Minimized energy used to determine displacements at an internal boundary

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ABSTRACT

A method for calculation of internal boundary conditions, as opposed to edge boundary conditions, is explained. The method of minimizing energy within the rigid zone below a water bottom is developed, and the sequence of matrix equations required is presented in some detail.

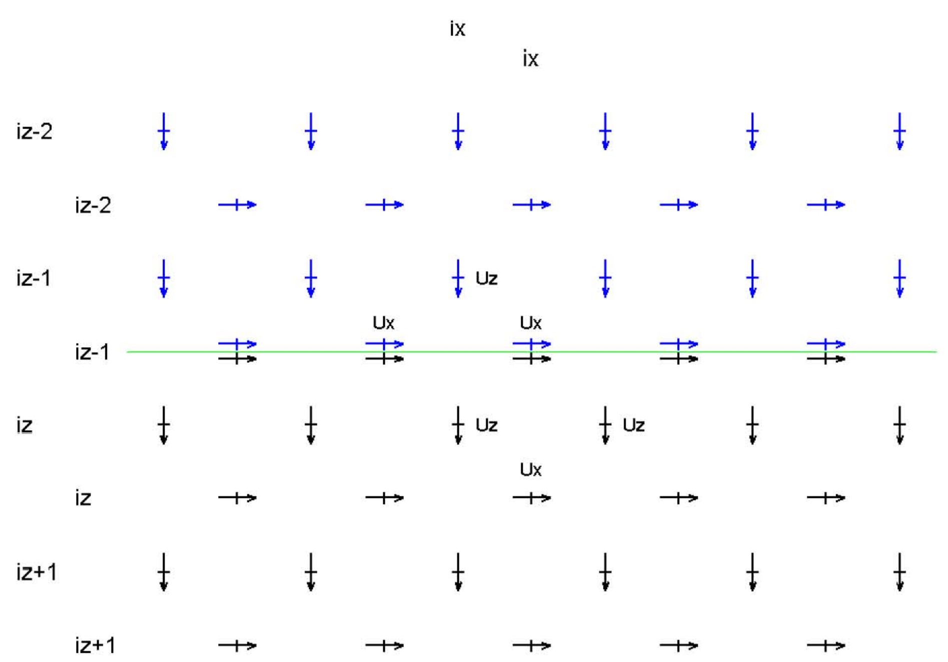

Figure 1: The displacements in the vicinity of the green line depicting the water bottom. It is assumed that at this point within a time step, all the Z-displacements have been calculated, and also all the X-displacements above and for one sample below the water bottom and lower. The X-displacements just below the water bottom must then be made consistent with the appropriate boundary conditions.

$$\begin{bmatrix} \mu_1 & 0 & 0 & 0 & 0 & \mu_1 Xs_1 \\ 0 & \mu_2 & 0 & 0 & 0 & \mu_2 Xs_2 \\ 0 & 0 & \mu_3 & 0 & 0 & \mu_3 Xs_3 \\ 0 & 0 & 0 & \mu_4 & 0 & \mu_4 Xs_4 \\ 0 & 0 & 0 & 0 & \mu_5 & \mu_5 Xs_5 \\ L_1 & 0 & 0 & 0 & 0 & L_1 Zc_1 \\ -L_2 & L_2 & 0 & 0 & 0 & L_2 Zc_2 \\ 0 & -L_3 & L_3 & 0 & 0 & L_3 Zc_3 \\ 0 & 0 & -L_4 & L_4 & 0 & L_4 Zc_4 \\ 0 & 0 & 0 & -L_5 & L_5 & L_5 Zc_5 \\ 0 & 0 & 0 & 0 & -L_6 & L_6 Zc_6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ 1 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_9 \\ f_{10} \\ f_{11} \end{bmatrix}$$

Figure 2: An abbreviated version of a matrix equation which shows how the Ux displacements immediately below the water bottom contribute to the shear and compressional stresses within the water bottom. The X vector specifies input displacement strains. Outputs f_1 through f_5 represent shear stresses, and f_6 through f_{11} represent compressional stresses.

