The P-SV conversion point in constant-gradient media

Michael A. Slawinski† and Gary F. Margrave

ABSTRACT

An analysis of the computation of the P-SV conversion point for the case of constant-gradient (velocity increasing linearly with depth) media is presented. The well known result that raypaths are circular arcs in such media is exploited to derive an implicit sixth order polynomial for r, the distance from the source to the conversion point. An exact expression is then obtained for the asymptotic limiting form as depth of the reflector becomes much larger than the source-receiver separation. The result is formally similar to that for constant velocity case except that the ration of vs/vp is replaced by the ratio of the velocity gradients. It is anticipated that a numerical solution for the nonasymptotic case will yield more significant departures from constant velocity theory.

INTRODUCTION

In exploration geophysics, for a significant range of depth, the velocity increases approximately linearly with depth. Slotnick (1959) states that "experience has shown that the velocity of seismic wave propagation in Tertiary basins can be closely approximated by expressing it as a linear function of depth".

The use of a constant-gradient velocity function yields more accurate results than a constant-velocity function while maintaining the elegance of expressions and the mathematical ease of manipulations. It also removes a confining assumption of straight rays. The introduction of curved rays is important in considering the location of the reflection point and the angles of incidence, e.g., in AVO/AVA studies. The consequences of curved raypaths are particularly important in the study of converted waves due to asymmetry introduced by different functions describing downgoing and upgoing rays.

This paper provides an implicit analytic equation for the reflection point of a converted raypath between a surface source and a surface receiver and an explicit expression for this reflection point in the limit a $z \to \infty$. The method could be easily extended to the VSP case where the receiver is located in the wellbore. Also, it provides an analytic method for estimating the angle of incidence.

METHODS AND RESULTS

Geometrical considerations

Consider a reflection from a horizontal interface at depth, H, in an isotropic medium where velocity has a constant vertical gradient and no lateral variations. The downgoing wave is subject to the velocity field, v, increasing linearly with depth, z, such that instantaneous velocity at point z is

† Department of Mechanical Engineering, The University of Calgary

$$v(z) = a + bz\,, \tag{1}$$

where a is the speed at the free surface and b is a positive constant. In the plains of Western Canada the value of b = 0.8 appears to agree well with experimental data (Goodway (1997) pers. comm.). The upgoing wave is subject to the velocity field, w, increasing linearly with depth, z, such that instantaneous velocity at point z is

$$w(z) = c + dz\,, \tag{2}$$

where c is the speed at the free surface and d is a positive constant. The ray trajectories for both downgoing and upgoing waves are circular arcs. The centre of the circular arc, C, corresponding to the trajectory of the downgoing wave has following coordinates (x,z) (Slotnick, 1959, or Krebes 1985),

$$C\left(\frac{1}{pb}\sqrt{1-(pa)^2}, \frac{a}{b}\right), \tag{3}$$

where p is the ray parameter (Figure 1). The centre of the circular arc, D, corresponding to the trajectory of the upgoing wave has following coordinates (x,z)

$$D\left(\frac{1}{pd}\sqrt{1-(pc)^2}, \frac{c}{d}\right). \tag{4}$$

Consider the source, S, with coordinates

$$S(0,0)\,, \tag{5}$$

and the receiver, R, with coordinates

$$R(X,0)\,. \tag{6}$$

(Note that either source or receiver can be located away from the surface. For instance, in the VSP case, the receiver would have a non-zero depth coordinate, i.e., receiver at depth Z. Hence, there would be another non-zero parameter in the expressions shown below.)

Let the reflection point, Q, have coordinates

$$Q(r,-H)\,, \tag{7}$$

where r is the lateral distance between the source, S, and the reflection point, Q.

The distance CS must be equal to CQ because they are both radii of the same circle and similarly, DR must be equal to DQ. Those equalities form the kernel of useful equations.

