Analytical inversion for Thomsen's γ parameter in weakly anisotropic layered media

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SUMMARY

An analytic scheme for traveltime inversion in horizontally layered media exhibiting transverse isotropy with a vertical symmetry axis (TIV) is presented. It pertains to SH waves, i.e., characterized by a single Thomsen parameter, γ, or to other waves exhibiting approximately elliptical velocity dependence. The proposed method incorporates ray bending at interfaces.

The suggested approach is particularly suitable for vertical seismic profiling as best providing required information. The input information consists of traveltime for obliquely travelling downgoing waves which can be measured by an offset survey, vertical wave speed from the zero-offset survey, and layer thicknesses directly from the wellbore.

Correct results necessitate high accuracy of all input parameters. The proposed inversion method is particularly sensitive to errors in measured traveltime.

INTRODUCTION (RUDIMENTARY EXPRESSIONS)

In a medium composed of *n* layers, the time, *t*, of transmission of a signal across the given medium between a point source and point receiver can be written as:

$$t = \sum_{i=1}^{i=n} \frac{l_i}{V_i}, \tag{1}$$

where l_i is a distance travelled in the *i*th layer, while V_i is a corresponding group velocity. If the medium is anisotropic, the value of the group velocity depends on the angle of propagation.

Group velocity, V, is related to phase velocity, v, by the following formula (Berryman, 1979):

$$V[\theta(\vartheta)] = \sqrt{v^2(\vartheta) + \left(\frac{dv}{d\vartheta}\right)^2}, \tag{2}$$

where θ and ϑ are group and phase angles respectively. The relation between group and phase angles is given by (Berryman, 1979) as an equation for the slope of the normal to the phase slowness curve expressed in polar coordinates:

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$$\tan[\theta(\vartheta)] = \frac{\tan\vartheta + \dfrac{1}{v(\vartheta)}\dfrac{dv}{d\vartheta}}{1 - \dfrac{\tan\vartheta}{v(\vartheta)}\dfrac{dv}{d\vartheta}}. \tag{3}$$

The expression for phase velocity of an SH wave in a transversely isotropic medium using the Thomsen parameter γ is:

$$v(\vartheta) = \beta\sqrt{1 + 2\gamma\sin^2\vartheta}\,, \tag{4}$$

where β is the wave speed along the symmetry axis. Inserting equation (4) into equation (2) gives:

$$V[\theta(\vartheta)] = \beta\sqrt{\frac{1 + 2\gamma(1+\gamma)(1-\cos 2\vartheta)}{1 + \gamma(1-\cos 2\vartheta)}}. \tag{5}$$

Similarly, inserting equation (4) into equation (3) gives:

$$\tan[\theta(\vartheta)] = (1 + 2\gamma)\tan\vartheta\,. \tag{6}$$

Solving equation (6) for the phase angle yields:

$$\vartheta = \arctan\frac{\tan\theta}{1 + 2\gamma}. \tag{7}$$

All equations above are exact. Inserting equation (7) into (5) constitutes an expression for group velocity as a function of group angle. Developing this expression as the Maclaurin series in γ, and keeping only the linear term gives:

$$V(\theta) \approx \beta\left[1 + \gamma\sin^2\theta\right]. \tag{8}$$

The development into a series and subsequent truncation is based on the approach proposed by Thomsen (1986). It assumes that the anisotropic parameter is much smaller than unity. In the context of the proposed inversion scheme, reasonably reliable results can be expected if $\gamma < 0.2$. Notice that in spite of its simplicity, equation (6) is exact while in Thomsen's (1986) presentation it is implied to result from full linearization. Furthermore, one can state that the first-order approximation yields:

$$V[\theta(\vartheta)] \approx v(\vartheta)\,, \tag{9}$$

i.e., group and phase velocities are equal to first order, since developing equation (4) into a Maclaurin series and truncating higher-order terms, gives:

$$v(\vartheta) \approx \beta\left[1 + \gamma\sin^2\vartheta\right]. \tag{10}$$

A graphical illustration of Figure 1 emphasizes the reliability of the approximation. The two outer (almost coincident) curves correspond to equation (5) with the angle defined by equation (7) and to equation (8). The inner curve is an isotropic, i.e., circular wavefront.

