Bulk properties of composite Cosserat media

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ABSTRACT

The following is a review of continuum mechanics in elastic media, where couple-stresses are included in the analysis. This is then followed by a proposal to study the properties of a composite of this type of material. The purpose of the analysis is to determine the importance of this type of description, in composites, as compared to normal analysis where couple-stresses are ignored.

INTRODUCTION

This note represents basically a rewriting of a portion of Mindlin and Tiersten's (1962) paper. I will put the work in index notation which seems to be clearer. An attempt will be made to keep the development in general curvilinear coordinates (which does not necessarily add to the clarity). The sections of concern are the development of the Cosserat equations which govern the behavior of continua where couple-stresses are included alongside the more familiar force-stresses, the development of constitutive relations (Toupin, 1962) which in the present analysis concerns the response of a Cosserat medium to deformation, the linearization of these relations, and the analysis of wave propagation through such a medium with emphasis on plane waves. Finally a proposal will be given to study the effects of a composite of such materials and determine the possibility of using Cosserat equations as a better description of bulk properties than the ordinary equations. The first section will be developed in detail, in the remaining important results will be stated without detailed development. Of special interest is the case where in each of the constituents the effects of couple-stress may be weak, and therefore the mechanics are adequately described by the normal couple-stress-free equations. But the bulk properties are effected by the micro-geometry of the inclusions, which may gives rise to the possibility of Cosserat equations being a better set of desciptive tools.

Force and Couple Stresses in Elasticity (Cosserat Equations)

It can be shown that any system of forces acting on a rigid body can be broken down into a single force which acts on an arbitrary point of the body plus an appropriate couple (Symon, 1971). A couple can be represented as a system of two forces which are equal in magnitude but opposite in direction and do not act directly against each other (i.e.. they do not have the same line of action). Otherwise, one could just as well replace these two forces by a force with magnitude zero, which is not very informative in any way. Couples provide a twisting effect to a mechanical system.

In continuum mechanics the assumption has been made that a similar scheme can be use to describe the forces acting on a material volume V, enclosed by an orientable closed surface S' as shown in Figure 1.

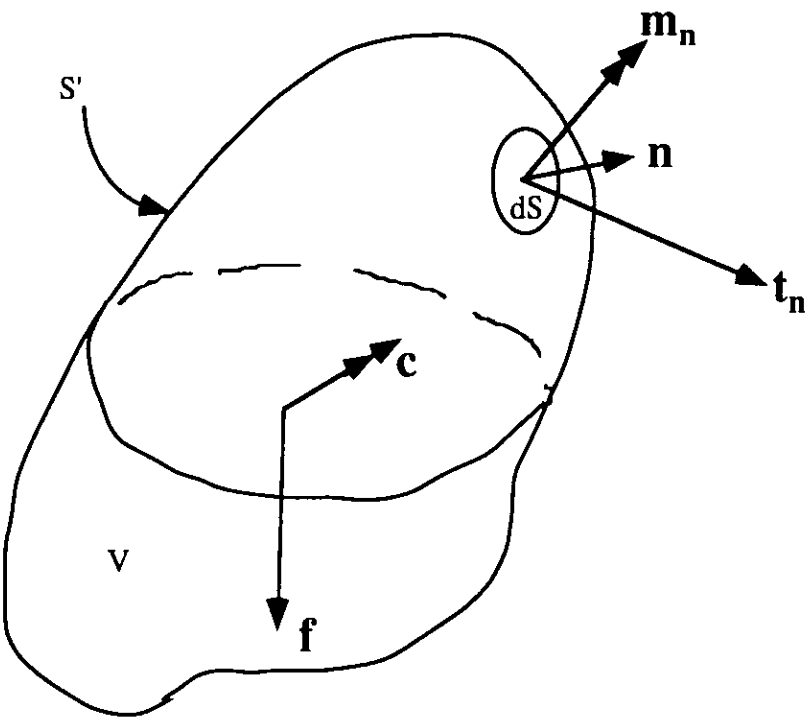

Figure 1

The material outside of V exerts forces on the material inside V through the surface S'. If we concentrate on a small region of the surface as outlined by the oval in figure 1, the net influence of the material in the direction of the outward unit normal vector $\mathbf{n}$ consists of a force per unit area $\mathbf{t_n}$, and a couple per unit area $\mathbf{m_n}$. In the interior of S' action-at-a-distance type of forces can influence the material in V. These forces will be assumed to be proportional to the mass acted upon; again we will simplify these forces at each point in V into the now familiar force per unit mass $\mathbf{f}$, and a couple per unit mass $\mathbf{c}$. This is represented in figure 1 where forces are illustrated as line segments ending in a single arrowhead and couples are portrayed with two arrowheads.

We shall now consider the motion of the material in the reference volume V due to these forces and couples. The motion will be governed by the equations of conservation of mass, balance of momentum and moment of momentum and the conservation of mechanical energy. The equations in index notation of these properties are respectively :

$$\frac{d}{dt}\int_V \rho \, dV = 0 \qquad , \tag{1.1}$$

$$\frac{d}{dt}\int_V v^i \rho \, dV = \int_S t^i_n \, dS + \int_V f^i \rho \, dV \qquad , \tag{1.2}$$

$$\frac{d}{dt}\int_V \varepsilon^i_{.jk} x^j v^k \rho \, dV = \int_S (\varepsilon^i_{.jk} x^j t^k_n + m^i_n) \, dS + \int_V (\varepsilon^i_{.jk} x^j f^k + c^i)\rho \, dV \qquad , \tag{1.3}$$

and

$$\frac{d}{dt}\int_V (\frac{1}{2} g_{ij} v^i v^j + U)\rho \, dV = \int_S (g_{ij} t^i_n v^j + \frac{1}{2} m^i_n \varepsilon_{ijk} g^{lk} v^j_{,l}) \, dS \tag{1.4}$$

$$+ \int_V (g_{ij} v^i f^j + \frac{1}{2} c^i \varepsilon_{ijk} g^{lk} v^j_{,l})\rho \, dV \qquad ,$$

where

$\frac{d}{dt} \equiv$ material time derivative,

$\rho \equiv$ mass density,

$\mathbf{n} \equiv$ outward normal of surface S'

$x^i \equiv$ position vector component,

$g_{ij} \equiv$ metric tensor,

$\varepsilon_{ijk} = \sqrt{g}\, e_{ijk} \equiv$ permutation tensor,

such that, $g = \det[g^{lk}] = |g^{lk}|$ and $e_{ijk} \equiv$ permutation symbols,

$v^i \equiv \frac{d}{dt} x^i = \dot{x}^i \equiv$ material velocity,

and $U \equiv$ internal energy per unit mass.

