From group or phase velocities to the general anisotropic stiffness tensor

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ABSTRACT

Two numerical inversions were designed to calculate the 21 independent stiffnesses that define, in general, an anisotropic medium from either group- or phase-velocity data. The accuracy, robustness and computational complexity of the two inversion procedures – group velocity to stiffnesses and phase velocity to stiffnesses – were then compared.

This group-velocity inversion overcomes the difficulty of calculating group velocity in a prescribed direction and can calculate group velocities accurately even in directions near shear-wave singularities. Although phase velocities are easier to calculate than group velocities, the group-velocity inversion performed better in laboratory tests because group velocities are easier to measure.

INTRODUCTION

Some laboratory measurements yield group velocities and others yield phase velocities (Vestrum and Brown, 1993). The velocities calculated from ray-tracing in VSP surveys are group velocities and the velocity calculated from time delays between receivers – downhole or on the surface – are phase velocities. This inversion from either group or phase velocities is specifically designed for the inversion of laboratory-measured group or phase traveltimes and may be useful in the case of the multicomponent VSP survey where there are velocity measurements at a broad range of angles through, for example, a fractured reservoir.

The latest numerical inversion methods for calculating the 21 independent elastic stiffnesses from velocities have been proposed by Jech (1991), Arts et al. (1991) and Arts (1993). Jech's (1991) method is a least-squares inversion of qP-wave group velocities for stiffnesses and Arts's (1993) method involves a similar type of inversion using qP, qS_1 and qS_2 phase velocities to perform a generalized linear inversion (GLI) for stiffnesses. These two authors use similar linear-inversion techniques. Other stiffness-determination procedures have been proposed by Neighbours and Schacher (1967) and Hayes (1969).

Arts (1993) performs his inversion for the stiffnesses of what is referred to as a general anisotropic medium, meaning a medium were nothing is assumed about the 21 independent elastic stiffnesses. He identifies the major problem involved in inverting from group velocities. In discussing why he chose to invert from phase velocities, he points out that the group velocities, in general, cannot be calculated directly in a prescribed direction, so an iterative procedure must be used to find the group velocity in a particular direction.

In Arts's inversion, which is very similar to the inversion by van Buskirk et al. (1986), he solves Christoffel's equation for the stiffnesses in terms of the phase velocities, the wavefront normals and the polarizations. He acknowledges, however, that it is very difficult, if not impossible, to obtain accurate measurements of polarizations. An iterative procedure is then performed to improve the estimate of the polarization vectors by minimizing the difference between the phase velocities calculated from the inverted stiffnesses and the observed phase velocities. This iterative procedure is the improvement added by Arts et al. (1991) to the van Buskirk et al. (1986) inversion, which requires an accurate particle-displacement measurement.

Because his inversion requires phase-velocity observations, Arts (1993) makes traveltime measurements between large opposing faces of a truncated cube with the assumption that he can generate plane waves across the sample between the faces. This limits his measurement of traveltimes to certain directions where faces have been cut into the sample. The upper limit on the number of observations for the inversion imposed by his experiment is 27, nine measurements of each of the three wave phases (qP, qS_1, qS_2).

Jech (1991) has overcome the difficulty involved in calculating the group velocity in a prescribed direction by using an iterative method to find the group velocity given a particular direction. He uses only qP-wave velocities in his inversion technique, likely because of the difficulty in finding qS-wave velocities. Jech (1991) says "a problem arises for quasi-shear [qS] waves, as there is a danger that we may not follow the right value of the normal [phase] velocity of two quasi-shear waves in regions where the normal velocity surfaces of two quasi-shear waves intersect." He goes on to propose that if one kept track of the particle polarizations during a search for a group velocity, one could discriminate between the two qS waves. This would not, in general, work because when two qS phase-velocity surfaces intersect, the polarizations usually change dramatically in any type of anisotropic medium.

Jech's inversion method is a straighforward GLI which is somewhat similar to the method outlined here. The main difference between the methods is the use of the qS_1 and qS_2 velocities.

In the method proposed here, a standard least-squares inversion is employed to find the stiffnesses which yield the best fit to the observed velocities. The same method is used for the group-velocity inversion as for the phase-velocity inversion, but there is an additional step involved in finding the group velocity in a prescribed direction when performing the group-velocity inversion.

No regard is given to the polarization vectors in the inversion developed in this paper; they can't effectively be measured and they behave unpredictably at times, so they are not considered in these inversion procedures. The criterion for establishing which velocity belongs to which phase is the order of highest to lowest velocity or from first to last arriving wave phase. The first arriving phase is qP, next is qS_1 and last arriving wave phase with the slowest velocity is defined as qS_2.The numerical scheme for inverting for the stiffness tensor used here is general in the sense that there is no assumption made of the symmetry class of the medium or the orientation of the symmetry axes. From group-velocity traveltime measurements of qP, qS_1 and qS_2 waves the stiffnesses for the medium are estimated using a generalized linear inversion (GLI).

Laboratory experiments were performed to test the inversion algorithm. Material used in this physical modelling is Phenolic CE, an industrial laminate. This material consists of canvas layers which are saturated and bonded together by phenolic resin. The layers are woven in such a way that the canvas fibers are straight in one direction whereas the fibers in the direction orthogonal to the straight fibers weave or curl over and under the straight fibers. There has already been work done on this material (Brown et al. 1991, 1993; Cheadle et al. 1991) where the material is assumed to have orthorhombic symmetry.

The symmetry planes are assumed to be oriented with one in the plane of the layers and two orthogonal to this plane, one parallel to the curly fibers and one parallel to the straight fibers. The Cartesian coordinate system used here for this material has the z axis normal to the canvas layers, the y axis in the direction of the straight fibers and the x axis in the direction of the curly fibers. The x axis is the direction of maximum qP-wave velocity, the y axis that of intermediate qP-wave velocity and the z axis that of slowest qP-wave velocity. This is due to the layering and the woven nature of the fabric. When convenient, spherical coordinates are used with Θ being the angle of colatitude measured from the z axis and Φ the angle of azimuth from the x axis.

