+2s+2}, \qquad \frac{2(s+1)}{(s^2+2s+2)^2}, \qquad \frac{2(3s^2+6s+2)}{(s^2+2s+2)^3}$$

Observing that the s-transform has roots and poles only in the left half plane confirms that these signals are minimum phase.
In general, any causal, finite energy signal has a minimum phase equivalent. Smoother waveforms are better for numerical simulations. The Ricker wavelet $w(t) = (1-t^2)\exp(-t^2)$ is zero-phase, not minimum phase.



A Ricker Wavelet

Curiously, the Ricker wavelet has no exact minimum phase equivalent, as its Fourier transform decreases too rapidly at infinity. Any calculation of a minimum phase equivalent is necessarily only an approximation.

Min phase preserving operators, and Q-atten

Our key result is the following:

Theorem

Let $A : H^2(\mathbb{C}_+) \to H^2(\mathbb{C}_+)$ be a bounded linear operator that preserves the set of shifted outer functions. Then A is a product-composition,

$$A = M_\psi C_\phi,$$

where M_ψ is multiplication by shifted outer function ψ and C_ϕ is right composition with a shifted outer function ϕ on the right half plane.

The depth of this result is that a minimum phase preserving operator in continuous time is necessarily of this form. In particular, minimum phase preserving implies Q-attenuation.
That is, in the s-domain, transform $F(s)$ is mapped to

$$\psi(s)F(\phi(s)) = \psi(s)\int_0^\infty f(t)e^{-t\phi(s)}\,dt.$$

The physical interpretation is that the operator contains a stationary filter determined by ψ, with an attenuation term determined by ϕ.
With $s = iy$ on the imaginary axis, at frequency y, the function $f(t)$ is locally transformed to

$$f(t)e^{-t\phi(iy)}.$$

This is Q-attenuation, with the real part of the exponent identified as

$$\Re(-t\phi(iy)) = -\pi ty/Q(y),$$

to give the frequency-dependent Q parameter.

Identification of operators

A min phase preserving operator is necessarily a product-composition

$$A = M_\psi C_\phi.$$

Recovering the two functions ψ, ϕ will specify the operator entirely.
Causal signal $f(t) = e^{-t}$ has Laplace transform $F(s) = \frac{1}{s+1}$. Thus

$$(AF)(s) = \psi(s)\frac{1}{\phi(s)+1}.$$

Causal signal $g(t) = te^{-t}$ has Laplace transform $G(s) = \frac{1}{(s+1)^2}$. Thus

$$(AG)(s) = \psi(s)\frac{1}{(\phi(s)+1)^2}.$$

The ratio recovers ϕ:

$$\frac{AF}{AG}(s) = \phi(s) + 1,$$

and the product of AF with this ratio recovers ψ:

$$(AF)(s)\frac{AF}{AG}(s) = \psi(s).$$

By measuring the output of operator A on only two causal signals

$$f(t) = e^{-t}, \qquad g(t) = te^{-t},$$

we completely recover operator $A = M_\psi C_\phi$.

Conclusions

Previous work on minimum phase preserving linear operators in discrete time is extended to the continuous time case. The geophysical problem of modelling the propagation of seismic energy through the earth is represented by linear operators which preserves the minimum phase property of signals, identified as outer functions in the Laplace transform domain. Such operators are necessarily product-composition operators, and as such, are uniquely determined by two analytic functions. The first function, ψ, represent a stationary filter, and the second, ϕ, represents a frequency dependent Q-attenuation. Such an operator is uniquely characterized by its action on two specific causal signals, the decaying exponential $\exp(-t)$ and its polynomial counterpart $t\exp(-t)$. That is, the Q-attenuation and filtering are determined by geophysical measurements of two received signals.

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