Minimum-phase signal calculation using the real cepstrum

Adrian D. Smith, Robert J. Ferguson
adrsmith@ucalgary.ca

Abstract

The concept of minimum phase in geophysics is an important one, especially for processes such as statistical deconvolution which assume the condition in the source wavelet. We wish to have an alternative method to the Hilbert transform to convert a signal of arbitrary phase to its minimum phase equivalent, while retaining the same amplitude spectrum. We implement a minimum-phase reconstruction based on the real cepstrum developed for a finite-impulse response (FIR) filters by treating the signal as a filter. We demonstrate that the algorithm is able to handle signals with ill-conditioned amplitude spectra and still give minimum-phase outputs through analysis of pole-zero plots, along with a simple deconvolution test. We also introduce two metrics: the Pole-Zero Ratio (PZR) and Pole-Zero Distance (PZD) as potential quantitative descriptions of how close a signal is to being minimum phase.

Cepstrum Properties

We list several cepstrum properties below and how they relate to the minrceps algorithm presented in Pei and Lin 2006 that we implement (see Oppenheim and Schafer 2009).

Complex Cepstrum:

$$\hat{C}(e^{iw}) = \ln\left[X(e^{iw})\right] = \ln\left|X(e^{iw})\right| + i\arg[X(e^{iw})] \quad (1)$$

$$\hat{c}(n) = \mathcal{F}^{-1}\{\hat{C}(e^{iw})\} \quad (2)$$

Real Cepstrum:

$$\hat{H}(e^{iw}) = Re\{\hat{C}(e^{iw})\} = \ln\left|X(e^{iw})\right| \quad (3)$$

$$\hat{h}(n) = \mathcal{F}^{-1}\{\hat{H}(e^{iw})\} \quad (4)$$

Minimum-Phase Real Sequence from Complex Cepstrum:

$$x(n) = \begin{cases} e^{\hat{c}(0)} & \text{if } n = 0 \\ \displaystyle\sum_{k=0}^{n} \left(\frac{k}{n}\right) \hat{c}(k)x(n-k) & \text{if } n > 0 \end{cases} \quad (5)$$

Reconstruction of Causal Sequence from Even Part:

$$x(n) = x_e(n)u_+(n), \qquad \text{where } u_+(n) = \begin{cases} 0 & \text{if } n < 0 \\ 1 & \text{if } n = 0 \\ 2 & \text{if } n > 0 \end{cases} \quad (6)$$

Relationship between Complex and Real Cepstra:

$$\hat{c}_e(n) = \mathcal{F}^{-1}\{\hat{H}(e^{iw})\} = \hat{h}(n) \quad (7)$$

Minrceps algorithm

1. Perform an L-point FFT on $x_\alpha(n) = \alpha^n x(n)$, where $n = 0, 1, \ldots, (N-1)$, to get $X_\alpha(k)$, $k = 0, 1, \ldots, (L-1)$, where $L \gg 8N$. ($\alpha \leq 1$, $\alpha \cong 1$)
2. Perform an IFFT on $\ln|X_\alpha(k)|$ to get $\hat{h}_\alpha(n)$, which is equal to $\hat{h}_{\alpha,min}(n)$.
3. Construct $\hat{c}_{\alpha,min}(n)$ from $\hat{h}_{\alpha,min}(n)$ using Equations 7 and 6.
4. Calculate $x_{\alpha,min}(n)$ from $\hat{c}_{\alpha,min}(n)$ using the recursion formula in Equation 5.
5. Rescale $x_{\alpha,min}(n)$ to get $x_{min}(n) = x_{\alpha,min}(n)\alpha^{-n}$.

(Pei and Lin, 2006).