Since the inversion developed here is general, i.e., assuming no symmetry, the assumption of orthorhombic symmetry can be investigated. In this investigation, an inversion for the nine independent orthorhombic stiffnesses is performed and the results are compared to the general inversion to determine whether or not the inversion incorporating the assumption of orthorhombic symmetry produces a solution as good as the inversion for the general anisotropic stiffness tensor.

METHOD FOR CALCULATING GROUP VELOCITIES

With a given set of stiffnesses the magnitude of the group or ray velocity cannot be explicitly calculated in a particular direction. On the other hand, given the stiffnesses and a wave normal, the phase velocity may be calculated analytically. With this phase-velocity information, the group velocity associated with that particular phase velocity may be calculated. The group velocity does not, in general, lie in the same direction as the wave normal or the phase-velocity direction. The group velocity is dependent on the phase-velocity magnitude, v, and the phase-velocity direction or wavefront normal vector, $\boldsymbol{n}$. If the group velocity is to be calculated in a prescribed direction, a search must be performed to find a phase-velocity vector which will yield a group velocity in the desired direction.

This is the first task performed by the least-squares inversion method. In the search for a group velocity in the prescribed direction, a guess is made for the phase-velocity direction, in the spherical coordinates Θ (colatitude relative to the z axis) and Φ (azimuth relative to the x axis) associated with the desired group-velocity direction. In the first-order approximation, the observed group-velocity vector, $\boldsymbol{g}^{obs}$, and the calculated group-velocity vector, $\boldsymbol{g}^{calc}$, from the guessed Θ and Φ are related by:

$$g_i^{obs} = g_i^{calc} + \frac{\partial g_i}{\partial \Theta}\Delta\Theta + \frac{\partial g_i}{\partial \Phi}\Delta\Phi. \qquad 1$$

In this equation, the only unknowns are $\Delta\Theta$ and $\Delta\Phi$, the errors in the Θ and Φ estimates. The partial derivatives are calculated numerically.

If matrix notation is used, such that

$$\vec{\delta g} = \begin{bmatrix} g_x^{obs} - g_x^{calc} \\ g_y^{obs} - g_y^{calc} \\ g_z^{obs} - g_z^{calc} \end{bmatrix}, \quad \tilde{\alpha} = \begin{bmatrix} \frac{\partial g_x}{\partial \Theta} & \frac{\partial g_x}{\partial \Phi} \\ \frac{\partial g_y}{\partial \Theta} & \frac{\partial g_y}{\partial \Phi} \\ \frac{\partial g_z}{\partial \Theta} & \frac{\partial g_z}{\partial \Phi} \end{bmatrix}, \quad \text{and } \vec{\Delta} = \begin{bmatrix} \Delta\Theta \\ \Delta\Phi \end{bmatrix}, \qquad 2$$

then we get an equation in the form:

$$\vec{\delta g} = \tilde{\alpha}\vec{\Delta} \qquad 3$$

which has a solution in the standard least-squares form (Twomey, 1977) given by

$$\vec{\Delta} = \left[\tilde{\alpha}^T\tilde{\alpha}\right]^{-1}\tilde{\alpha}^T\,\vec{\delta g}. \qquad 4$$

Once values for $\Delta\Theta$ and $\Delta\Phi$ are estimated using this technique, they are then added to the initial guesses and the process is repeated. The iterations continue until a group velocity is calculated in a direction within some allowable deviation from the desired direction or the changes in the angles Θ and Φ decrease to within an acceptable tolerance level.

In performing this least-squares inversion, group velocities, g^{calc}, are calculated using Kendall and Thomson's (1989) equation and the stiffnesses and density of the material.

Because of the unpredictable behaviour of the polarizations of the different wave phases, the decision as to which wave phase is associated with a particular velocity is made using the which-came-first criterion. The wave with the highest velocity is assumed to be the qP, the second fastest phase is considered the qS_1 and the slowest wave phase will then be the qS_2.

The only problem with this method is in the difficulty in finding a solution close to a shear-wave singularity. At a point singularity or conical point (see Brown et al. 1993), the first derivative of the velocity with respect to Φ or Θ is discontinuous, which can make searching around these point singularities unstable. Near singularities or other problem areas on the wave surface where this search method fails, the program switches into a recursive random search in an attempt to find the appropriate group velocity.

Computationally intensive and limited in accuracy, this method pays no attention to the shape of the surface and can get the search algorithm within a degree or two of the desired group direction. The routine calculates 500 points chosen in a Gaussian distribution around an initial-guess direction. The point with a group-velocity direction closest to the desired direction is taken as the new mean and the standard deviation of the Gaussian-distributed random search is decreased by a factor of five and the search starts over. This process continues until a solution is found or there is little improvement.

In most cases, the GLI method will converge if the random search can get close enough to the desired group direction. Occasionally the GLI will still diverge and the program has to settle for a group velocity that is usually in a direction less than 1° from the desired direction.

METHOD FOR CALCULATING STIFFNESSES

Stiffnesses from group velocities

The previous application of this linear-inversion technique was merely to calculate a group-velocity vector in a prescribed direction. In the inversion for the stiffnesses, the approach taken is similar to the method used in calculating the group velocity.