We are using the material (Lagrangian) frame of reference, which means the material time derivative can be taken within the integral sign. One direct consequence of this is equation (1.1) can be written as:

$$\frac{d}{dt}\int_V \rho \, dV = \int_V \frac{d\rho}{dt} dV = 0 = \dot{\rho} \quad . \tag{1.1a}$$

By the tetrahedron argument, as summarized in appendix A, we can recast the force per unit area and couple per unit area respectively in the following form:

$$t_n^i = t^{ij} n_j \tag{1.5}$$

and

$$m_n^i = m^{ij} n_j . \tag{1.6}$$

The surface integral in equation (1.2) can be transformed by use of equation (1.5) and the divergence theorem in the following manner:

$$\int_s t_n^i \, dS = \int_s t^{ij} n_j \, dS = \int_V t^{ij}_{,j} \, dV \quad , \tag{1.7}$$

where

$$t^{ij}_{,j} = \frac{\partial t^{ij}}{\partial x^j} + \left\{ {i \atop rj} \right\} t^{rj} + \left\{ {j \atop rj} \right\} t^{ir}$$

$$= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^r} \left[\sqrt{g} \, t^{ir} \right] + \left\{ {i \atop rj} \right\} t^{rj} \equiv \text{covariant derivative,}$$

$$\left\{ {l \atop ij} \right\} = g^{lk} [ij,k] \, ,$$

and

$$[ij,k] = \frac{1}{2} \left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right) .$$

Substitution of equation (1.7) into equation (1.2) and using equation (1.1) results in the following, after rearrangement of terms:

$$\int_V t^{ij}_{,j} + f^i \rho - \dot{v}^i \rho \, dV = 0 \quad .$$

Since the volume is arbitrary, we have the result:

$$t^{ij}_{,j} + f^i \rho - \dot{v}^i \rho = 0 \, , \tag{1.8}$$

or

$$t^{ij}_{,j} + f^i \rho = \dot{v}^i \rho$$

this is the usual force-stress equation of motion. In the same spirit of casting the integral equation (1.2) into its differential form (1.8) we shall do the same for the rest of the equations. First we shall concentrate on the separate parts of equation (1.3) as follows,

$$\frac{d}{dt}\int_V \varepsilon^{i}_{.jk} x^j v^k \rho \, dV = \int_V \varepsilon^{i}_{.jk} (v^j v^k \rho + x^j \dot{v}^k \rho + x^j v^k \dot{\rho}) \, dV$$

$$= \int_V \varepsilon^{i}_{.jk} x^j \dot{v}^k \rho \, dV \quad , \tag{1.3a}$$

the last equality is due to the antisymmetric nature of the permutation tensor and equation (1.1a), also

$$\int_S \varepsilon^{i}_{.jk} x^j t^k_n \, dS = \int_S \varepsilon^{i}_{.jk} x^j t^{kl} n_l \, dS = \int_V (\varepsilon^{i}_{.jk} x^j t^{kl} n_l)_{,l} \, dV$$

$$= \int_V \varepsilon^{i}_{.jk}(x^j t^{kl}_{,l} + x^j_{,l} t^{kl}) \, dV = \int_V \varepsilon^{i}_{.jk} x^j t^{kl}_{,l} + \varepsilon^{i}_{.jk} x^j_{,l} t^{kl} dV \quad , \tag{1.3b}$$

the first equality uses equation (1.5), the second uses the divergence theorem, and finally

$$\int_S m^i_n \, dS = \int_S m^{ij} n_j \, dS = \int_V m^{ij}_{,j} \, dV \quad , \tag{1.3c}$$

where equation (1.6) and the divergence theorem is used respectively. Substitution of equations (1.3a-c) into the conservation of momentum equation (1.3) yields:

$$\int_V \varepsilon^{i}_{.jk} x^j (t^{kl}_{,l} + f^k - \dot{v}^k \rho) \, dV + \int_V \varepsilon^{i}_{.jk} x^j_{,l} t^{kl} + c^i \rho + m^{ij}_{,j} \, dV = 0 \quad ;$$

the first integral of which is identically zero, as can be verified by direct comparison with equation (1.8). Since again the volume is arbitrary the integrand of the second integral must equal zero. This results in the equation:

$$\varepsilon^{i}_{.jk} x^j_{,l} t^{kl} + c^i \rho + m^{ij}_{,j} = 0 \quad ,$$

or

$$\varepsilon^{i}_{.jk} t^{kj} + c^i \rho + m^{ij}_{,j} = 0 \quad . \tag{1.9}$$

Equation (1.9) is the couple-stress equation of motion, and it provides an alternate expression of the anti-symmetric part of the force stress tensor as:

$$t^{[ij]} = -\frac{1}{2} \varepsilon^{i}_{de} x^{d,j} (\varepsilon^{e}_{fg} t^{gf}) = \frac{1}{2} \varepsilon^{i}_{de} x^{d,j} (c^e \rho + m^{ef}_{,f}) \quad . \tag{1.10}$$