Four test cases

Case A: Minimum-phase wavelet

Case B: Mixed-phase wavelet with linear phase

Case C: Mixed-phase wavelet with linear phase and a number of isolated zeroes in the amplitude spectrum

Case D: Mixed-phase wavelet with linear phase and an interval of zeroes in the amplitude spectrum

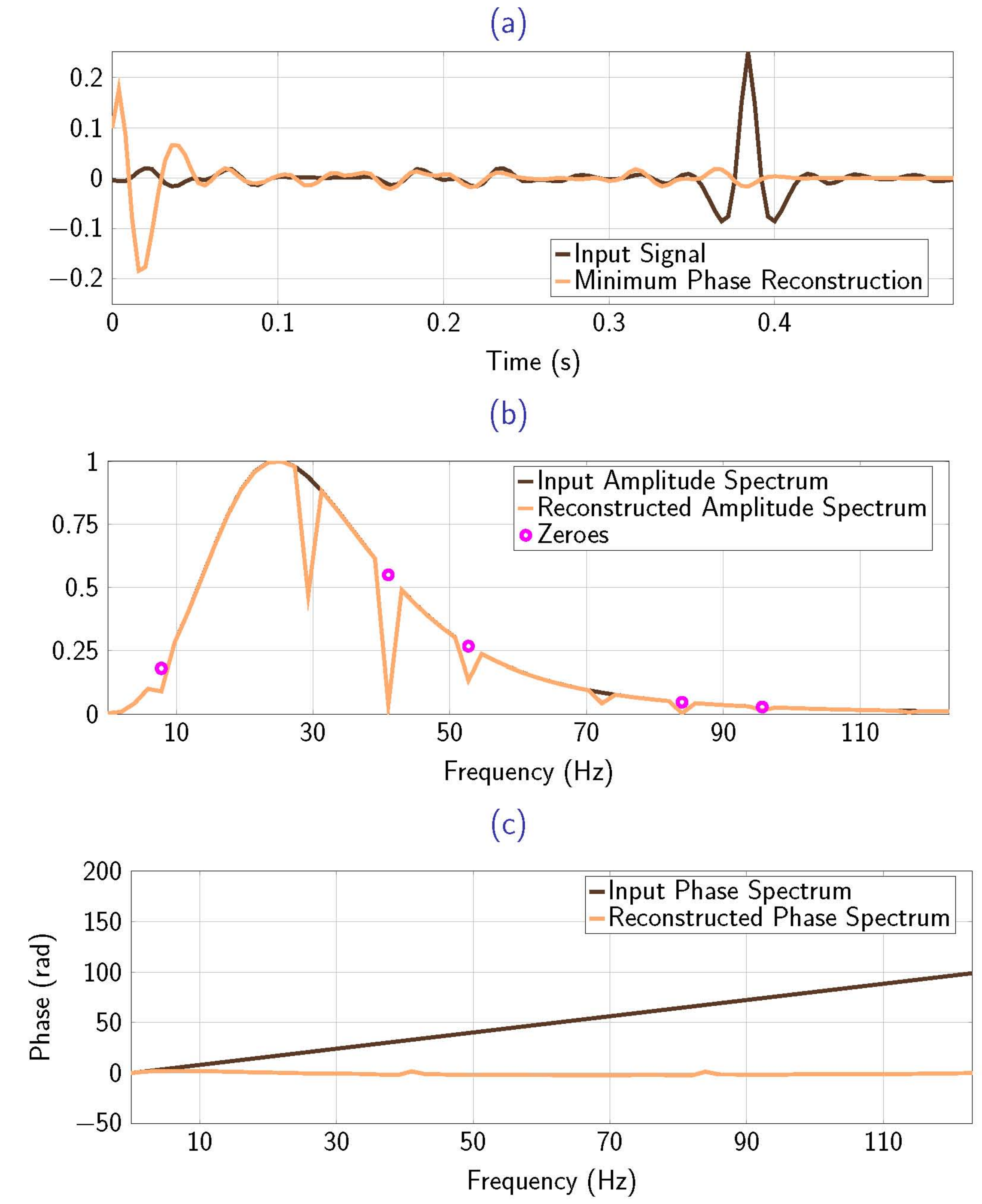

Figure : 1. Case C: Mixed (linear) phase input with isolated zeroes. (a) Input signal along with minimum-phase reconstruction created using minrceps. (b) Amplitude spectra of input and output, with isolated zeroes in input spectrum shown in magenta circles. (c) Phase spectra of input and output.