It is similar because the problem is essentially the same. The goal is to minimize the difference between the observed group velocities and the calculated group velocities. The relationship between the ith observed group-velocity magnitude and the ith calculated group-velocity magnitude is:

$$g_i^{obs} = g_i^{calc} + \frac{\partial g_i}{\partial C_j}\Delta C_j \qquad 5$$

where C_j is the jth independent stiffness, $j = 1, 21$. Using the matrix notation:

$$\delta g_i = g_i^{obs} - g_i^{calc}, \quad \alpha_{ji} = \frac{\partial g_i}{\partial C_j}, \quad \text{and}\, \Delta_i = \Delta C_i, \qquad 6$$

the difference between the observed and calculated group velocities can be expressed in an equation of the same form as equation 4:

$$\vec{\Delta} = [\tilde{\alpha}^T \tilde{\alpha}]^{-1} \tilde{\alpha}^T \vec{\delta g}.$$

This mathematical problem is similar to that of calculating group velocities in a prescribed direction, except that now the g_i^{calc} and g_i^{obs} are the ith magnitudes of the calculated and observed group velocities, respectively, instead of the ith components of a group-velocity vector as defined in equation 2. Also, the variables that are to be obtained by the inversion are the stiffnesses and not the phase angles or wave normals associated with a desired group-velocity direction. Another difference between the problems is that this inversion is attempting to solve for 21 independent parameters instead of two. Since we are now solving for many more variables, 21 instead of two, there may be some instability in the solution. To stabilize the inversion process, a small scalar quantity λ, is added to the diagonals of the matrix to be inverted in order to dampen the solutions. This damping factor is added into equation 4 to give

$$\vec{\Delta} = [\tilde{\alpha}^T \tilde{\alpha} + \lambda I]^{-1} \tilde{\alpha}^T \vec{\delta g}. \qquad 7$$

Because of all of the numerical calculation of derivatives and the numerical inversion of a large matrix, this operation is computationally fairly cumbersome. What makes this process extremely computer-intensive is the inversion used to calculate the group velocity at a prescribed direction every time a group velocity or a derivative of the group velocity with respect to a stiffness parameter needs to be calculated.

Stiffnesses from phase velocities

The method discussed here for calculating the 21 independent stiffnesses is a least-squares inversion for stiffnesses from group velocities. In the case where phase velocities are measured, like in the experiments done at the Institut Français du Pétrole (Arts, 1993), the same scheme is utilized for the inversion. The exception is that the phase velocities may be calculated directly so the effort and computer time involved in finding a velocity in the appropriate direction is not required.

The calculation of stiffnesses from phase velocities works like the same calculation from group velocities. The only change required in the algorithm is the substitution of the phase-velocity magnitude v for the group-velocity magnitude g in equation 7. After the appropriate substitutions are made, this equation for the corrections to the stiffnesses becomes

$$\vec{\Delta} = \left[\tilde{\alpha}^T \tilde{\alpha} + \lambda I\right]^{-1} \tilde{\alpha}^T \vec{\delta v} \qquad 8$$

where

$$\delta v_i = v_i^{obs} - v_i^{calc}, \quad \alpha_{ji} = \frac{\partial v_i}{\partial C_j}, \quad \text{and}\, \Delta_i = \Delta C_i. \qquad 9$$

This inversion is 200 to 500 times faster than the same inversion from group velocities due to the absence of the search for a group velocity at each step of the program. Also, these calculations are able to yield a more numerically accurate result because the velocities are calculated directly and exactly instead of being estimated from a least-squares inversion. An additional bonus in this method is that there are no limitations in the calculation of velocities even at or near singularities, where there can be problems when calculating the group velocity in a prescribed direction.

Error analysis

In performing the inversion, it is desirable to know how well the velocities that are calculated from the stiffnesses match the observed velocities, i.e. how well the model fits the data. A statistical velocity error is defined to quantify the goodness of fit. Also, once a set of stiffnesses is determined, a statistical estimate of the uncertainty in each stiffness will need to be calculated.

The velocity error for each iteration is simply defined here as the standard deviation between the measured and calculated velocities σ, which, modified from Kanasewich (1985), is

$$\sigma = \sqrt{\frac{1}{N-(M+1)}\left(\vec{\delta g}^T \vec{\delta g} - \vec{\Delta}^T \tilde{\alpha}^T \vec{\delta g}\right)}, \qquad 10$$

where N is the number of measurements and M is the number of model parameters that are to be solved for; $M = 21$ in this general case.

Once the standard deviation has been calculated for the experimental observations, the uncertainties for the calculated model parameters, in this case stiffnesses, are calculated from the diagonal elements of the covariance matrix, $C_{M,}$

which is defined (Jenkins and Watts, 1968; Kanasewich, 1985) as:

$$\tilde{C}_M = \left[\tilde{\alpha}^T \tilde{\alpha}\right]^{-1} \sigma^2. \qquad 11$$

Each diagonal element of the covariance matrix is the variance of the respective inverted stiffness. By assuming a near-Gaussian distribution of error, the square root of each variance is used as an estimate of the uncertainty in the respective stiffness.

APPLICATION OF THE INVERSION TO LABORATORY MEASUREMENTS

Two experiments were performed to obtain each of phase and group velocities for the two methods of inversion for material properties. Velocity data were acquired in each experiment for all three wave phases, qP, qS_1, and qS_2, in Phenolic CE, the industrial laminate.

The first experiment, designed for the acquisition of phase velocities, involved the measurement of traveltimes between large transducers on flat parallel faces of a small (9.6 cm) bevelled cube of phenolic. Nine measurements were made on the 18-sided block for each wave phase yielding 27 velocity measurements for input into the inversion.

