Galerkin methods for numerical solutions of acoustic, elastic and viscoelastic wave equations

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Overview

This work is a summary of highlights from the 2012 MSc thesis of the first author, which aims to bridge the gap between the development of accurate physical models and the implementation of modern numerical techniques for the accurate solutions of partial differential equations. We apply state-of-the-art numerical methods based on domain-decomposition combined with local pseudospectral spatial discretization, to three physically realistic models of seismic waves, in acoustic, elastic, and viscoelastic media.

Galerkin method

The Galerkin method seeks to find a solution $\mathbf{u}$ to differential equation $L[\mathbf{u}] = \mathbf{f}$ as a linear combination of basis functions ϕ_i,

$$\mathbf{u} = \sum_i a_i \phi_i.$$

The measure of the residual $R[u] = L[\mathbf{u}] - \mathbf{f}$ should then be zero, thus

$$\int_\Omega R[\mathbf{u}]\phi_j \, dx = 0, \text{ for all } j,$$

or,

$$\sum_i a_i \int_\Omega L[\phi_i]\phi_j \, dx = \int_\Omega \mathbf{f}\phi_j \, dx, \text{ for all } j.$$

For certain problems, this is a Rayleigh-Ritz minimization technique. It reduces to a large N-dimensional system of equations to be solved,

$$K\mathbf{a} = \mathbf{f}.$$

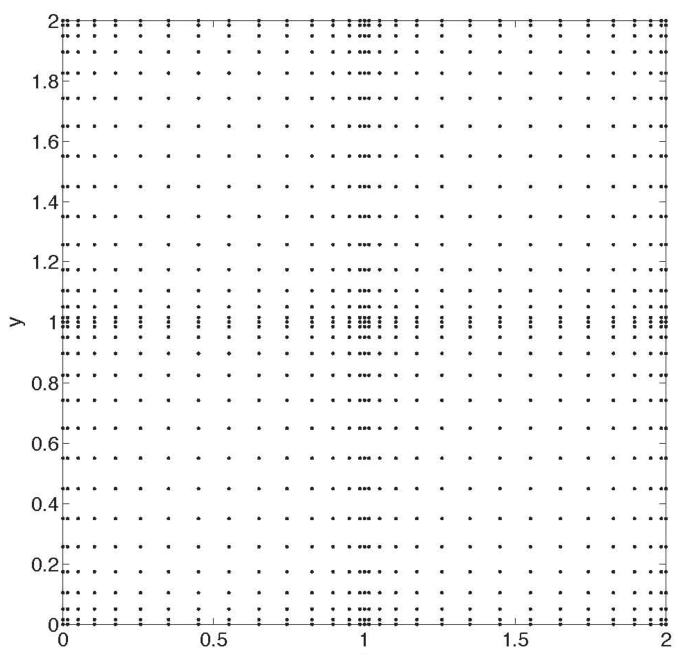
For the time evolution of a wave propagating through a medium, time and space are treated separately. A large system of ODEs results,

$$M\ddot{\mathbf{u}} + K\mathbf{u} = 0, t > 0,$$
$$\mathbf{u}(0) = \mathbf{u}_0,$$
$$\dot{\mathbf{u}}(0) = \mathbf{u}_1,$$

where M is the called the mass matrix, and K the stiffness matrix.

Nodes

The basis functions and their derivatives/integral are determined by values at nodes in the domain. For optimal performance, the nodes are not on a uniform grid, but based on a selection of Gauss-Lobato points. It can be advantageous to select a clustering of grid points near boundaries and interfaces.



A sample 2D grid of nodes for Galerkin.

A good selection of grid points avoids the Runge phenomena and leads to higher accuracy with fewer grid points.

Wave Equations

The elastic wave equation is expressed with a vector valued displacement function $\mathbf{u}$ and forcing function $\mathbf{f}$. The differential system is

$$\rho(\mathbf{x})\ddot{u}_i(\mathbf{x}, t) = \sum_j \frac{\partial}{\partial x_j}\sigma_{ij}(\mathbf{u}) + f_i, \quad \mathbf{x} \in \Omega, t > 0, i = 1, \ldots, d,$$

where the stresses for the isotropic medium are

$$\sigma_{ij}(\mathbf{u}) = \lambda(\nabla \cdot \mathbf{u})\delta_{ij} + \mu(\frac{\partial}{\partial x_i}u_j + \frac{\partial}{\partial x_j}u_i),$$