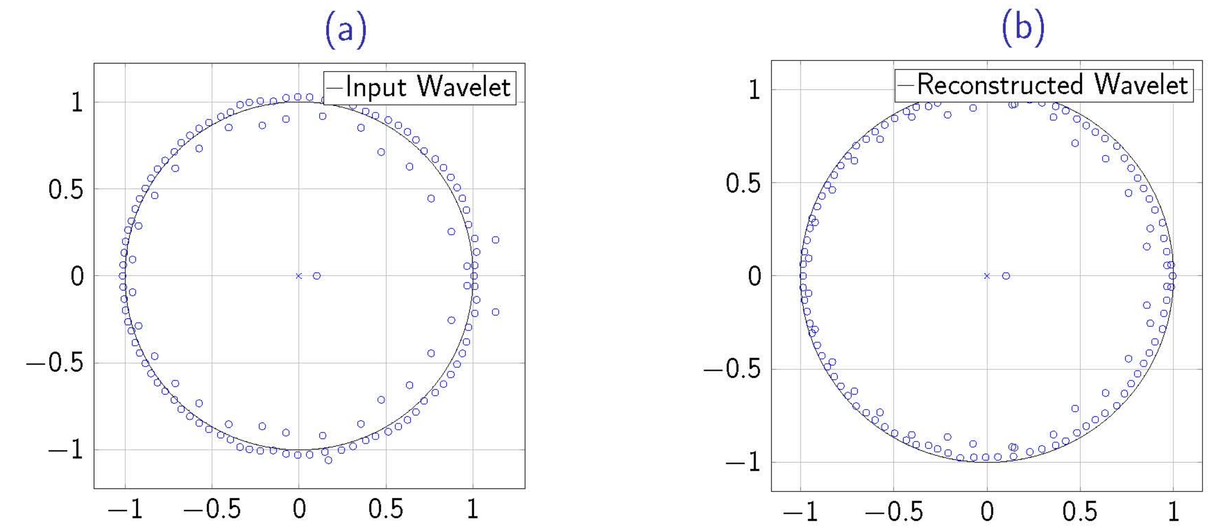

Figure : 2. Case C: Pole-zero plots showing location of poles and zeroes of signals. Poles are denoted by x's and zeros by o's. (a) Input signal. (b) Minimum phase reconstruction using minrceps.

Minimum-phase metrics

Pole-Zero Ratio (PZR):

$$PZR = \frac{(\text{Number of P/Z on or outside } |z| = 1)}{(\text{Total Number of P/Z})}$$

Pole-Zero Distance (PZD):

$$PZD = \frac{\sum_{(|P/Z| \geq 1)}(|P/Z| - 1)}{(\text{Number of P/Z on or outside } |z| = 1)}$$