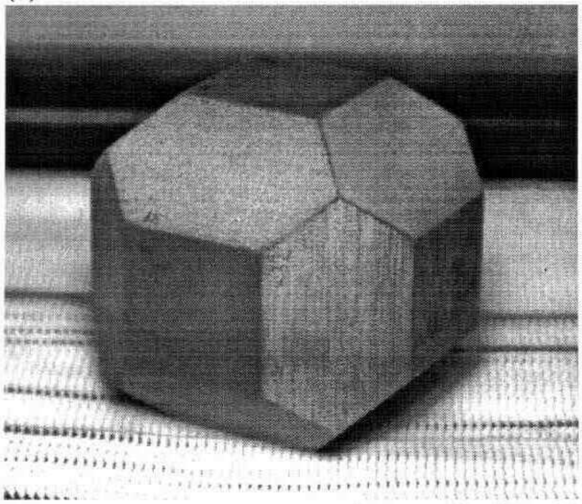
Group velocities were calculated from the traveltimes of the second experiment which involved smaller transducers transmitting ultrasonic waves through a larger (23-cm diameter) sphere of the industrial laminate. Traveltimes were acquired on the sphere with transducers at antipodes. Measurements were made at 15° spacing along lines of equal azimuth which were spaced 45° apart, so that data were collected in the *yz* plane, the *zx* plane and in a plane bisecting the angle between the *yz* and *zx* planes. The resulting 105 data points were used for the inversion from group velocity to stiffnesses.

Phase-velocity measurements

Figure 1 shows photographs of the cube with the bevelled edges and the large transducers used in the experiment. The measurements of traveltimes were across the sample between opposite faces which are approximately parallel. The 18 faces of the cube enabled the experimenters to make nine such traveltime measurements for each wave phase, including three in the principal-axis directions and six measurements at angles bisecting each axis pair.

When designing this experiment, large transducers were chosen to try to ensure that phase traveltimes would be measured across the sample. The shear-wave transducers have a diameter of 3.8 cm and the compressional-wave transducers are 5 cm in diameter. The qP sources and receivers cannot be considered to be this large, however, because the faces on the sample are only 4 cm across. The maximum group-minus-phase angle, δ_m, that allows determination of phase velocity (from Vestrum and Brown, 1993) is 21.6° for qS waves and 22.6° for qP waves, if phase velocities are to be acquired. Vestrum and Brown (1993) have shown that the largest angle between the group- and phase-velocity vectors is 11.6° for this medium which is close to half of δ_m; so there should be no doubt that the measured traveltimes are phase traveltimes.

(a)



(b)

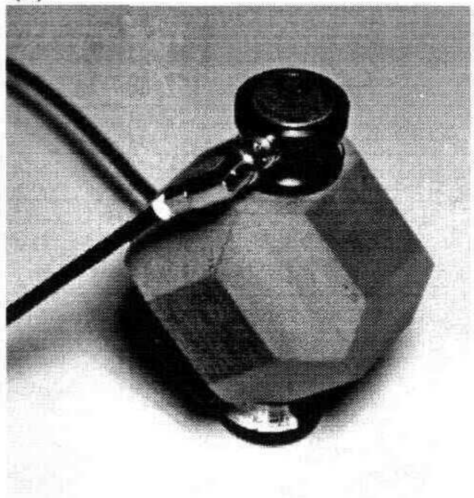


FIG. 1 (a) Bevelled cube of Phenolic CE. (b) The large transducers used to obtain phase traveltimes.

Traveltimes were measured as the time elapsed between pulse initiation and the arrival of the first discernible energy in the waveform on the oscilloscope. The time is determined to three significant figures, the first arrivals being picked to the nearest 0.1 μs, with an uncertainty in the pick of around ±0.5 μs. The distances have an uncertainty of a few tenths of a millimeter because of the thin layer of coupling agent between the transducers and the cube. The uncertainty in the calculated laboratory velocities based on uncertainties in distance and time measurements is estimated at ±2%.

The algorithm was applied to the velocity data and the inversion converged at a velocity error (defined by equation 10) of 18.0 m/s, less than 1% of the average velocity which is well within the estimated experimental uncertainty of ±2%. The resulting stiffnesses and their statistical uncertainties are listed in table 1. If the statistical error estimates in the stiffnesses appear high, this may be caused by the inversion being only somewhat overdetermined. With 27 measurements to determine 21 parameters, there are only 1.28 times more measurements than the minimum required.

	Final inverted stiffnesses (GPa)						Error in the stiffnesses (GPa)					
	n=1	n=2	n=3	n=4	n=5	n=6	n=1	n=2	n=3	n=4	n=5	n=6
m=1	15.91	6.48	6.10	0.06	-0.08	0.21	0.14	0.13	0.13	0.34	0.14	0.23
m=2	6.48	14.43	5.72	-0.15	-0.05	-0.12	0.13	0.13	0.14	0.18	0.28	0.22
m=3	6.10	5.72	10.88	0.07	0.05	0.15	0.13	0.14	0.12	0.23	0.18	0.26
m=4	0.06	-0.15	0.07	3.05	0.01	0.01	0.34	0.18	0.23	0.05	0.06	0.06
m=5	-0.08	-0.05	0.05	0.01	3.44	-0.02	0.14	0.28	0.18	0.06	0.05	0.08
m=6	0.21	-0.12	0.15	0.01	-0.02	3.76	0.23	0.22	0.26	0.06	0.08	0.05

TABLE 1 Stiffnesses estimated by phase-velocity inversion and their associated uncertainties.

Velocities from those stiffnesses and the velocities from the experiment are plotted (Wessel and Smith, 1991) in figures 2a-c for the *xy*, *yz* and *zx* planes, respectively. These graphs show the agreement between the velocities computed from the inverted stiffnesses and the observed velocities.