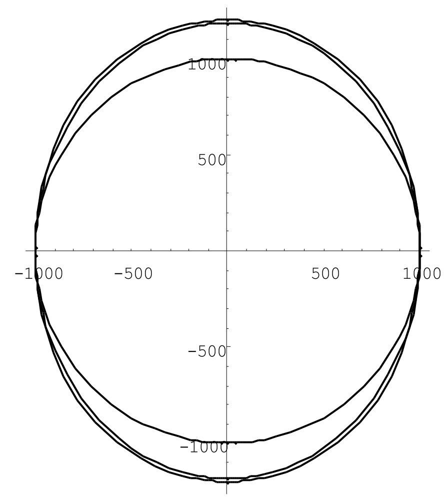

FIG.1. Wavefront plots constructed using equation (5) with the angle defined by equation (7) (exact representation) and using equation (8), (approximate representation) almost coincide for weak anisotropy (γ = 0.2 and β = 1000). The inner circular wavefront corresponds to an isotropic case (γ = 0). The symmetry axis is horizontal.

The use of the fully linearized approximation neglects certain physical attributes of wave propagation in anisotropic media and thus has to be used with proper understanding of assumptions and consequences. Notably, for P and SV waves, which are characterized by two anisotropic parameters, the first-order approximation does not yield as good a fit as it does for SH waves, which always exhibit elliptical velocity dependence. In the subsequent development the linearization serves to give a traveltime inversion scheme for a single anisotropic parameter γ characterizing SH waves.

Traveltime inversion (mathematical formulation)

Consider transmission through a two-layer medium illustrated in Figure 2. Let the upper (surface) layer be isotropic and the lower (buried) layer be anisotropic, belonging to the symmetry class of transverse isotropy with a vertical symmetry axis. Based on traveltime measurements, we would like to infer the anisotropic parameter of the buried layer. We shall employ the geometry of the vertical seismic profile and focus our attention on the downgoing wave.

Using quantities defined in Figure 2 and equation (8), we can write equation (1) as follows:

$$t = \frac{\sqrt{(X-r)^2 + H_1^2}}{V_1} + \frac{\sqrt{r^2 + H_2^2}}{\beta\left(1+\gamma \dfrac{r^2}{r^2 + H_2^2}\right)} \qquad (11)$$

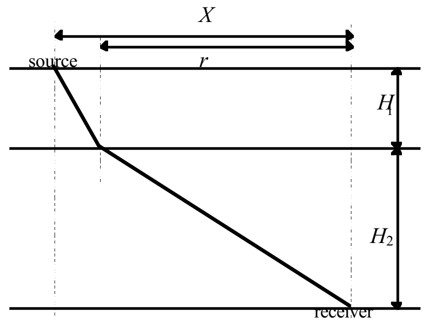

FIG.2. Two-layer isotropic/anisotropic model.

We can also impose the condition of Fermat's principle of stationary time, namely:

$$\frac{dt(r)}{dr} = 0. \tag{12}$$

It can be proven using the implicit function theorem (see Appendix 1) that having measured the traveltime $t(r;\gamma) = T$, the appropriate value of anisotropic parameter can be uniquely determined as a solution of the simple equation:

$$\frac{d[\gamma(r)]}{dr} = 0. \tag{13}$$

Let us solve equation (11) for γ, and denote the measured time as T. We obtain:

$$\gamma = \frac{V_1\left(r^2 + H_2^2\right)^{\frac{3}{2}}}{\beta r^2\left[TV_1 - \sqrt{(X-r)^2 + H_1^2}\right]} - \frac{r^2 + H_2^2}{r^2} \tag{14}$$

The appropriate value of r must lie between 0 and X (see Figure 2). since it is found it can be inserted into equation (14) and corresponding value of γ calculated. The entire process can be easily accomplished with a mathematical software package (see Appendix 2 for a Mathematica® program).

Traveltime inversion (geometrical visualization)

The solution can be visualized geometrically (Slawinski, 1996) by plotting equation (11) as a surface spanned by r and γ on the horizontal axes. The measured traveltime is illustrated by a horizontal plane intersecting the surface. The curve of intersection can be viewed as an expression of γ in terms of r at a constant value of $t = T$, i.e., $\gamma(r;T)$. The solution is illustrated by the nadir of the intersection curve (see Figure 3) whose coordinates correspond to the appropriate values of γ and r. Hence an extremum of this curve is found by equation (13).

Further justification of the process can be obtained by plotting the derivative surface of equation (11) spanned by r and γ on the horizontal axes and intersected by a horizontal plane corresponding to equation (12).