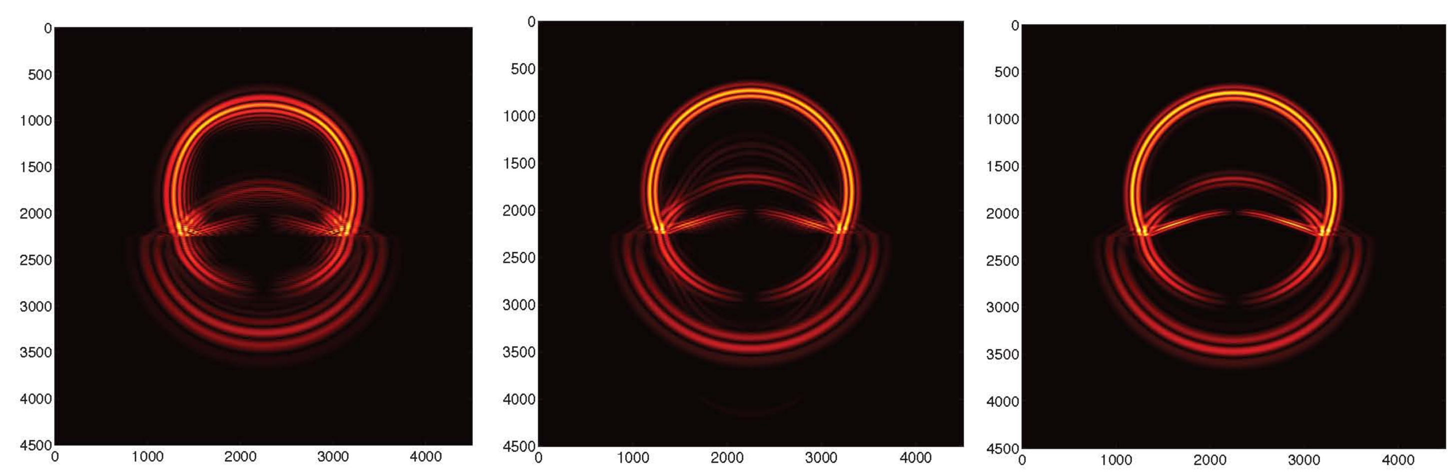
with λ, μ the elastic parameters for the medium.
In the viscoelastic case, the material response for stresses becomes

$$\sigma_{ij}(\mathbf{u}) = \lambda(\nabla \cdot \mathbf{u})\delta_{ij} + \mu(\frac{\partial}{\partial x_i}u_j + \frac{\partial}{\partial x_j}u_i) + \lambda'\nabla \cdot \dot{\mathbf{u}}\delta_{ij} + \mu'\frac{\partial}{\partial t}(\frac{\partial}{\partial x_i}u_j + \frac{\partial}{\partial x_j}u_i),$$

where λ', μ' are additional viscoelastic parameters.

Elastic solutions

A nodal Galerkin method is compared to second and fourth order finite difference methods on a 501 by 501 node grid. A forcing term with Ricker wavelet time-component and conservative spatial component is used to propagate a 15 Hz wavelet in a 4500m square bipartite medium with properties $\rho = 2.064 g/cm^3$, $V_p = 2305 m/s$, $V_s = 997 m/s$ in the first layer, and $\rho = 2.14 g/cm^3$, $V_p = 4500 m/s$, $V_s = 2600 m/s$ in the second layer.

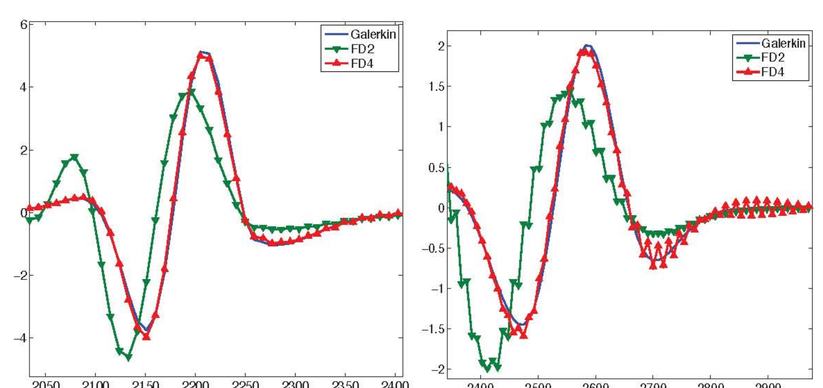


2nd order FD, 4th order FD, Galerkin solutions.

The figures show the norm of the displacement for the three models propagated to one second and then normalized and clipped to exaggerate the dispersion effects.

Galerkin and Finite Difference

The above numerical solutions are compared at time $t = 1$, near different regions in the domain.