$$\begin{bmatrix} \mu_1 & 0 & 0 & 0 & 0 & L_1 & -L_2 & 0 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 & 0 & 0 & L_2 & -L_3 & 0 & 0 & 0 \\ 0 & 0 & \mu_3 & 0 & 0 & 0 & 0 & L_3 & -L_4 & 0 & 0 \\ 0 & 0 & 0 & \mu_4 & 0 & 0 & 0 & 0 & L_4 & -L_5 & 0 \\ 0 & 0 & 0 & 0 & \mu_5 & 0 & 0 & 0 & 0 & L_5 & -L_6 \\ \mu_1 Xs_1 & \mu_2 Xs_2 & \mu_3 Xs_3 & \mu_4 Xs_4 & \mu_5 Xs_5 & L_1 Zc_1 & L_2 Zc_2 & L_3 Zc_3 & L_4 Zc_4 & L_5 Zc_5 & L_6 Zc_6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & Xs_1 \\ 0 & 1 & 0 & 0 & 0 & Xs_2 \\ 0 & 0 & 1 & 0 & 0 & Xs_3 \\ 0 & 0 & 0 & 1 & 0 & Xs_4 \\ 0 & 0 & 0 & 0 & 1 & Xs_5 \\ 1 & 0 & 0 & 0 & 0 & Zc_1 \\ -1 & 1 & 0 & 0 & 0 & Zc_2 \\ 0 & -1 & 1 & 0 & 0 & Zc_3 \\ 0 & 0 & -1 & 1 & 0 & Zc_4 \\ 0 & 0 & 0 & -1 & 1 & Zc_5 \\ 0 & 0 & 0 & 0 & -1 & Zc_6 \end{bmatrix}$$

Figure 3: The components of the Toeplitz energy matrix. The matrix from Figure 2 is transposed and appears on the left of this Figure. The matrix on the right has the same form as the one in Figure 2, but the elastic constants are replaced by a +1 or -1, so that it may calculate displacements instead of stresses.

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & 1 \end{bmatrix} \begin{bmatrix} \mu_1+L_{12} & -L_2 & 0 & 0 & 0 & C_1 \\ -L_2 & \mu_2+L_{23} & -L_3 & 0 & 0 & C_2 \\ 0 & -L_3 & \mu_3+L_{34} & -L_4 & 0 & C_3 \\ 0 & 0 & -L_4 & \mu_4+L_{45} & -L_5 & C_4 \\ 0 & 0 & 0 & -L_5 & \mu_5+L_{56} & C_5 \\ C_1 & C_2 & C_3 & C_4 & C_5 & \Sigma C_i^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ 1 \end{bmatrix} = E$$

$$L_{ij} = L_i + L_j$$

$$C_i = \mu_i Xs_i + L_i Zc_i - L_{i+1} Zc_{i+1}$$

Figure 4: The matrix equation which sums the total elastic energy (E) in the water bottom. The square Toeplitz matrix here in the center results from multiplying the two matrices of Figure 3, and there is now room to display the two displacement vectors to complete the equation.

$$\partial/\partial x_4 \begin{bmatrix} + a_{11}x_1x_1 & + a_{21}x_2x_1 & + a_{31}x_3x_1 & + a_{41}x_4x_1 & + a_{51}x_5x_1 & + a_{61}x_1 \\ + a_{12}x_1x_2 & + a_{22}x_2x_2 & + a_{32}x_3x_2 & + a_{42}x_4x_2 & + a_{52}x_5x_2 & + a_{62}x_2 \\ + a_{13}x_1x_3 & + a_{23}x_2x_3 & + a_{33}x_3x_3 & + a_{43}x_4x_3 & + a_{53}x_5x_3 & + a_{63}x_3 \\ + a_{14}x_1x_4 & + a_{24}x_2x_4 & + a_{34}x_3x_4 & + a_{44}x_4x_4 & + a_{54}x_5x_4 & + a_{64}x_4 \\ + a_{15}x_1x_5 & + a_{25}x_2x_5 & + a_{35}x_3x_5 & + a_{45}x_4x_5 & + a_{55}x_5x_5 & + a_{65}x_5 \\ + a_{16}x_1 & + a_{26}x_2 & + a_{36}x_3 & + a_{46}x_4 & + a_{56}x_5 & + a_{66} \end{bmatrix} = 0$$

Figure 5: The energy sum of Figure 4 must be differentiated with respect to each X variable to get the value where the energy is minimum. Only the highlighted terms contribute to the particular derivative equation in X_4.

$$\begin{bmatrix} 2\mu_1+2L_{12} & -2L_2 & 0 & 0 & 0 \\ -2L_2 & 2\mu_2+2L_{23} & -2L_3 & 0 & 0 \\ 0 & -2L_3 & 2\mu_3+2L_{34} & -2L_4 & 0 \\ 0 & 0 & -2L_4 & 2\mu_4+2L_{45} & -2L_5 \\ 0 & 0 & 0 & -2L_5 & 2\mu_5+2L_{56} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = - \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{bmatrix}$$

$$L_{ij} = L_i + L_j$$

$$C_i = \mu_i Xs_i + L_i Zc_i - L_{i+1} Zc_{i+1}$$

Figure 6: The equation which results by substitution of the particular terms into the general terms of Figure 5 and by translation back into matrix format. The matrix is essentially identical to the original core matrix in Figure 4 because of the symmetry of the off diagonal terms. Since the X's are the unknowns, the matrix must be inverted.

CONCLUSIONS

Internal boundary conditions are more complex than they at first appear. A better understanding should lead to better water bottom models, and the principles may help with other boundary condition problems.