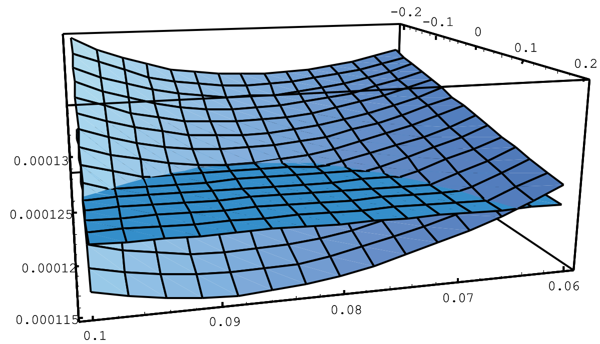

FIG.3. Geometrical illustration of solution which corresponds to the nadir of the intersection curve between the surface and the plane. Horizontal axis (parallel to the page) corresponds to $r/10X$; range (0.06 to 0.1). Horizontal axis perpendicular to the page corresponds to γ; range (-0.2 to 0.2). Vertical axis corresponds to traveltime.

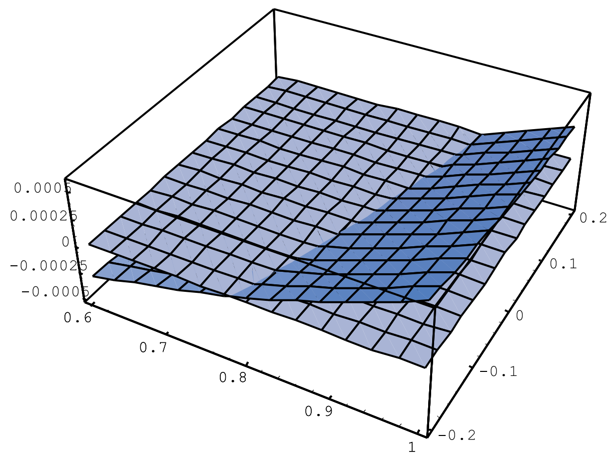

FIG. 4. The inclined surface corresponding to the derivative $dt(\gamma,r)/dr$ and the horizontal plane representing $dt/dr = 0$.

The line of intersection between the surface and the plane is a set of points obeying Fermat's principle (see Figure 4). In the projection onto the $r\gamma$ plane, the solution corresponds to the common point of two intersection curves from Figures 3 and 4. The intersection of the steeply dipping Fermatian curve with the U-shaped curve always occurs in the nadir of the latter (see Figure 5).

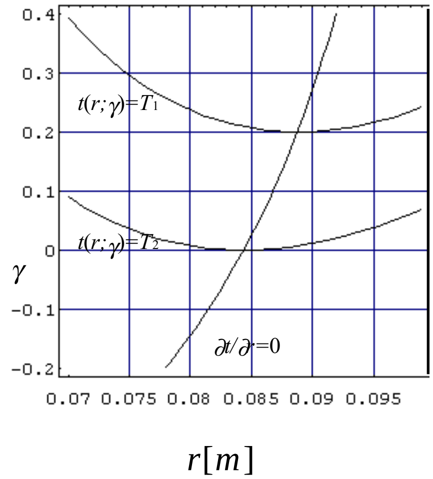

FIG.5. The projections of the curves $dt/dr = 0$ and $\gamma(r,T)$ on the $r\gamma$ plane. The two chosen values of T correspond to the results of forward modelling obtained with $\gamma = 0$, and $\gamma = 0.2$. There is a good agreement in this illustration of the inverse solution.

Computational example

Consider an isotropic layer of thickness 355 meters and SH-wave velocity of 1030 m/s, superposed on an anisotropic (TIV) layer of thickness 1045 meters, vertical wave speed of 1609 m/s and the anisotropic parameter $\gamma = 0.096$. Both layers are horizontal and separated by a planar interface.

The source is placed on top of the isotropic layer, while the receiver is positioned on the bottom of the anisotropic layer. The traveltime for a range of horizontal source-receiver offsets calculated using the linearized method is given in Table 1. Also, the value of anisotropic parameter, γ, calculated using the traveltime inversion scheme is shown.

Offset (m)	Traveltime (s)	Inverted γ
0	1.25	N/A
90	1.25104	*0.0959772*
190	1.27558	*0.0961738*
290	1.3484	*0.0958288*
390	1.02475	*0.0959165*
490	1.04204	*0.0959752*
590	1.06286	*0.0959918*
690	1.08699	*0.0959842*
790	1.11419	*0.0959669*
890	1.1442	*0.0959865*
990	1.17678	*0.0959947*

Table 1. Computational results of traveltime inversion

CONCLUSIONS

The proposed inversion scheme allows one to calculate the value of Thomsen's anisotropic parameter γ for horizontally layered TIV media. The information required for inversion consists of layer thicknesses and vertical wave speeds. The acquisition context of vertical seismic profiles is particularly suitable for this inversion scheme. Layer thicknesses can be obtained directly from wellbore information, while the vertical wave speed can be reliably established from the zero-offset survey.