Exact equation for conversion point

For algebraic reasons it is more convenient to equate the squares of the above distances. Thus

$$(CS)^2 = (CQ)^2, \tag{8}$$

and

$$\left(DR\right)^2 = \left(DQ\right)^2 . \tag{9}$$

In terms of coordinates one gets

$$\left(\frac{1}{pb}\sqrt{1-\left(pa\right)^2}\right)^2 + \left(\frac{a}{b}\right)^2 = \left(\frac{1}{pb}\sqrt{1-\left(pa\right)^2} - r\right)^2 + \left(\frac{a}{b} + H\right)^2 , \tag{10}$$

and

$$\left(\frac{1}{pd}\sqrt{1-\left(pc\right)^2} - X\right)^2 + \left(\frac{c}{d}\right)^2 = \left(\frac{1}{pd}\sqrt{1-\left(pc\right)^2} - r\right)^2 + \left(\frac{c}{d} + H\right)^2 . \tag{11}$$

Both equations (10) and (11) can be solved for the ray parameter, p, to yield respectively

$$p = \frac{2r}{\sqrt{\left(H^2 + r^2\right)\left(4a^2 + 4abH + b^2\left(H^2 + r^2\right)\right)}}, \tag{12}$$

and

$$p = \frac{2\left(r - X\right)}{\sqrt{4c^2\left(H^2 + \left(r - X\right)^2\right) + 4cdH\left(H^2 + r^2 - X^2\right) + d^2\left(H^2 + r^2 - X^2\right)^2}}. \tag{13}$$

The ray parameter, p, is preserved upon the reflection from a horizontal interface. Thus right-hand sides of equations (12) and (13) can be equated. Hence, the only unknown is the horizontal coordinate, r, of the reflection point $Q(r, -H)$. The resulting algebraic equation

$$\frac{r}{\sqrt{\left(H^2 + r^2\right)\left(4a^2 + 4abH + b^2\left(H^2 + r^2\right)\right)}} - \frac{\left(r - X\right)}{\sqrt{4c^2\left(H^2 + \left(r - X\right)^2\right) + 4cdH\left(H^2 + r^2 - X^2\right) + d^2\left(H^2 + r^2 - X^2\right)^2}} = 0, \tag{14}$$

implicitly determines r. It can be squared and manipulated to yield an implicit sixth order polynomial in r (see equation (16)). It can be conveniently solved numerically. For instance, one can use the “FindRoot” command in the Mathematica® software. Inserting the appropriate value of r into equation (12) yields the ray parameter, p, from which the angle of incidence, θ, can be easily calculated, i.e.,

$$\theta = \arcsin\left(p(a + bH)\right). \tag{15}$$

Asymptotic equation for conversion point

Placing two fractions of equation (14) on either side of the equality sign, squaring both sides of the equation and cross multiplying yields

$$r^2\left[4c^2\left(H^2+(r-X)^2\right)+4cdH\left(H^2+r^2-X^2\right)+d^2\left(H^2+r^2-X^2\right)^2\right]= \\ =(r-X)^2\left(H^2+r^2\right)\left(4a^2+4abH+b^2\left(H^2+r^2\right)\right) \tag{16}$$

Rearranging and equating the limit of both sides one gets,

$$\lim_{H\to\infty}\{r^2\left[4c^2\left(H^2+(r-X)^2\right)+4cdH\left(H^2+r^2-X^2\right)+d^2\left(H^2+r^2-X^2\right)^2\right]- \\ (r-X)^2\left(H^2+r^2\right)\left(4a^2+4abH+b^2\left(H^2+r^2\right)\right)\}=0. \tag{17}$$

Taking a limit of the expression (17) as H tends to infinity, i.e., at a very large depth, yields (after tedious algebra)

$$d^2r^2-b^2(r-X)^2=0\,. \tag{18}$$

Solving for r one gets,

$$r=\frac{X}{1\pm\frac{d}{b}}\,. \tag{19}$$

For the P-SV conversion one uses the sum in the denominator. Numerical examples performed on equation (14) confirm expression (19). Equation (19) has a similar form to the asymptotic equation for the conversion point in constant velocity media, i.e.,

$$r=\frac{X}{1+\frac{v_{SV}}{v_P}}, \tag{20}$$

where v_{sv} and v_P are shear- and compressional-wave velocities, respectively. The similarity of form between equations (19) and (20) can, perhaps, be explained by the fact that for a very large depth, rays are almost straight, thus resembling constant-velocity medium. There are, however, significant differences. Equation (19) depends on the velocity's rate of change, while equation (20) depends on the velocity itself.