The proof that equation (1.10) actually furnishes the anti-symmetric part of a tensor is given in appendix B. Note, the anti-symmetric part of the force stress tensor is identically zero if both the body couple and divergence of couple-stress are zero; therefore, under this condition the force stress tensor is totally symmetrical. The total tensor can be expresses as the sum of symmetric and anti-symmetric parts as follows:

$$t^{ij} = t^{(ij)} + t^{[ij]}, \tag{1.11}$$

where the term with superscripts in round brackets is the symmetric part of the tensor . Substitution of equation (1.10) into equation (1.11) and subsequently into the equation of motion (1.8) results in the alternate form of the equation of motion:

$$t^{(ij)}_{\ \ ,j} + \frac{1}{2}\varepsilon^{ij}_{\ \ e}m^{ef}_{\ \ ,fj} + f^i\rho + \frac{1}{2}\varepsilon^{ij}_{\ \ e}c^e_{\ ,j}\rho = \dot{v}^i\rho \tag{1.12}$$

Note, in equation (1.12) if the body couple and couple stress term are zero we revert to the standard equation of motion which is the start of most analysis. A further reduction can be achieved on equation (1.12) by considering the scalar of the couple stress given by:

$$\widetilde{m} = m^{ij}\, x_{i,j}, \tag{1.12a}$$

and the deviator of the couple stress of the form:

$$m^{\{ij\}} = m^{ij} - \frac{1}{3}\widetilde{m}\, x^{i,j} \tag{1.12b}$$

We will now show that equation (1.12a) has no contribution to equation (1.12), thus only the deviator given by equation (1.12b) will have any effect in equation (1.12). Consider the expression:

$$\begin{aligned}
\varepsilon^{ij}_{..e}[\widetilde{m}\, x^{e,f}]_{,fj} &= \varepsilon^{ij}_{..e}\left[\widetilde{m}_{,f}\, x^{e,f} + \widetilde{m}\, x^{e,f}_{...,f}\right]_{,j} \\
&= \varepsilon^{ij}_{..e}\left[\widetilde{m}^{,e}\right]_{,j} \\
&= \varepsilon^{ije}\,\widetilde{m}_{,ej} = 0\ ,
\end{aligned}$$

where the last equality is due to the fact that covariant differentiation of invariants (scalars) is commutative, in other words, we have the following symmetry property:

$$\widetilde{m}^{,ej} = \widetilde{m}^{,je}.$$

We can thus substitute equation (1.12b) into equation (1.12) to arrive at a form which more truly represents the independent parameters constrained by this equation. Equation (1.12) becomes:

$$t^{(ij)}_{\ \ ,j} + \frac{1}{2}\varepsilon^{ij}_{\ \ e}\, m^{\{ef\}}_{\ \ ,fj} + f^i\,\rho + \frac{1}{2}\varepsilon^{ij}_{\ \ e}\, c^e_{\ ,j}\,\rho = \dot{v}^i\,\rho \tag{1.13}$$

We will now cast the equation of conservation of mechanical energy (1.4) into its differential form, to achieve this end we shall manipulate separate terms in the equation independently and after recombine them into our desired form. Start by considering:

$$\frac{d}{dt}\int_V \left(\frac{1}{2}\, g_{ij}\, v^i v^j + U\right)\rho\, dV = \int_V (g_{ij}\, v^i \dot{v}^j + \dot{U})\rho\, dV \quad , \qquad (1.4a)$$

where the dot represents material time derivatives, secondly:

$$\begin{aligned}\int_S g_{ij}\, t^i_n\, v^j\, dS &= \int_S (t^{ij} n_j)\, v_i\, dS \\ &= \int_V (t^{ij} v_i)_{,j}\, dV \\ &= \int_V t^{ij}_{\;,j}\, v_i + t^{ij}\, v_{i,j}\, dV \quad . \qquad (1.4b)\end{aligned}$$

The first equality is from equation (1.6), the second is just the divergence theorem and the last is just the distributive law of covariant differentiation; finally we consider:

$$\begin{aligned}\int_S \frac{1}{2}\, m^i_n\, \varepsilon_{ijk}\, g^{lk}\, v^j_{,l}\, dS &= \int_S \frac{1}{2}\, m^{ir}\, n_r\, \varepsilon_{ijk}\, g^{lk}\, v^j_{,l}\, dS \\ &= \int_V \frac{1}{2}\, (m^{ir}\, \varepsilon_{ijk}\, g^{lk}\, v^j_{,l})_{,r}\, dV \\ &= \int_V \frac{1}{2}\, (\varepsilon_{ijk} m^{ir}_{\;,r}\, v^{j,k} + \varepsilon_{ijk} m^{ir}\, v^{j,k}_{\;,r})\, dV \quad . \qquad (1.4c)\end{aligned}$$

The steps taken above are almost identical to the previous and will not be elaborated upon. We can now incorporate equations (1.4a) - (1.4c) into equation (1.4) and upon rearrangement we get:

$$\int_V \dot{U}\, \rho\, dV = \int_V \left\{ (t^{ij}_{\;,j} + \rho\, f^i - \rho\, \dot{v}^i)\, v_i + \varepsilon_{ijk}\left(\frac{1}{2}\, m^{ir}_{\;,r} + \rho\, c^i\right) v^{j,k} + t^{ij}\, v_{j,i} + \frac{1}{2}\, \varepsilon_{ijk} m^{ir}\, v^{j,k}_{\;,r} \right\} dV \quad ,$$

which upon comparison to equations (1.8) and (1.9) simplifies to:

$$\int_V \dot{U}\, \rho\, dV = \int_V \left\{ t^{ij}\, v_{j,i} - \frac{1}{2}\, \varepsilon_{ijk} (\varepsilon^i_{.rs}\, x^r_{,l}\, t^{sl})\, v^{j,k} + \frac{1}{2}\, \varepsilon_{ijk} m^{ir}\, v^{j,k}_{\;,r} \right\} dV \quad ,$$