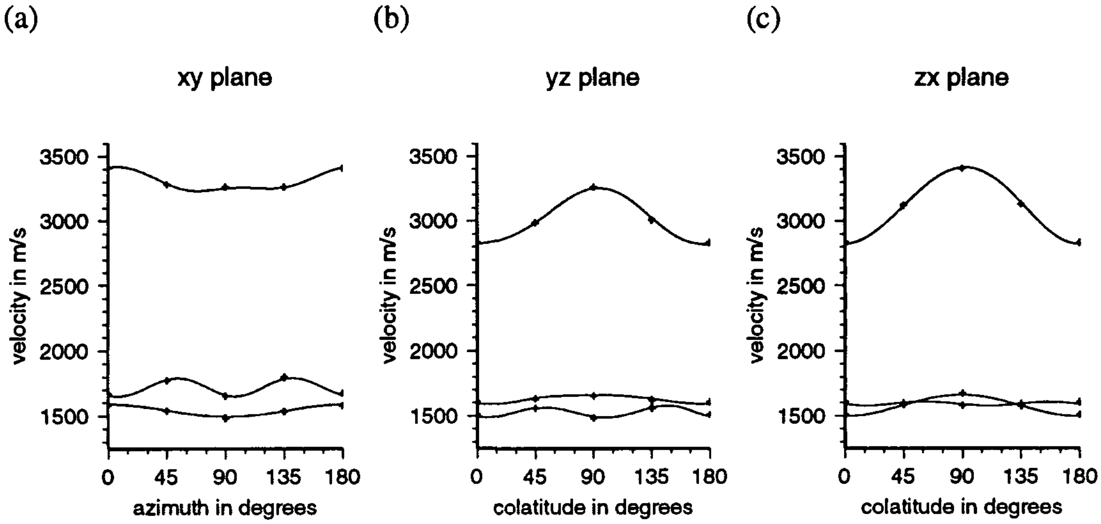


FIG. 2 Phase-velocity plots for the phenolic block. The crosses represent the observed velocity data and the solid curves are the phase velocities computed from the inverted stiffnesses.

Collecting the data for this inversion went quickly since only 27 measurements could be made of traveltimes between flat faces with parallel normal vectors without further carving up the sample. The inversion took a few seconds of computer time making this method of determination of stiffnesses fast and easy. The small velocity error (< 1%), less than the estimated relative experimental error (2%), indicates that this is also an effective method.

Group-velocity measurements

For the second experiment performed on the phenolic material, a 23-cm diameter sphere of the material was used. This large sphere was manufactured out of two cylinders of the material glued together and machined into a sphere. The glue seam is at the equator of the experimental coordinate system, the *xy* plane, and holds the top canvas layer of one piece to the bottom canvas layer of the other. It was desirable to make a sphere large enough to gain more clear separation between the two shear arrivals as well as separation between the shear arrivals and the reflected or refracted qP arrivals.

One of the advantages of performing an inversion from group velocities is realized here. Since flat faces and large transducers are not required or even desired in these experiments, a sphere is used and velocity measurements may be taken anywhere on the sphere. This freedom from taking measurements on flat faces has allowed the measurement of traveltimes yielding 105 velocities for the inversion. The sphere was placed in a jig that holds both transducers and has a protractor surrounding the sphere to make it easy to determine where on the sphere the measurements are taken (figure 3). Using the protractor, measurements may be taken as closely spaced as every half-degree around the sphere.

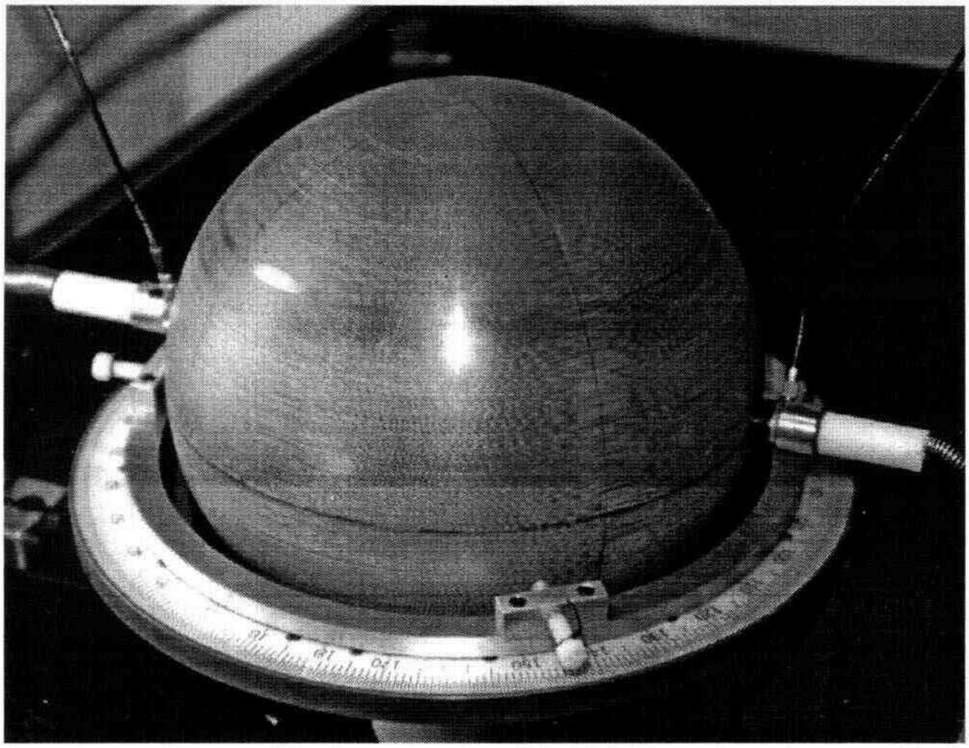


FIG. 3 Experimental set-up for measuring group velocities on the sphere. In this jig, the sphere is clamped to a ring inside the protractor and can be rotated about a vertical axis at an increment measured on the protractors.

