Composite media as generalized continua

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ABSTRACT

When the wavelength of a disturbance within a composite medium is large in comparison to the constituents of the composite, it becomes appropriate to look for average quantities which describe the relevant observable phenomena and to replace the composite with a generalized continuum which describes the evolution of these average quantities. One particular form of averaging will be investigated and a particular generalized continuum description will be developed for a linear elastic composite.

INTRODUCTION

Any material which at one scale looks to be homogeneous will at some other scale appear to be heterogeneous and even possibly discontinuous. To see this, one only needs to acknowledge the existence of molecules and atoms at some scale. Nevertheless, the continuum description has been so successful and the alternative of describing all the effects on all the constituent particles so daunting that we return to the continuum description with little hesitation. Rivlin (1968) showed how a system of particles each comprised of different mass points can be described as a generalized continuum and Eringen (1968) showed how to start from a granular composite and reach the same end, which he calls a micromorphic continuum. A composite is just a heterogeneous continuum which at some scale will also appear to be homogeneous, even though its behavior may be quite unique at that scale. Backus (1962) looked at a composite of layered media and through averaging was able to describe the static behavior as a transversely isotropic homogeneous medium. Others have attempted to push this basically static description to nonzero wavelengths with some success. The dynamic effects such as dispersion of the wavelet has not been dealt with in these cases. In this paper the basic thrust is to generalize Backus's (1962) averaging method and to mathematically develop a generalize continuum theory consistent with this method of averaging.

PERTURBATION TO BACKUS AVERAGING

The following averaging scheme was introduced in Backus's (1962) paper. He defined a linear averaging operator in the form:

$$\langle f \rangle(x) = \int_{-\infty}^{\infty} w(\xi - x)\, f(\xi)\, d\xi \,, \tag{1}$$

where

$$w(x) \geq 0 \,,\ w(\pm\infty) = 0$$

$$\int_{-\infty}^{\infty} w(x)\,dx = 1\ ,\ \int_{-\infty}^{\infty} x\,w(x)\,dx = 0 \ \text{ and } \int_{-\infty}^{\infty} x^2 w(x)\,dx = l^2\ .$$

For functions f that are approximately constant when x is varied by no more than l, Backus used the following approximation:

$$\langle fg\rangle = f\langle g\rangle\ . \tag{2}$$

As a first step one may generalize equation (1) to n dimensions by letting $\mathbf{x}$ represent an n-dimensional position vector and then recast the linear average as:

$$\langle f\rangle(\mathbf{x}) = \int_{V_\infty} w(\xi - \mathbf{x})\,f(\xi)\,dV_\xi\ , \tag{3}$$

such that,

$$w(\mathbf{x}) \geq 0\ ,\ \ w(\pm\infty) = 0\ ,$$

$$\int_{V_\infty} w(\mathbf{x})\,dV_{\mathbf{X}} = 1,\ \ \frac{1}{A_{\mathbf{t}}}\int_{-\infty}^{\infty} x\,w(x\,\mathbf{t})\,dx = 0\ ,$$

$$\frac{1}{A_{\mathbf{t}}}\int_{-\infty}^{\infty} x^2\,w(x\,\mathbf{t})\,dx = l_{\mathbf{t}}^2 \ \text{ and } \ A_{\mathbf{t}} = \int_{-\infty}^{\infty} w(x\,\mathbf{t})\,dx\ ,$$

where $\mathbf{t}$ is a unit vector, indicating the direction for which the appropriate property is calculated, and V_∞ represents the limit as a finite volume $V_{\mathbf{X}}$ is allowed to expand and cover all space.

Now we attempt to generalize the approximation. Our assumption is, for a neighborhood around $\mathbf{x}$ where $\mathbf{x}$ varies by no more than $l_{\mathbf{t}}$ in the direction of unit vector $\mathbf{t}$, the function f can be represented by a Taylor's series as follows:

$$f(\mathbf{y}) = f(\mathbf{x}) + f_{,i}(\mathbf{x})\,(y^i - x^i) + \frac{1}{2}f_{,ij}(\mathbf{x})\,(y^i - x^i)\,(y^j - x^j) + \cdots\ , \tag{4}$$

where y^i are the components of vector $\mathbf{y}$, x^i are the components of vector $\mathbf{x}$, $f_{,i}$ represents the partial derivative of f with respect to x^i and summation on repeated diagonal indices is enforced. We now turn our attention to the product of the function f with another function g and the average of the product:

$$\langle f\,g\rangle(\mathbf{x}) = \int_{V_\infty} w(\mathbf{y} - \mathbf{x})\,f(\mathbf{y})\,g(\mathbf{y})\,dV_{\mathbf{y}} \tag{5}$$

$$= \int_{V_\infty} w(\mathbf{y} - \mathbf{x})\,g(\mathbf{y})\left[f(\mathbf{x}) + f_{,i}(\mathbf{x})\,(y^i - x^i) + \frac{1}{2}f_{,ij}(\mathbf{x})\,(y^i - x^i)\,(y^j - x^j) + \cdots\right]dV_{\mathbf{y}}$$