The inversion is expressly formulated for SH waves since their anisotropic behaviour is characterized by a single Thomsen parameter, γ. The method can also be applied to any wave exhibiting approximately elliptical velocity dependence.

Although illustrated above for a two-layer case, the method can be used in a multilayer setting. In such a scenario the inversion is performed consecutively for deeper layers, in such a way that all parameters above the layer in question are known (Slawinski, 1996).

The results of inversion are highly sensitive to the traveltime measurement. This characteristic can be visualized by examining Figure 2 where, due to the gentle dip of the traveltime surface, a slight vertical shift results in large lateral displacement of the intersection curve.

APPENDIX 1

Suppose there exists a function of three variables, T, r, and γ:

$$F(T,r,\gamma) = t(r,\gamma) - T, \tag{A1}$$

for which all first derivatives exist in an open set D, $D \subset \Re^3$ (this is expected to hold true in all cases of physical interest); and that there exists a point $P(T_0,r_0,\gamma_0)$, $P \in D$, such that $F(P) = 0$, i.e., $t(r_0,\gamma_0) = T_0$, and that $\partial t/\partial\gamma \neq 0$ at P. Then by the implicit function theorem, there exists a function $g(T, r)$ possessing all first derivatives, that satisfies the equation:

$$F(T,r,g(T,r)) = t(r,g(T,r)) - T = 0, \tag{A2}$$

in an open neighbourhood $N \subset \Re^2$ of $(T_0,r_0) \in \Re^2$, such that $g(T_0,r_0) = \gamma_0$.

Geometrically, note that $F(T,r,\gamma) = 0$ implicitly defines a 2-D surface in $\Re^3$. Now, consider a curve on this surface defined by its intersection with the surface $T = T_0$. The implicit form of the equation of this curve, with r as the parameter, is:

$$F(T_0,r,g(T_0,r)) = t\,(r,g(T_0,r)) - T_0 = 0. \tag{A3}$$

Differentiating equation (A3) with respect to r gives:

$$\frac{dF}{dr} = \frac{d\left[t(r,g(T_0,r)) - T_0\right]}{dr} = \frac{\partial t}{\partial r} + \frac{\partial t}{\partial g}\frac{d\left[g(T_0,r)\right]}{dr} \tag{A4}$$

Equation (A4) is satisfied everywhere along the curve. Now consider the point defined by the intersection of the above curve with the surface implicitly defined by the equation $\partial t(r, \gamma)/\partial r = 0$. Then, at this point, the following equation holds:

$$\frac{\partial t}{\partial g}\frac{dg(T_0,r)}{dr}=0. \tag{A5}$$

Assume $\partial t/\partial g = \partial t/\partial \gamma \neq 0$; then:

$$\frac{dg(T_0,r)}{dr}=0. \tag{A6}$$

Hence, at the point defined by the intersection of the surfaces:

$$\begin{cases} t(r,\gamma)=T \\ \dfrac{\partial t}{\partial r}=0 \\ t(r,\gamma)=T_0 \end{cases} \tag{A7}$$

corresponding to ray traveltime for specific r and γ, Fermat's principle of stationary time, and a particular (measured) time, respectively, the following equation holds:

$$\frac{dg(T_0,r)}{dr}=0, \tag{A8}$$

where the function $g(T,r)$ is obtained by solving $F(T,r,\gamma)=0$ for γ.

APPENDIX 2

A Mathematica® program for the calculation of Thomsen's parameter by traveltime inversion:

X = *horizontal source - receiver distance*
HJ = *thickness of the upper, isotropic layer*
HD = *thickness of the lower anisotropic layer*
V = *wave speed in the upper layer*
B = *vertical wave speed in the lower layer*
TT = *one way traveltime*

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D[V*(r^2+HD^2)*Sqrt[r^2+HD^2]/(B*r^2*(TT*V-Sqrt[(X-r)^2+HJ^2]))-
(r^2+HD^2)/r^2,r]

FindRoot[% == 0, {r, 0.01*X, X/2, 0.99*X}]

N[V*((r/.%)^2+HD^2)*Sqrt[(r/.%)^2+HD^2]/(B*(r/.%)^2*(TT*V-Sqrt[(X-
(r/.%))^2+HJ^2]))
-((r/.%)^2+HD^2)/(r/.%)^2]
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REFERENCES

Berryman, J.G., Long-wave elastic anisotropy in transversely isotropic media: Geophysics, **44**, 896-917.

Slawinski, M.A., 1996, On elastic-wave propagation in anisotropic media: reflection/refraction laws, raytracing and traveltime inversion, Ph.D. Thesis, Univ. of Calgary.

Thomsen, L., 1986, Weak elastic anisotropy: Geophysics, **51**, 1954-1966.