After the first arrivals were picked for each wave phase and a few anomalous data points at the glued seam on the equator of the sphere were removed, 99 velocities were used in the final inversion – removing anomalous data is an affordable luxury when one has 105 data points. The velocity error after the final iteration is 7.7 m/s, approximately 0.3% of the average velocity, well within the estimated experimental uncertainty of ±2%. The resulting stiffnesses are listed in table 2 along with the statistical uncertainties in those stiffnesses which were calculated as described earlier. The uncertainties in the stiffnesses for this inversion are substantially lower than the statistical uncertainties calculated from the phase-velocity inversion (table 1). This difference in the error in the stiffnesses is probably a result of this inversion being much more overdetermined, with nearly five times as many measurements as number of required parameters.

The data points used for the inversion, as well as the velocity curves from the inverted stiffnesses are graphed in figure 4. In these velocity plots the data acquired in the experiment fit rather neatly on the velocity curves calculated from the inverted stiffnesses. The relatively high statistical accuracy of the inversion is likely due to the large number of data points. The consistency of these velocity data is likely also a factor in the low statistical error result and this is probably a positive side-effect of using a sphere in the experiment. The transducers are moved along the sphere as it is rotated in the jig (figure 3) so they stay in contact with the sphere from measurement to measurement with the same couplant between the transducer and the sphere. The experimental set-up is not disturbed as much as it is when removing the transducers from one pair of faces and attaching them to other faces, as done for the phase-velocity measurements.

	Final inverted stiffnesses (GPa)						Error in the stiffnesses (GPa)					
	n=1	n=2	n=3	n=4	n=5	n=6	n=1	n=2	n=3	n=4	n=5	n=6
m=1	16.85	7.88	6.81	0.07	-0.18	0.12	0.06	0.06	0.04	0.05	0.02	0.06
m=2	7.88	16.03	6.51	0.00	-0.26	-0.08	0.06	0.06	0.05	0.02	0.06	0.07
m=3	6.81	6.51	11.14	0.00	-0.05	-0.04	0.04	0.05	0.02	0.01	0.02	0.03
m=4	0.07	0.00	0.00	3.03	0.01	0.04	0.05	0.02	0.01	0.01	0.02	0.02
m=5	-0.18	-0.26	-0.05	0.01	3.40	-0.01	0.02	0.06	0.02	0.02	0.01	0.02
m=6	0.12	-0.08	-0.04	0.04	-0.01	3.89	0.06	0.07	0.03	0.02	0.02	0.02

TABLE 2 Final inverted stiffnesses and their respective uncertainties from the group-velocity inversion of the laboratory data.

The velocity curves appear to be nearly symmetrical on either side of the equatorial plane or 90° colatitude. The possibility of these symmetries existing is supported by the relatively small values that were calculated for the off-diagonal stiffnesses where *m* or *n* is greater than three. If the medium displayed orthorhombic symmetry, these values would be zero and each axis, *x*, *y*, and *z*, would be an axis of symmetry. The likelihood that this medium has orthorhombic symmetry will be discussed in a later section.

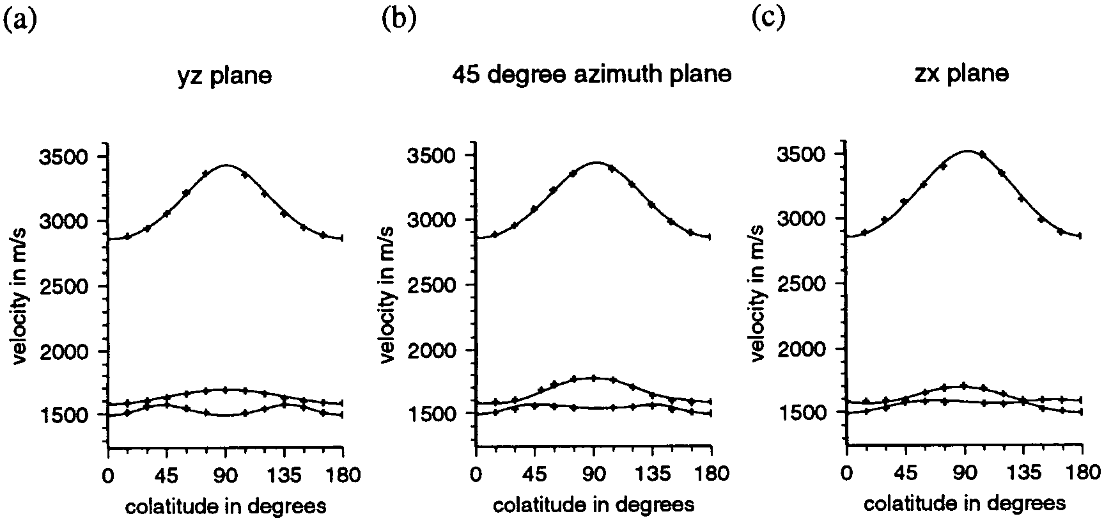


FIG. 4 Group-velocity plots for the phenolic sphere. The crosses represent the data points from the experiment and the solid lines are the group velocities from the inverted stiffnesses.

Despite the lack of robustness and accuracy in the group-velocity-inversion method that was apparent in the numerical example, the inversion yielded relatively accurate stiffnesses with an excellent fit of the model velocities to the velocities measured in the lab. Whatever this inversion lacks in numerical accuracy and stability, it appears to make up for when used on real data. In the measuring of group traveltimes, there is freedom to take several measurements from many different angles without the need to cut faces into the material whose normals are in the desired direction of wave propagation. It is this freedom to make as many measurements as are desired and the freedom to make the measurements as consistent as possible that redeems this inversion method.

Why bother with two separate inversions?

The difference between group and phase velocities is relatively small (≈ 2% in Vestrum and Brown (1993)) even for a medium of moderate (≈ 20%) anisotropy. Arts (1993) performs an experiment where traveltimes are measured through a small (6.5-cm diameter) sphere. He then performs his phase-velocity inversion on the velocities calculated from those traveltimes arguing that there is very little (< 1%) difference between group and phase velocities in a medium with weak (< 10%) anisotropy. The question is: will there be a significant difference a medium of moderate anisotropy between the phase-velocity-inverted stiffnesses and the group-velocity-inverted stiffnesses or is the additional effort involved in the group-velocity inversion a waste of computational time?