$$= f(\mathbf{x})\langle g(\mathbf{x})\rangle + f_{,i}(\mathbf{x})\left[\langle g(\mathbf{x})\,x^i\rangle - x^i\langle g(\mathbf{x})\rangle\right] +$$

$$+ \frac{1}{2}f_{,ij}(\mathbf{x})\left[\langle g(\mathbf{x})\,x^i\,x^j\rangle - x^j\langle g(\mathbf{x})\,x^i\rangle - x^i\langle g(\mathbf{x})\,x^j\rangle + x^i\,x^j\langle g(\mathbf{x})\rangle\right] + \cdots$$

As can be seen by direct comparison of equation (5) and equation (2) the first term of equation (5) is equation (2). It must be recognized at this point that equation (5) is only accurate as an infinite series and, if we truncate it, the function f and all its derivatives must be approximated by some averages as well. This point seemed to have been glossed over in Backus' (1962) paper. The use of Taylor-series truncations also results in the assumption of a smooth neighborhood of the function in question.

POTENTIAL APPLICATION IN LINEAR ELASTICITY

First consider the constitutive relation of a linear elastic material below:

$$\sigma^{ij} = c^{ij}_{..kl}\, \varepsilon^{kl} , \tag{6}$$

keeping in mind the difficulties above. The average of equation (6), by using the first two terms in relation (5), can be written as:

$$\begin{aligned} \langle\sigma^{ij}\rangle &= \langle c^{ij}_{..kl}\, \varepsilon^{kl}\rangle \\ &= C^{ij}_{..kl}\, \varepsilon^{kl} + B^{ij..m}_{..kl}\, \varepsilon^{kl}_{..,m} , \end{aligned} \tag{7}$$

where

$$C^{ij}_{..kl} = \langle c^{ij}_{..kl}\rangle \text{ and } B^{ij..m}_{..kl} = \langle c^{ij}_{..kl}\, x^m\rangle - x^m \langle c^{ij}_{..kl}\rangle .$$

The linear momentum equation in the absence of body forces is:

$$\sigma^{ij}_{..,j} - \rho\, \ddot{u}^i = 0. \tag{8}$$

Before considering the average of equation (8), we shall derive a couple of intermediate results. Consider:

$$\langle\sigma^{ij}_{..,j}\rangle = \int_{V_\infty} w(\xi^i - x^i)\, \sigma^{ij}_{..,j}(\xi^i)\, dV_\xi , \tag{9}$$

which can be cast in another form by the following reasoning:

$$\begin{aligned} \frac{\partial[w(\xi^i - x^i)\, \sigma^{ij}(\xi^i)]}{\partial\xi^j} &= \frac{\partial w(\xi^i - x^i)}{\partial\xi^j}\, \sigma^{ij}(\xi^i) + w(\xi^i - x^i)\, \frac{\partial\sigma^{ij}(\xi^i)}{\partial\xi^j} \\ &= -\frac{\partial w(\xi^i - x^i)}{\partial x^j}\, \sigma^{ij}(\xi^i) + w(\xi^i - x^i)\, \sigma^{ij}_{.,j}(\xi^i). \end{aligned} \tag{10}$$

When relation (10) is used in equation (9) we have:

$$\langle \sigma^{ij}_{..,j} \rangle = \int_{V_\infty} \frac{\partial [w(\xi^i - x^i)\, \sigma^{ij}(\xi^i)]}{\partial \xi^j} \, dV_\xi + \int_{V_\infty} \frac{\partial w(\xi^i - x^i)}{\partial x^j} \sigma^{ij}(\xi^i) \, dV_\xi$$

$$= \int_{S_\infty} [w(\xi^i - x^i)\, \sigma^{ij}(\xi^i)]\, n_j \, dS_\xi + \frac{\partial}{\partial x^j} \int_{V_\infty} w(\xi^i - x^i)\, \sigma^{ij}(\xi^i) \, dV_\xi$$

$$= \langle \sigma^{ij} \rangle_{,j}, \tag{11}$$

where we have used the divergence theorem and the fact that $w$goes to zero on S_∞. We shall now use equation (5) to recast the average of the product of density and particle acceleration as:

$$\langle \rho \ddot{u}^i \rangle = \ddot{u}^i \bar{\rho} + \ddot{u}^i_{.,m} \mu^m + \ddot{u}^i_{.,mn} \nu^{mn} , \tag{12}$$

where

$$\bar{\rho} = \langle \rho \rangle \ , \ \mu^m = \langle \rho x^m \rangle - x^m \langle \rho \rangle .$$

and

$$\nu^{mn} = \frac{1}{2} \left(\langle \rho x^m x^n \rangle - x^m \langle \rho x^n \rangle - x^n \langle \rho x^m \rangle + x^m x^n \langle \rho \rangle \right)$$