we make further note that:

$$\begin{aligned}\frac{1}{2}\, \varepsilon_{ijk} \varepsilon^i_{.rs}\, x^r_{,l}\, t^{sl}\, v^{j,k} &= \left(\frac{1}{2}\, \varepsilon_i^{\;.jk}\, \varepsilon^i_{.ls}\, t^{sl}\right) v_{j,k} \\ &= t^{[jk]}\, v_{j,k} \quad ,\end{aligned}$$

the last equality can be seen by direct comparison with equation (1.10), hence:

$$\int_V \dot{U}\,\rho\, dV = \int_V \left\{(t^{ij} - t^{[ij]})\, v_{j,i} + \frac{1}{2}\,\varepsilon_{ijk} m^{ir}\, v^{j,k}_{\ ,r}\right\} dV$$
$$= \int_V \left\{t^{(ij)}\, v_{j,i} + \frac{1}{2}\,\varepsilon_{ijk} m^{ir}\, v^{j,k}_{\ ,r}\right\} dV \quad ,$$

or in differential form:

$$\dot{U}\,\rho = t^{(ij)}\, v_{j,i} + \frac{1}{2}\,\varepsilon_{ijk} m^{ir}\, v^{j,k}_{\ ,r} \; . \tag{1.14}$$

Note:

$$x^i_{,r}\varepsilon_{ijk} v^{j,kr} = \varepsilon_{ijk} v^{j,ki} = 0 \, ,$$

which is a trivial consequence in rectangular Cartesian coordinates. This means the scalar of the couple-stress again has no effect in equation (1.14), a result which can be seen by substitution of (1.12b) into equation (1.14), therefore we can rewrite (1.14) in the following more informative way:

$$\dot{U}\,\rho = t^{(ij)}\, v_{j,i} + \frac{1}{2}\,\varepsilon_{ijk} m^{\{ir\}}\, v^{j,k}_{\ ,r} \; . \tag{1.15}$$

Equations (1.9), (1.13) and (1.15), the Cosserat equations, leaves the antisymmetric part of the force-stress and the scalar of the couple-stress indeterminate. This means the number of independent variables controlled by these equations are (9-3)=6 from the symmetric part of the force-stress and (9-1)=8 from the deviator of the couple-stress, for a total of 14 independent variables. This becomes important when we try to relate the response of such a medium to deformation in the next section.

Toupin's constitutive relations

Constitutive relations here will be concerned with the response of a Cosserat medium to deformation, which mean other effect such as temperature and temperature gradient will not be considered. The deformation will be described with respect to some initial reference configuration called the material frame. The material frame position will be designated by the capital letter X and all references to that frame in terms of subscripts and superscripts will also be in capital letters. The arc distance in the spatial and material (Eulerian and Lagrangian) frames are respectively:

$$(ds)^2 = g_{kl}\, dx^k\, dx^l \tag{2.1a}$$

and

$$(dS)^2 = G_{KL}\, dX^K\, dX^L \tag{2.1b}$$

where the metric tensors are self defining in each of the equations (frames). Now consider an infinitesimal directed line segment in the material frame and its image in the spatial frame, they will be related by the following formula:

$$dx^k = x^k_{;K}\, dX^K \tag{2.2a}$$

or

$$dX^K = X^K_{;k}\, dx^k, \tag{2.2b}$$

where the semicolon designates ordinary partial differentiation. Taking the inner product of equations (2.2a) and (2.2b) we arrive at:

$$(ds)^2 = g_{kl}\, X^k_{;K}\, X^l_{;L}\, dX^K\, dX^L \tag{2.3a}$$

and

$$(dS)^2 = G_{KL}\, X^K_{;k}\, X^L_{;l}\, dx^k\, dx^l. \tag{2.3b}$$

The difference of the squared length of line elements containing the same material points in the deformed and undeformed bodies gives a measure of length change due to deformation. This can then be used as a variable to determine the amount of energy stored during deformation (I will not be considering systems which generate energy during deformation). With this in mind we write the following formula:

$$\begin{aligned}(ds)^2 - (dS)^2 &= (g_{kl}\, X^k_{;K}\, X^l_{;L} - G_{KL})\, dX^K\, dX^L \\ &= (g_{kl} - G_{KL}\, X^K_{;k}\, X^L_{;l})\, dx^k\, dx^l.\end{aligned} \tag{2.4}$$

The bracketed terms after the two equal signs provides a measure of length change in the material and spatial frames respectively. We shall give these quantities (Lagrangian and Eulerian strain tensors) special symbols, as defined below:

$$2E_{KL} = 2E_{LK} = g_{kl}\, x^k_{;K}\, x^l_{;L} - G_{KL} \tag{2.5a}$$

and

$$2e_{kl} = 2e_{lk} = g_{kl} - G_{KL}\, X^K_{;k}\, X^L_{;l}, \tag{2.5b}$$

the symmetry is obvious, due to the symmetry of the metric tensor. When couple-stress is not taken into account the strain tensors as defined by equations (2.5a) and (2.5b) are sufficient to describe the specific energy during deformation, but as can be seen this provides only 6 independent components while we have seen in the case when couple-stresses are considered there exists 14 degrees of freedom. R.A. Toupin (1962) has shown that an appropriate second variable is:

$$K_{IJ} = -\epsilon_I^{KL}\, E_{JK,L}, \tag{2.6}$$

the scalar of which is zero giving a total of 8 independent components. The 8 independent components from equation (2.6) and the 6 independent components from equation (2.5a) is sufficient for our purpose. We now assume the specific energy can be expressed as a function of these two tensors as:

$$U = U(E_{IJ}, K_{rs}), \tag{2.7}$$

which implies:

$$\rho\,\dot{U} = \rho\,\frac{\partial U}{\partial E_{IJ}}\,\dot{E}_{IJ} + \rho\,\frac{\partial U}{\partial K_{RS}}\,\dot{K}_{RS} \quad . \tag{2.8}$$