Deconvolution test

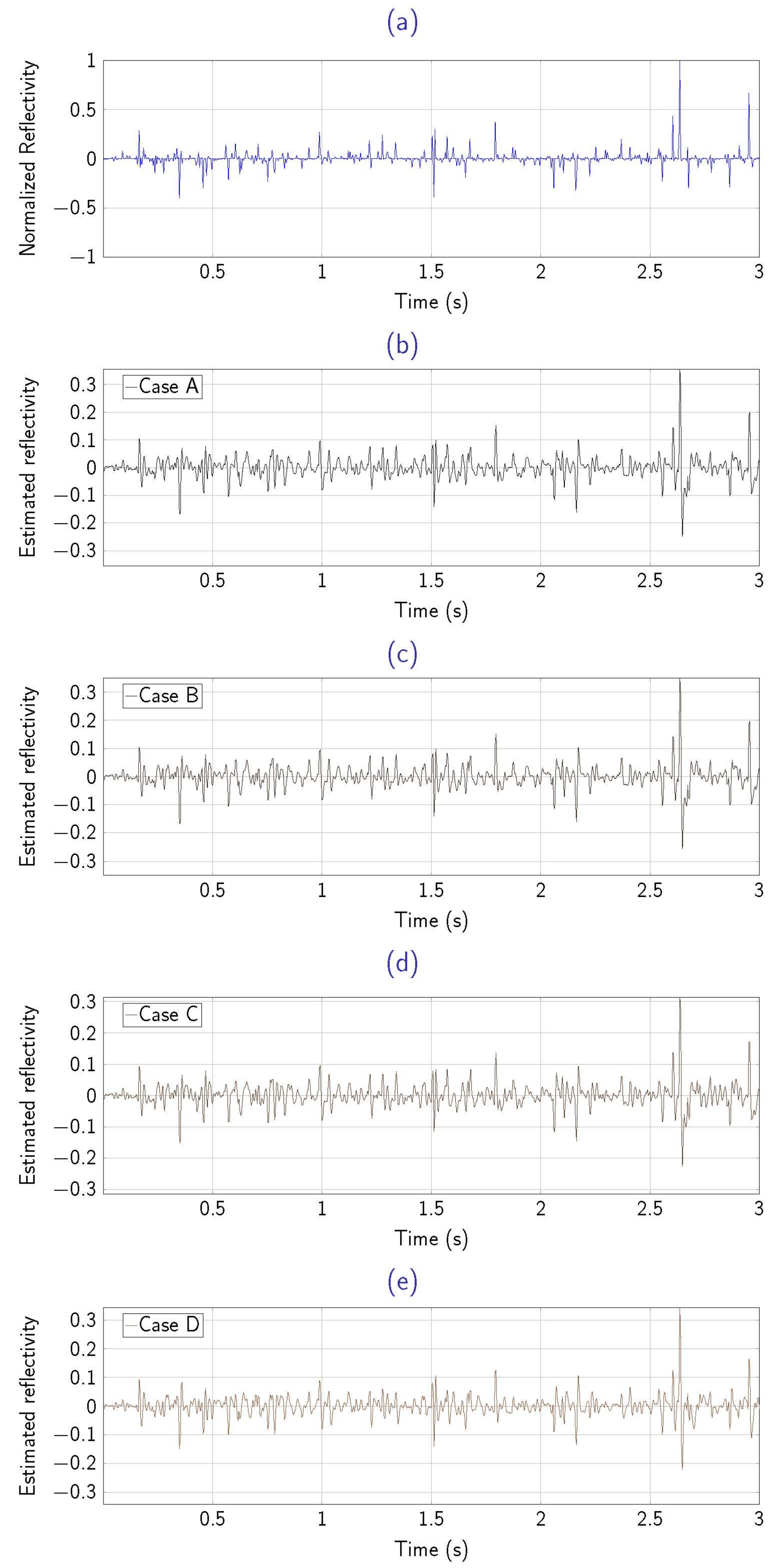

Figure : 3. Original reflectivity (a) along with reflectivities estimated from stationary Wiener deconvolution of synthetic traces generated by minimum-phase reconstructions Cases A-D. (b-e).

Conclusions

- Minrceps algorithm is based upon calculation of the real cepstrum
- Algorithm appears to be able ill-conditioned inputs
- Deconvolution test reveals acceptable minimum-phase outputs
- Metrics introduced to measure "minimum-phaseness" (PZR) and (PZD)

Future Work

- Linearization of minrceps algorithm
- Develop method to quantify accuracy of deconvolution results
- Analyze PZR and PZD directly against deconvolution performance

Acknowledgements

We thank the sponsors of CREWES for their support. We also gratefully acknowledge support from NSERC (Natural Science and Engineering Research Council of Canada) through the grant CRDPJ 379744-08.