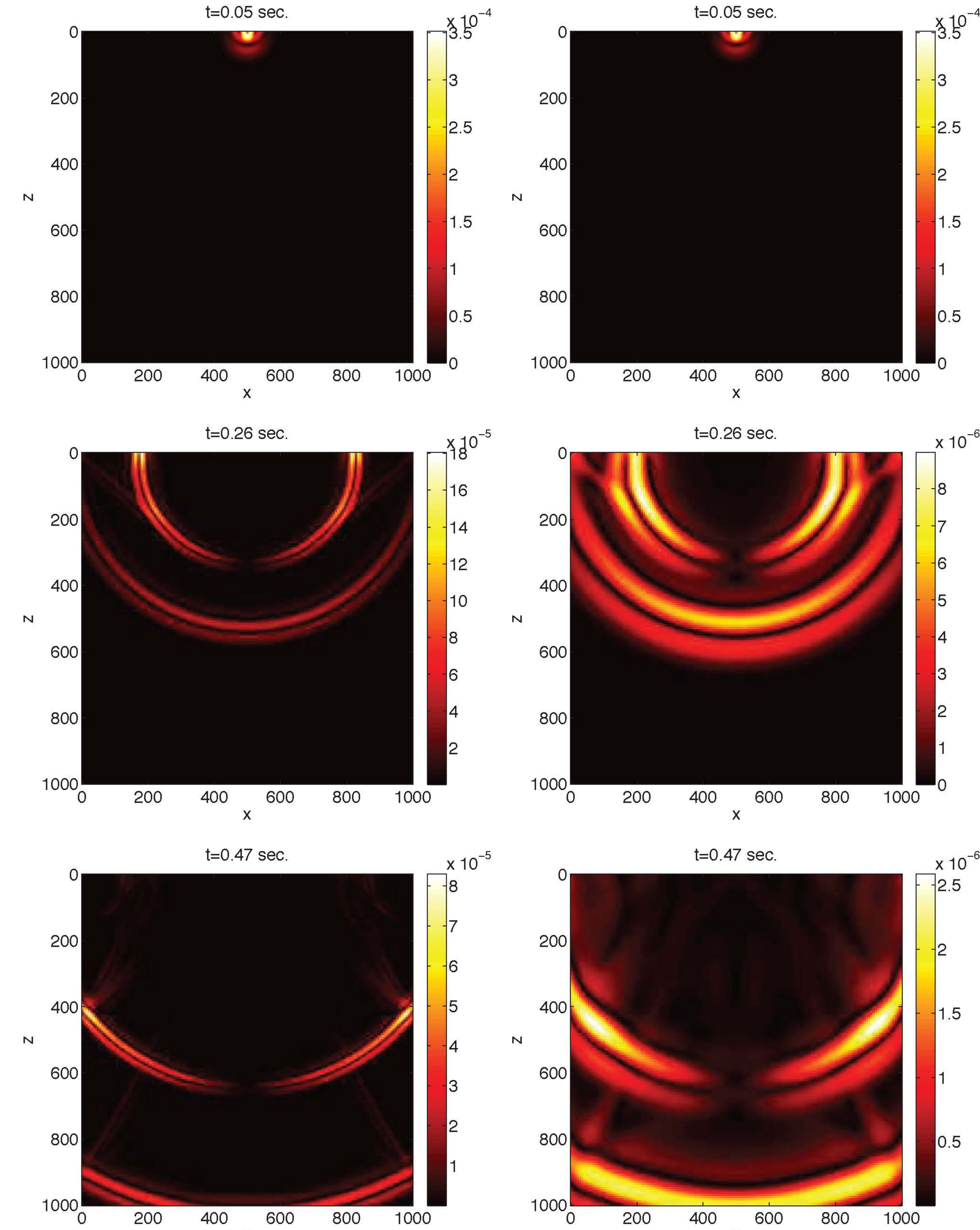


Comparison of waveforms at smooth, and abrupt interfaces.

The amplitude error associated with the second-order stencil is apparent, as is the dispersion of all three methods near a jump. Note the ringing that is apparent for both finite difference results.

Viscoelastic solution

A demonstration of the Galerkin method used with accurate physical models is given. Left and right show wave propagation of a single impulse through a two-layer elastic medium and through a two-layer viscoelastic medium with damping parameters $Q_p = 24$, $Q_s = 16$. Note physically relevant effects: of wavelength broadening with propagation.



Propagation in 2-layer elastic, viscoelastic media.

Conclusions

A Galerkin method is a feasible and computationally efficient method for the numerical modelling of several types of seismic waves. Also in the thesis is the treatment of numerically imposed boundaries and interfaces using the weak form of the dynamic equilibrium equations.

Bibliography

McDonald, M. A., 2012, Numerical methods in seismic wave propagation: University of Calgary MSc thesis.

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