The object is to put the conservation of energy equation (1.15) into the same double inner product form. After some troublesome manipulations this can be done giving rise to:

$$\rho\,\dot{U} = \phi^{IJ}\,\dot{E}_{IJ} + \psi^{RS}\,\dot{K}_{RS}\,, \tag{2.9}$$

where

$$\psi_{IJ} = \left|X^{I}_{,i}\right| X_{I,i}\, m^{\{ij\}}\, X_{J,j}\,, \tag{2.9a}$$

and

$$\phi_{IJ} = X_{I,i}\, t^{(ij)}\, X_{J,j} - \left|X^{I}_{,i}\right| A_{(IJ)}\,, \tag{2.9b}$$

such that

$$A_{IJ} = \varepsilon_{IKL}\, X^{K}_{,k}\, m^{\{kr\}}\, X^{R}_{,r}\, X^{L}_{,j}\, x^{j}_{,RJ}\,. \tag{2.9c}$$

Before proceeding we have to note that:

$$\alpha\, X^{I,J}\dot{K}_{IJ} = 0, \tag{2.10}$$

where α is an arbitrary constant, this is due again to the fact that the scalar of K is zero and therefore not all nine components are independent. We can therefore add equation (2.10) to equation (2.8) without changing anything then subtract the result from equation (2.9) to get:

$$\left[\phi^{IJ} - \rho\,\frac{\partial U}{\partial E_{IJ}}\right]\dot{E}_{IJ} + \left[\psi^{RS} - \rho\,\frac{\partial U}{\partial K_{RS}} - \alpha\, X^{R,S}\right]\dot{K}_{RS} = 0 \quad . \tag{2.11}$$

If we assume all term within the square brackets of equation (2.11) are independent of the terms they are dot producted with then we can write:

$$\phi^{IJ} = \rho\,\frac{\partial U}{\partial E_{IJ}}\,, \tag{2.12a}$$

and

$$\psi^{RS} = \rho\,\frac{\partial U}{\partial K_{RS}} + \alpha\, X^{R,S} \quad . \tag{2.12b}$$

By noting that all terms in equation (2.12b) have zero scalars we can now finally set the constant α to zero. Now we can solve for the symmetric part of the force-stress tensor and the deviator of the couple-stress tensor from equations (2.12a) and (2.12b), giving:

$$t^{(ij)} = \rho\, x^{i}_{,I}\,\frac{\partial U}{\partial E_{IJ}}\, x^{j}_{,J} + \rho\, G_{NL}\, \varepsilon^{INM}\,\frac{\partial U}{\partial K_{JL}}\, x^{(i}_{,I}\, x^{j)}_{,JM}\,, \tag{2.13a}$$

and

$$m^{(ij)} = \rho \left|x^{i}{}_{,I}\right| x^{i}{}_{,I} \frac{\partial U}{\partial K_{IJ}} x^{j}{}_{,J} \quad . \tag{2.13b}$$

Equations (2.13a) and (2.13b) are one form of Toupin's constitutive equations.

Linearization

We will now linearize the equations previously developed, this linearized set will form the basis from which all future development will stem. We shall begin by assuming the specific energy can be expresses in terms of a Taylor series in powers of E and K and also allowing the undeformed state to be one of zero stress and density with subscript 0:

$$\rho_0 U = \frac{1}{2} a^{QRST} K_{QR} K_{ST} + b^{QRST} E_{QR} K_{ST} + c^{QRST} E_{QR} E_{ST} + \cdots , \tag{3.1}$$

where a,b and c are constant material tetradics and the higher order terms are understood to have higher order polyadics for coefficients. The linear terms are zero from our zero stress assumption above. We will define the material displacement as:

$$\mathbf{u} = \mathbf{x} - \mathbf{X}, \tag{3.2}$$

which we will constrain to have a small material gradient in the following sense:

$$\left|\mathbf{u}_{,K}\right| << 1 .$$

This implies the material gradients and spatial gradient can be taken as approximately equal and the density remains approximately constant, the distinction between material and spatial frame can then be ignored. In this regime we can use the follow approximation for the material strain dyadic:

$$E_{ij} \approx \frac{1}{2} (u_{i,j} + u_{j,i}) = \in_{ij} , \tag{3.3a}$$

which in turn gives the approximation for:

$$\begin{aligned} K_{ij} &\approx -\varepsilon_i^{\;lm} \in_{lj,m} = -\frac{1}{2} (\varepsilon_i^{\;lm} u_{l,jm} + \varepsilon_i^{\;lm} u_{j,lm}) \\ &= \frac{1}{2} \varepsilon_i^{\;lm} u_{m,jl} = \chi_{ij} . \end{aligned} \tag{3.3b}$$

Substitution of approximations in equation (3.3a) and (3.3b) into equation (3.1) yields:

$$\rho_0 U = W \approx \frac{1}{2} a^{qrst} \chi_{qr} \chi_{st} + b^{qrst} \in_{qr} \chi_{st} + \frac{1}{2} c^{qrst} \in_{qr} \in_{st} + \cdots , \tag{3.4}$$

which can be used with the general constitutive equations (2.13a) and (2.13b) to give:

$$m^{(ij)} \approx \frac{\partial W}{\partial \chi_{ij}} \tag{3.5a}$$

and

$$t^{(ij)} \approx \frac{\partial W}{\partial \epsilon_{ij}} + \varepsilon^{i}_{.mn} \frac{\partial W}{\partial \chi_{lm}} u_{l}{}^{,nj},$$

which upon the further assumption that $u_{l}{}^{,nj}$ is negligible can be written as:

$$t^{(ij)} \approx \frac{\partial W}{\partial \epsilon_{ij}}. \tag{3.5b}$$

By ignoring all terms of higher order we can write the following set of linearized equations:

$$W = \frac{1}{2} a^{qrst} \chi_{qr} \chi_{st} + b^{qrst} \epsilon_{qr} \chi_{st} + \frac{1}{2} c^{qrst} \epsilon_{qr} \epsilon_{st}, \tag{3.6a}$$

$$t^{(ij)} = \frac{\partial W}{\partial \epsilon_{ij}} = c^{ijst} \epsilon_{st} + b^{ijst} \chi_{st}, \tag{3.6b}$$

and

$$m^{(ij)} = \frac{\partial W}{\partial \chi_{ij}} = a^{ijst} \chi_{st} + b^{qrij} \epsilon_{qr}. \tag{3.6c}$$