DISCUSSION AND CONCLUSIONS

To investigate equation (14) one can generate a three-dimensional plot (Figure 2) with horizontal axes corresponding to the reflector depth, H, and the horizontal coordinate, r, of the reflection point $Q(r, -H)$ The solution, r, corresponds to the intersection of the surface generated by the left-hand side of equation (14) and the horizontal plane at zero. One observes that for large values of, H, the value of r approaches an asymptote, as expected. The range of the ratio of values of d and b is limited. To obey the physical principles of wave propagation one must require that, for isotropic media,

$$0\le w(z)\le\frac{v(z)}{\sqrt{2}}, \tag{21}$$

or

$$0 \le c + dz \le \frac{a + bz}{\sqrt{2}} . \qquad (22)$$

For very large values of z, it implies that

$$0 \le d \le \frac{b}{\sqrt{2}} , \qquad (23)$$

or from equation (19)

$$0.6X \approx \frac{\sqrt{2}X}{\sqrt{2}+1} \le r \le X . \qquad (24)$$

The same condition applies to constant velocity medium, i.e., the lateral range of the asymptotic vertical line is the same for constant-gradient and constant-velocity media. Furthermore, it would appear that at large depths, where the asymptotic approach applies, both asymptotic approaches might yield similar results. A significant difference would be exhibited in the non-asymptotic approach which plays critical role for shallower horizons. For specific applications one can compare results of the constant-gradient approach, i.e., equation (14) to a constant velocity approach, i.e.,

$$\frac{d}{dr}\left(\frac{\sqrt{H^2 + r^2}}{v} + \frac{\sqrt{(X-r)^2 + H^2}}{w} \right) = 0, \qquad (25)$$

which is an expression of Fermat's principle of stationary time where v and w stand for constant compressional- and shear-wave velocities, yielding

$$\frac{r}{v\sqrt{r^2 + H^2}} = \frac{X - r}{w\sqrt{(X-r)^2 + H^2}} . \qquad (26)$$

Equation (26) is a quartic equation in r and thus can always be solved analytically.

From experimental data it is known that the value of the ratio of compressional- to shear-wave velocities can, between the surface and deeper targets, change by an order of magnitude. In such cases, equation (14) should provide better results than either constant-velocity or asymptotic approach. Particularly, for the case of large lateral offsets between source and receiver, encountered in VSP data, the proper use of equation (14) plays a significant role. Further analysis, e.g., AVO/AVA studies can be performed more reliably by removing the asymptotic and constant velocity assumptions.

Furthermore, the initial investigation suggests that in the constant-gradient case the function approaches the asymptotic value of r less rapidly than in the simpler, but less accurate, constant-velocity approach. This concepts needs to be further investigated. It, appears, however, that one should carefully perform error analysis while using an asymptotic approximation in constant-gradient approach for moderately deep targets.

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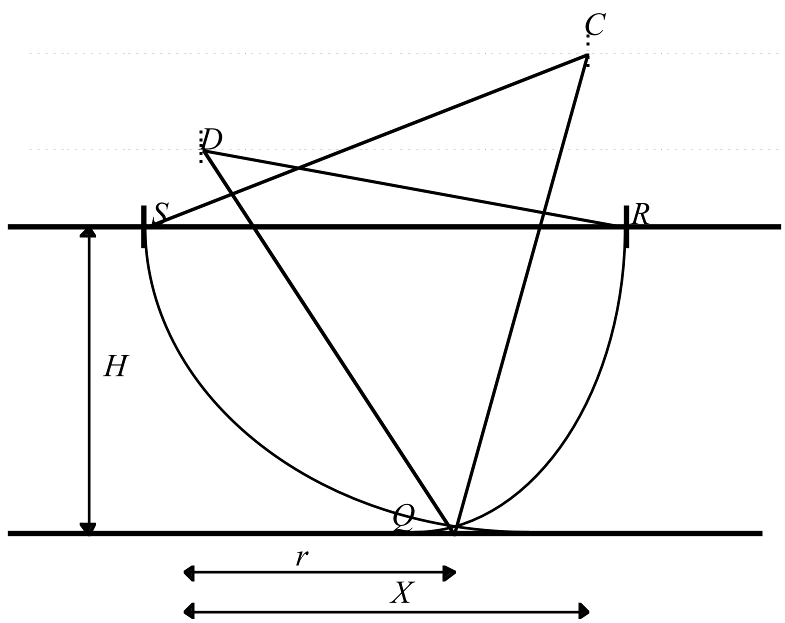

FIG. 1. Illustration of parameters for converted wave scenario in a constant gradient medium. All ray trajectories are arcs of circles whose centres are above the source-receiver surface.