	Group-velocity inverted stiffnesses (GPa)						Phase-velocity inverted stiffnesses (GPa)					
	n=1	n=2	n=3	n=4	n=5	n=6	n=1	n=2	n=3	n=4	n=5	n=6
m=1	16.85	7.88	6.81	0.07	-0.18	0.12	16.37	7.47	6.44	0.04	-0.16	-0.01
m=2	7.88	16.03	6.51	0.00	-0.26	-0.08	7.47	15.58	6.20	-0.00	-0.26	-0.11
m=3	6.81	6.51	11.14	0.00	-0.05	-0.04	6.44	6.20	11.03	0.01	-0.03	-0.15
m=4	0.07	0.00	0.00	3.03	0.01	0.04	0.04	-0.00	0.01	3.00	0.04	0.03
m=5	-0.18	-0.26	-0.05	0.01	3.40	-0.01	-0.16	-0.26	-0.03	0.04	3.35	0.01
m=6	0.12	-0.08	-0.04	0.04	-0.01	3.89	-0.01	-0.11	-0.15	0.03	0.01	3.88

TABLE 3 Stiffnesses from the group- and phase-velocity inversions of the same data.

In trying to answer this question, an inversion was performed on the velocity data from the large sphere assuming that the velocities are phase velocities. These stiffnesses are then compared with the stiffnesses already derived from the group-velocity inversion. Table 3 shows the stiffnesses from the group-velocity inversion and the phase-velocity inversion of the same data. The differences between corresponding elements in the stiffness matrix are listed in table 4.

	Difference between the stiffnesses (GPa)					
	n=1	n=2	n=3	n=4	n=5	n=6
m=1	0.48	0.41	0.38	0.02	-0.02	0.14
m=2	0.41	0.44	0.31	0.00	0.00	0.03
m=3	0.38	0.31	0.11	-0.00	-0.02	0.11
m=4	0.02	0.00	-0.00	0.03	-0.03	0.01
m=5	-0.02	0.00	-0.02	-0.03	0.04	-0.01
m=6	0.14	0.03	0.11	0.01	-0.01	0.02

TABLE 4 Differences between the group-velocity inverted stiffnesses and the phase-velocity inverted stiffnesses.

There appears to be a fairly significant difference in stiffnesses between these two inversions and these differences are not uniform over the entire matrix. For example, the difference between the values for C_{44}, C_{55}, and C_{66} are very small ($\leq$ 0.04 GPa or $\leq$ 1%) compared to the differences between the values for C_{12} and C_{13} which are over 5%.

This exercise shows that there can be a fairly significant difference between the stiffnesses derived from the group-velocity inversion and from the phase-velocity inversion. This difference emphasizes the value of using the correct inversion method with the appropriate type of data. It is theoretically incorrect to perform a phase-velocity inversion on group-velocity data and there can be significant differences between stiffnesses when the wrong type of velocity is assumed in the inversion when dealing with velocities from moderately or highly anisotropic materials.

There are also significant differences between the stiffnesses of table 1 (from phase velocities) and table 2 (from group velocities). However, the velocities for these two cases were measured on two different samples of phenolic and we believe these differences to be real. The internal inconsistency and low error level of each inversion support this conclusion.

Inversion for a not-so-general elastic tensor

The previous work on the phenolic laminate cited here (Brown et al. 1991, 1993; Cheadle et al. 1991) was done assuming that the anisotropy of the material belonged to the orthorhombic symmetry class. In order to investigate whether or not the velocity data obtained in the laboratory may be interpreted as velocities from an orthorhombic medium, an inversion was attempted in which only the nine independent stiffnesses required to define an orthorhombic medium were free to vary.

In the orthorhombic case, the nine independent non-zero stiffnesses in the general anisotropic stiffness tensor are as follows:

$$C_{mn} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix}. \qquad 12$$

The inversion procedure is identical to the general inversion with the exception that all 12 of the stiffnesses not included in equation 12 are held at zero throughout the inversion process. This ensures that the inversion program will derive the best solution for stiffnesses with orthorhombic – or higher order – symmetry.

When the program was run on the group-velocity data from the sphere, the velocity error was minimized at 16.1 m/s. This is more than double the velocity error found in the inversion of the same data for the general elastic tensor, but this is still less than 1% (≈0.6%) of the average velocity.

Table 5 contains the stiffnesses and uncertainties for the inversion that assumes that the medium is orthorhombic. Note that the stiffnesses are not very different from those calculated in the general inversion (table 2), but the uncertainties are almost double the uncertainties associated with the general inversion. This is not surprising, since the velocity error for this inversion is twice that of the previous inversion.

The match of the laboratory velocities to the model velocities does not appear to be quite as good in this case, as seen in figure 5, as in the general inversion, shown in figure 4. The most significant discrepancy between the model and observed velocities appears to be in the qP-wave velocity. The observed velocities appear to be shifted slightly to the right in figures 4b and 4c. This shift of a few degrees in colatitude could be due to an error made while aligning the transducers at the estimated z-axis or zero-colatitude points.

	Final inverted stiffnesses (GPa)						Error in the stiffnesses (GPa)					
	n=1	n=2	n=3	n=4	n=5	n=6	n=1	n=2	n=3	n=4	n=5	n=6
m=1	17.00	7.85	6.65	0	0	0	0.11	0.09	0.06	0	0	0
m=2	7.85	15.97	6.56	0	0	0	0.09	0.10	0.06	0	0	0
m=3	6.65	6.56	11.16	0	0	0	0.06	0.06	0.04	0	0	0
m=4	0	0	0	3.03	0	0	0	0	0	0.02	0	0
m=5	0	0	0	0	3.35	0	0	0	0	0	0.02	0
m=6	0	0	0	0	0	3.88	0	0	0	0	0	0.03

TABLE 5 Final inverted stiffnesses and their respective uncertainties from the group-velocity inversion of the laboratory data assuming an orthorhombic medium.