Equations (3.6a),(3.6b) and (3.6c) are the general linearized constitutive equations. If one considers all the symmetries imposed on the material teradics we will find that "a" has 36 independent components, "b" has 48 and "c" has the usual 21 independent components. If we now restrict the material to be centrosymmetric-isotropic we wind up with the following much simpler system of equations:

$$W = 2\eta\, \chi^{ij} \chi_{ij} + 2\eta' \chi^{ij} \chi_{ji} + \lambda \left(\epsilon_{ii}\right)^2 + \mu\, \epsilon^{ij} \epsilon_{ij}, \tag{3.7a}$$

$$t^{(ij)} = \frac{\partial W}{\partial \epsilon_{ij}} = \lambda\, \epsilon_{kk} g^{ij} + 2\mu\, \epsilon^{ij}, \tag{3.7b}$$

and

$$m^{(ij)} = \frac{\partial W}{\partial \mu_{ij}} = 4\eta\, \chi^{ij} + 4\eta' \chi^{ji}, \tag{3.7c}$$

where λ and μ are the familiar Lame constants while η and η' are new constants due to the introduction of couple-stresses. If we now substitute equations (3.3a) and (3.3b) into equations (3.7b) and (3.7c) we arrive at:

$$t^{(ij)} = \lambda\, u^{k,k} g^{ij} + 2\mu \left(u^{ij} + u^{ji}\right), \tag{3.8a}$$

and

$$m^{(ij)} = 4\eta\, \varepsilon_i^{\,lm} u_{m,jl} + 4\eta' \varepsilon_i^{\,lm} u_{m,jl}. \tag{3.8b}$$

Now we can substitute equations (3.3a),(3.3b),(3.8a) and (3.8b) into the equation of motion (1.13) which takes on the form:

$$\mu\, u^{i,j}_{..,j} + (\lambda+\mu)\, u^{j}_{.,ji} + \eta\, \varepsilon^{i}_{.jk}\varepsilon^{j}_{.lm}\, u^{l,mks}_{....,s} + \rho\, f^{i} + \frac{1}{2}\varepsilon^{i}_{.jk}\, c^{k}_{.,j}\,\rho = \rho\, \ddot{u}^{i}, \tag{3.9}$$

this equation will be the starting point for further analysis.

Wave motion

The equation of motion (3.9) in the absence of body forces and body couples has the form:

$$\mu\, u^{i,j}_{..,j} + (\lambda+\mu)\, u^{j}_{.,ji} + \eta\, \varepsilon^{i}_{.jk}\varepsilon^{j}_{.lm}\, u^{l,mks}_{....,s} = \rho\, \ddot{u}^{i}. \tag{4.1}$$

We now proceed in the usual manner by taking divergence and curl of the equation of motion (4.1), resulting in:

$$c_1^2\, \varphi_{,jj} = \ddot{\varphi}, \tag{4.2}$$

and

$$c_2^2\, \psi^{i}_{,jj} - c_2^2\, l^2\, \psi^{i}_{,jj} = \ddot{\psi}^{i}, \tag{4.3}$$

where

$$\varphi = u^{i}_{,i}, \qquad \psi^{i} = \varepsilon^{ijk} u_{j,k},$$

$$l^2 = \frac{\eta}{\mu}, \quad c_1^2 = \frac{(\lambda + 2\mu)}{\rho}, \quad c_2^2 = \frac{\mu}{\rho}.$$

As can be seen the dilatational wave given by equation (4.2) is identical to couple-stress free case while the rotational case is quite different. If $l = 0$ we would recover the usual rotational wave equation. To examine the effects of this extra term on wave propagation consider a plane rotational wave of the form:

$$\psi^{i} = d^{i} A \;\exp[i\, k\, (n_i x^i - c\, t)] = d^{i} A \;\exp[i(k_i x^i - \omega\, t)], \tag{4.4}$$

where

$d^i \equiv$ unit vector,

$A \equiv$ scalar amplitude,

$k \equiv$ wave number,

$n_i \equiv$ unit wave normal,

$c \equiv$ phase velocity,

and $\omega \equiv$ angular frequency.

Substitution of (4.4) into (4.3) results in the following two equations:

$$\omega^2 = c_2^2 k^2 (1-l^2k^2)\ , \quad c^2 = c_2^2 (1-l^2k^2); \tag{4.5}$$

which we can use to solve for k^2, giving us two roots, which we will denote as:

$$k_1^2 = \frac{1}{2} l^{-2}\left[\sqrt{1 + \frac{4\, l^2 \omega^2}{c_2^2}} - 1\right], \tag{4.6a}$$

and

$$k_2^2 = -\frac{1}{2} l^{-2}\left[\sqrt{1 + \frac{4\, l^2 \omega^2}{c_2^2}} + 1\right]. \tag{4.6b}$$

Since l is real we can see that k_1 is real while k_2 is purely imaginary, this means there are two rotational plane waves one propagating and the other non-propagating and both are dispersive. The propagating wave will have a group velocity given by:

$$\frac{d\omega}{dk_1} = c_2 \frac{1 + 2\, l^2 k_1^2}{\sqrt{1+ l^2 k_1^2}}. \tag{4.7}$$

Formula (4.7) shows the group velocity as a monotonically increasing function of $|k_1$.