Note that we have chosen the highest order of spatial differentiation in the displacement to be consistent with the terms kept for linear deformation in equation (7). The equation of motion (6) can be averaged in the following manner:

$$\langle \sigma^{ij}_{..,j} \rangle - \langle \rho \ddot{u}^i \rangle = 0,$$

which, upon substitution of relations (11) and (12) becomes:

$$\langle \sigma^{ij} \rangle_{,j} - \ddot{u}^i \bar{\rho} - \ddot{u}^i_{.,m} \mu^m - \ddot{u}^i_{.,mn} \nu^{mn} = 0. \tag{13}$$

Using equation (7) in equation (13) results in:

$$[C^{ij}_{..kl} \varepsilon^{kl} + B^{ij..m}_{..kl} \varepsilon^{kl}_{..,m}]_{,j} - \ddot{u}^i \bar{\rho} - \ddot{u}^i_{.,m} \mu^m - \ddot{u}^i_{.,mn} \nu^{mn} = 0 . \tag{14}$$

Equation (14) can be put into a more useful form by using the following relations:

$$\varepsilon^{kl} = \frac{1}{2} (u^{k,l} + u^{l,k}) ,$$
$$C^{ij}_{..kl} = C^{ij}_{..lk} \text{ and } B^{ij..m}_{..kl} = B^{ij..m}_{..lk} ,$$

which upon introduction to equation (14) is transformed into:

$$[C^{ij}_{..kl} \varepsilon^{kl} + B^{ij..m}_{..kl} \varepsilon^{kl}_{..,m}]_{,j} - \ddot{u}^i \bar{\rho} - \ddot{u}^i_{.,m} \mu^m - \ddot{u}^i_{.,mn} \nu^{mn} = 0 \tag{15}$$

I have been able to gain some headway in deriving equation (15) as a direct consequence of Hamilton's principle as outlined in Bedford's (1985) book. This is an interesting mathematical exercise, but how can we use the new generalized continuum description to give us more information? As a step towards answering that question, we shall look at the simple situation of a plane wave propagating through this medium.

PLANE-WAVE SOLUTION

Consider the possibility that there exist plane-wave solutions to equation (15) of the following form:

$$u^r = Ap^r \exp\left(i\left[k_s x^s - \omega t\right]\right) = Ap^r \exp\left(i\, k\left[n_s x^s - v\, t\right]\right). \qquad (16)$$

where

$u^r \equiv$ displacement vector,
$A \equiv$ amplitude,
$p^r \equiv$ unit polarization vector,
$k_s \equiv$ propagation vector,
$x^s \equiv$ spatial coordinate vector,
$\omega \equiv$ angular frequency,
$t \equiv$ time variable,
$k \equiv$ magnitude of propagation vector $|\mathbf{k}| = \sqrt{k_s k^s}$

and $n_s \equiv$ unit vector in direction of phase propagation.

On substituting equation (16) in equation (15) and assuming **C** and **B** are constant tensors, we arrive at:

$$\left[- C_{abcd}\, k^d\, k^b - i\, B_{abcde}\, k^d\, k^b\, k^e + \omega^2 \bar{\rho}\, \delta_{ac} + i\, \mu_e\, k^e\, \omega^2 \delta_{ac} - v_{ef}\, k^e\, k^f\, \omega^2 \delta_{ac}\right] p^c = 0, \qquad (17)$$

such that

$\delta_{ac} \equiv$ Kronecker delta.

Using the identity $k^d = k\, n^d$ we can recast equation (17) in the following form:

$$\left[v^2 \delta_{ac} - C^*_{ac}\right] p^c = 0\ , \qquad (18)$$

where

$$C^*_{ac} = \frac{\left(C_{abcd}\, n^d\, n^b\right) + i\left(k\, B_{abcde}\, n^d\, n^b\, n^e\right)}{\left(\bar{\rho} - k^2\, v_{ef}\, n^e\, n^f\right) + i\left(k\, \mu_e\, n^e\right)}\ .$$

In order that equation (18) have nontrivial solutions, the following condition must be satisfied:

$$\det\left[v^2 \delta_{ac} - C^*_{ac}\right] = 0\ . \qquad (19)$$

Equation (19) is of the form of an eigenvalue problem. The eigenvalues resulting from its solution will be the phase velocities. It is interesting to note that when $k = 0$, equation (18) takes on the same form as the velocity equations in standard elastic media. These velocities will in general be complex, indicating attenuation and dispersion are possibilities. Once the eigenvalues are obtained, equation (18) can then be use to solve for the polarization vector (eigenvector).

CONCLUSION

This preliminary investigation into the use of generalized continua to describe the average behavior of a composite has resulted in a series of equations which is meant to describe the dynamic behavior of the composite in an averaged sense. A plane wave solution was tried in the equations and conditions for the existence of plane waves was derived. It must be stressed at this point that the indication of a dynamic expression, quite different from the one governing the individual component of the composite, is probably the most important feature of the exercise.

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