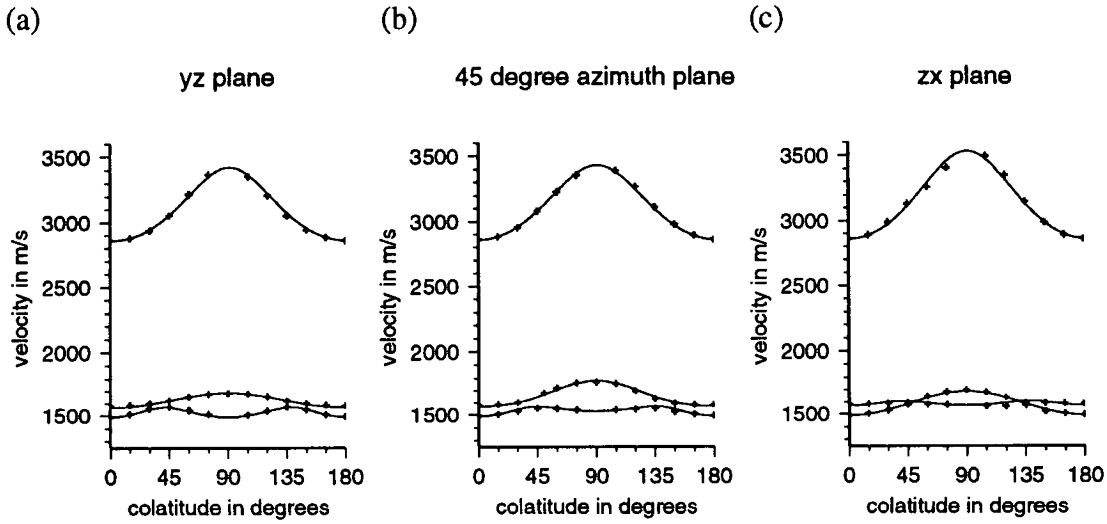


FIG. 5 Group-velocity plots for the phenolic sphere. The crosses represent the data points from the experiment and the solid lines are the group velocities from the orthorhombic stiffnesses.

The assumption that this material displays orthorhombic symmetry appears to be quite reasonable. A solution for the stiffnesses of the large sphere was found that fits the data within a reasonable margin of experimental error when orthorhombic symmetry was assumed. The inversion used here is a little faster than the inversion for the general elastic tensor and would be appropriate for media that display cubic, hexagonal or orthorhombic symmetry where the orientation of the principal axes is known.

CONCLUSIONS

A method has been developed to calculate the twenty-one independent elastic stiffnesses from either group or phase velocities. This method is general in the sense that it requires no prior knowledge of the symmetry class of the medium and no prior knowledge of the polarization of the individual wave phases. The which-came-first criterion is used to decide which velocity is associated with a particular wave phase in the sense that the qP is the first or primary phase, the qS_1 is the secondary phase and the qS_2 is the third phase.

An additional inversion step is necessary for the group-velocity inversion since the group velocity cannot be directly calculated in a prescribed direction. This search for a group velocity adds computational complexity and decreases the accuracy of the inversion. Even though the group-velocity inversion is not as robust or as accurate as the phase-velocity inversion, the inversion of the numerical-model data yielded stiffnesses nearly identical to the input model stiffnesses.

Each of the three inversions performed on the data from the two laboratory experiments generated stiffnesses that yield forward-model velocities that match up well with the velocities observed in the laboratory.

Despite the limitations imposed by the algorithm and shown in the numerical testing, the group-velocity inversion for the general anisotropic tensor yielded the best results in its inversion of the 99 velocity data-points from the sphere of phenolic. The smallest velocity error (7.7 m/s) and the least average statistical uncertainty in the stiffnesses (0.04 GPa) came out of this particular application of the algorithm. The excellent performance of this inversion is attributed to the freedom that the experimenter has to make as many measurements in whatever directions as are desirable, without having to cut the sample so that plane waves may be generated.

In contrast to the group-velocity method, the phase-velocity inversion was the most robust and most accurate method in the theory and numerical testing, but did not perform as well when faced with real laboratory data. The only drawback to the application of this method appears to be the limitation of inverting only plane-wave velocities. The model velocities fit the observed velocities with a statistical error of 18 m/s resulting in an average uncertainty in the stiffnesses of 0.16 GPa, statistical errors that are substantially higher than those for the group-velocity inversion. Despite this higher statistical error due to fewer measurements, the errors from the phase-velocity inversion are still well within the uncertainty estimates for the laboratory measurements.

It was shown here that if we assume that the velocities from the sphere are phase velocities so that we can use the faster and more-accurate phase-velocity inversion, the resulting stiffnesses can be significantly different than the stiffnesses resulting from a group-velocity inversion. Even though there isn't much difference between the group and phase velocities of this moderately anisotropic medium, it is important to use the appropriate inversion when dealing with moderately to highly anisotropic media if accurate stiffnesses are to be calculated.

One additional group-velocity inversion was performed on the data from the phenolic sphere to see how well the model would fit the data if the medium were constrained to be orthorhombic, with the symmetry axes coincident with the coordinate axes used in the experiment. This orthorhombic model fits very well to the observed velocity data. The velocity error of 16.1 m/s, 0.6% of the average velocity, is well within the experimental error estimate of ±2%. From this inversion exercise one may conclude that the assumption of orthorhombic symmetry appears to be valid for this medium.

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