Composite Cosserat media: A proposal

Now that we have looked at the development of continuum mechanics in Cosserat medium, and noted the differences this type of medium as compared to the case where couple-stresses are ignored, it is important to say that according to Mindlin and Thiersten (1962) there does not seem to be much experimental evidence that couple-stresses play an important role. This suggests that constants like l and η are probably small, therefor the ordinary theory is sufficient. If we now look at the bulk properties of a composite Cosserat medium the effects of microgeometry may make the effective l and η values less negligible. I propose to use the techniques brought to my attention by Dr. D.J. Bergman in a series of lectures on composite media given at C.O.F.R.C. (1991) to investigate this possibility.

REFERENCES

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Appendix A
The Cauchy Tetrahedron Argument

The result of the following reasoning is to show that if an arbitrary function φ is dependent only on the position x^i of the surface S' on which it is defined and the normal to that surface **n**,then this function can be recast in terms of an inner product between a tensor defined from φ and the unit normal. The function could also be dependent on time. Note this will not hold in many instances, such as when φ is dependent on the curvature of the surface as well. The quantity of concern is not the value of φ, but rather it is the differential φds,which can be approximated by the average value of φ in a small area Δs multiplied by the area; we will write this as $\tilde{\varphi}\Delta s$.

We shall now construct a tetrahedron by introducing locally an orthogonal right-handed system x^i with origin O, such that the area Δs consists of the region defined by the plane normal to **n** and the intersection of this plane with the orthogonal axis. The points of intersection are labeled A^i. The altitude from the origin O to the point P is of length h. Each of the surfaces bounded on two sides by the axes and the third by the plane normal **n** are labeled as Δs^i, where the superscript i determines the axis to which the plane surface is perpendicular. This is shown in figure A-1.

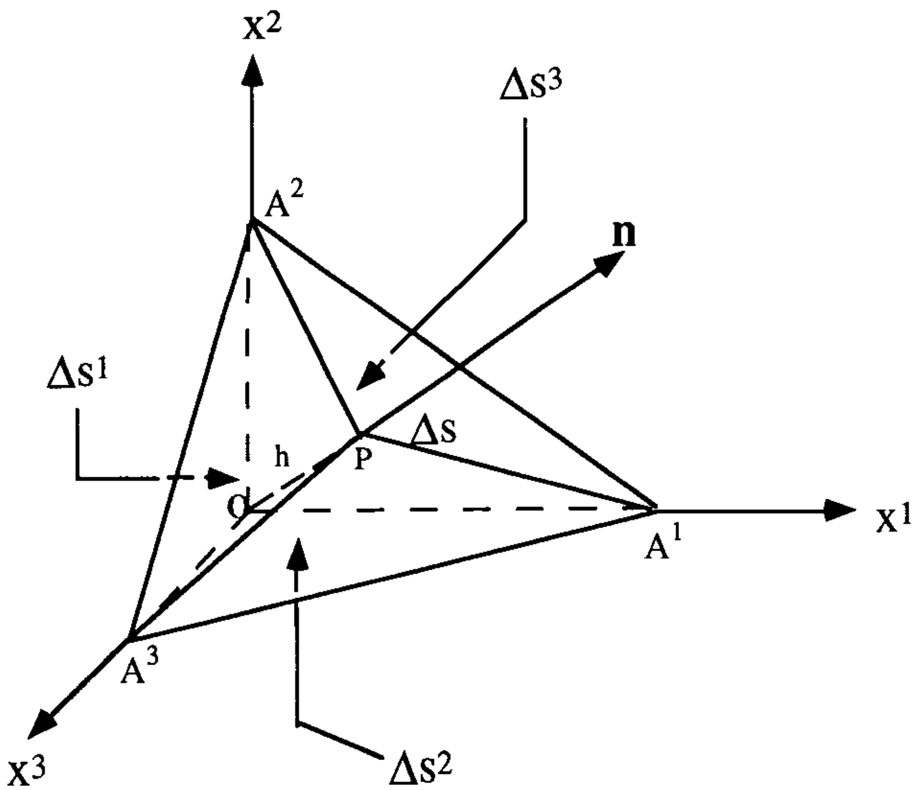

Figure A-1

The components of **n** in this local coordinate system is just the direction cosines given by:

$$n^i = \cos(\angle A^i OP)_. \qquad \text{(A-1)}$$

The altitude h is given by:

$$h = \overline{OA^{(i)}}\, n_{i.} \qquad \text{(A-2)}$$

(no summation implied as indicated by brackets)

The volume of the tetrahedron can be written as:

$$\Delta V = \frac{1}{3} h\, \Delta s = \frac{1}{3} \overline{OA^{(i)}}\, \Delta s^i_{\;.} \qquad \text{(A-3)}$$

Substitution of equation (A-2) into (A-3) and simplifying results in:

$$\Delta s^i = n_i\, \Delta s_. \qquad \text{(A-5)}$$

We can approximate the integral ψ of φ around the close surface of the tetrahedron by the sum of average values it takes at each plane face multiplied by the area of the face (note, the direction of the unit normal is crucial and determines the sign of the sum). This can be written as:

$$\psi \approx \tilde{\varphi}(\mathbf{n})\, \Delta s - \tilde{\varphi}(\widehat{\mathbf{x}}^1)\, \Delta s^1 - \tilde{\varphi}(\widehat{\mathbf{x}}^i)\, \Delta s^2 - \tilde{\varphi}(\widehat{\mathbf{x}}^3)\, \Delta s^3 \qquad \text{(A-6a)}$$

where $\widehat{\mathbf{x}}^i$ is a unit vector in the direction of coordinate axis x^i and this sum becomes exact as the volume goes to zero. Substitution of equation (A-5) into equation (A-6) results in:

$$\psi \approx \left[\tilde{\varphi}(\mathbf{n}) - \tilde{\varphi}(\widehat{\mathbf{x}}^1)\, n_1 - \tilde{\varphi}(\widehat{\mathbf{x}}^i)\, n_2 - \tilde{\varphi}(\widehat{\mathbf{x}}^3)\, n_3\right] \Delta s_. \qquad \text{(A-6b)}$$