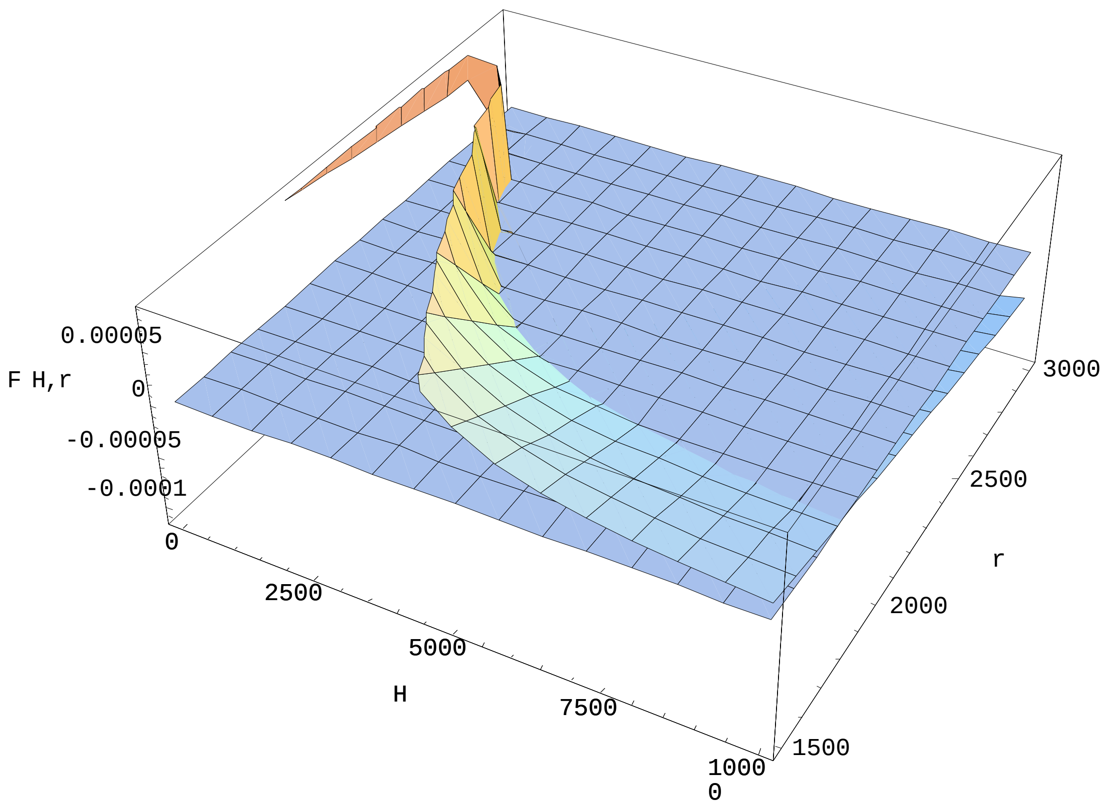

FIG. 2. Intersection of a surface with the horizontal plane corresponding to the solution for *r* of equation (14). The downgoing wave is subjected to the velocity field defined by $v(z)$ = 2,000+0.8z, while the upgoing wave is subjected to the velocity field defined by $w(z)$ = 1,000+0.6z. The horizontal source-receive offset, X = 3,000. The depth, H, varies between 0 and 10,000. Note that the application of given linear velocity functions is a subject to a condition $v(z) > \sqrt{2}w(z)$. This implies that the fundamental physical property derived from Poisson's ratio ($\sigma \in [0, 0.5]$), which is apparent in the ratio of compressional- and shear-wave velocities, is satisfied for $z < 12{,}070$ metres.