In many instance we can show that the integral ψ is also dependent on the volume enclosed by the surface, in particular if ψ is proportional to the volume ΔV, or

$$\psi \approx C\, \Delta V_, \qquad \text{(A-7)}$$

where C is the constant of proportionality.This is always true when tractions and couples per unit area are concerned. Combining equations (A-6b) and (A-7) then substitute in equation (A-3) yields:

$$\left[\tilde{\varphi}(\mathbf{n}) - \tilde{\varphi}(\widehat{\mathbf{x}}^1)\, n_1 - \tilde{\varphi}(\widehat{\mathbf{x}}^i)\, n_2 - \tilde{\varphi}(\widehat{\mathbf{x}}^3)\, n_3\right] \Delta s \approx C\, \Delta V = C\frac{1}{3} h\, \Delta s_,$$

upon division by Δs we get:

$$\left[\widetilde{\varphi}(\mathbf{n}) - \widetilde{\varphi}(\widehat{\mathbf{x}}^1)\, n_1 - \widetilde{\varphi}(\widehat{\mathbf{x}}^i)\, n_2 - \widetilde{\varphi}(\widehat{\mathbf{x}}^3)\, n_3\right] \approx C\, \frac{1}{3}\, h \quad . \tag{A-8}$$

In the limit as h approaches zero the formula becomes exact and the left hand side goes to zero which results in:

$$\varphi(\mathbf{n}) = \varphi(\widehat{\mathbf{x}}^1)\, n_1 + \varphi(\widehat{\mathbf{x}}^i)\, n_2 + \varphi(\widehat{\mathbf{x}}^3)\, n_3. \tag{A-9}$$

If we now define $\varphi^i = \varphi(\widehat{\mathbf{x}}^i)$, then we can rewrite equation (A-9) as:

$$\varphi(\mathbf{n}) = \varphi^i\, n_i, \tag{A-10}$$

which is our desired conclusion.

Appendix B
Anti-symmetric part of a Tensor

Here we shall show the anti-symmetric part of a tensor can be written in the form given by equation (1.10) in the main part of the paper. For convenience the formula is reproduce here with only index symbols altered, as follows:

$$t^{[mn]} = -\frac{1}{2}\,\varepsilon^{m}_{.qr}\,x^{n,q}\,(\varepsilon^{r}_{.jk}\,x^{k}_{.,p}\,t^{pj})\;k_1 \quad . \tag{B-1}$$

The proof consists of applying basic tensor manipulations to equation (B-1) to arrive at a simpler tensor formula, this formula is then shown to be the formula for the anti-symmetric part of a tensor in one particular coordinate system; since the original formula is in tensor form, if it is true in one coordinate system it is true in all curvilinear coordinate systems. To begin we rewrite (B-1) as:

$$\begin{aligned} t^{[mn]} &= -\frac{1}{2}\,g^{ms}\,\varepsilon_{sqr}\,g^{qt}\,x^{n}_{,t}\;g^{ir}\,\varepsilon_{ijk}\,x^{k}_{,p}\,t^{pj} \\ &= -\frac{1}{2}\,g^{ms}\,g^{qt}\,g^{ir}\,\varepsilon_{sqr}\,\varepsilon_{ijk}\,x^{n}_{,t}\;x^{k}_{,p}\,t^{pj} \\ &= -\frac{1}{2}\,\varepsilon^{mti}\,\varepsilon_{ijk}\,x^{n}_{,t}\;x^{k}_{,p}\,t^{pj} \quad . \end{aligned} \tag{B-2}$$

The permutation tensor and permutation symbol are related by the following formula:

$$\varepsilon_{ijk} = \sqrt{g}\;e_{ijk}, \tag{B-3}$$

and the permutation symbol has the following well known relationship expressed in terms of Kronecker delta symbols:

$$e^{mti}\,e_{ijk} = \delta^{mt}_{jk} = \delta^{m}_{j}\,\delta^{t}_{k} - \delta^{t}_{j}\,\delta^{m}_{k}, \tag{B-4}$$

so upon substitution of (B-3) into the equation (B-2) and then using equation (B-4) we arrive at:

$$\begin{aligned} t^{[mn]} &= -\frac{1}{2}\,g\,(\delta^{m}_{j}\,\delta^{t}_{k} - \delta^{t}_{j}\,\delta^{m}_{k})\;x^{n}_{,t}\;x^{k}_{,p}\,t^{pj} \\ &= -\frac{1}{2}\,g\,(t^{pm}\,x^{n}_{,k}\,x^{k}_{,p} - t^{pt}\,x^{n}_{,t}\;x^{m}_{,p}) \\ &= -\frac{1}{2}\,g\,(t^{pm}\,x^{n}_{,p} - t^{pt}\,x^{m}_{,p}) \\ &= -\frac{1}{2}\,g\,(t^{nm} - t^{mn}) \\ &= \frac{1}{2}\,g\,(t^{mn} - t^{nm}) \quad . \end{aligned} \tag{B-5}$$

Now we shall use a rectangular Cartesian coordinate system where g=1 and equation (B-5) takes on the familiar symmetric form:

$$t^{[mn]} = \frac{1}{2}\,(t^{mn} - t^{nm}) \quad , \tag{B-6}$$

which ends the proof.