

## Tutorial: hybrid finite difference – finite transform methods for hyperbolic partial differential equations

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### ABSTRACT

The use of integral transforms has long been a solution method for hyperbolic equations or systems. Finite integral transforms, which are discussed here, became popular with the advent of digital computers, with the first references in the literature appearing in the early 1960's. Combining these transforms with finite difference methods has been shown to provide highly accurate numerical solutions to problems of wave propagation in elastic media, as the reduced spatial dimensionality employed in the finite difference part of the numerical solution results in a reduction of grid dispersion. Also, for problems where elastic parameters are dependent upon only one spatial coordinate, a considerable saving of the space required is achieved.

In this report, the simple case of the *SH* potential equation in a medium whose elastic parameters are only dependent upon depth is considered. This is a fairly simplistic case but does demonstrate, for tutorial purposes, many of the concepts required for the practical, numerical computation of synthetic traces using this method. Apart from a fairly standard treatment of the finite-difference aspect of the problem, the subject of replacing the infinite series in the inverse transform by a truncated finite series is explored for a band limited source wavelet.

### INTRODUCTION

The *SH* potential equation in a vertically-inhomogeneous, radially-symmetric halfspace problem is reduced to a finite difference problem in the vertical-spatial coordinate and time coordinate, by removing the radial coordinate (in a cylindrical coordinate system) through the implementation of a finite Hankel transform. The resulting problem is usually computationally faster than a finite difference program in two spatial dimensions. This has the added benefit that grid dispersion may be reduced significantly (to within 2 or 3 percent of the exact solution) by utilizing a band-limited source pulse and choosing the proper number of terms to adequately approximate the infinite series which comprises the inverse finite Hankel transform. In addition, for this problem specification, out-of-plane spreading is included in the solution, which is generally not the case for finite-difference schemes for two-dimensional geological models. This method of computing synthetic seismograms has come to be known in the literature as the Alekseev-Mikhailenko method.

The drawback of the formulation presented here is that only vertical and not lateral inhomogeneities may be included in the model description. This limits usage to models with plane parallel interfaces. Other finite-transform methods exist and are documented for inhomogeneous two- and three-dimensional media. However, the exploration of these problems would obscure the tutorial purpose of this report.

## THE SCALAR POTENTIAL SH WAVE EQUATION

In a radially-symmetric, isotropic, vertically-inhomogeneous media, the scalar potential equation for *SH* wave propagation may be written in cylindrical coordinates,  $(r, \theta, z)$ , as

$$\frac{\mu(z)}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{\partial}{\partial z} \left( \mu(z) \frac{\partial \phi}{\partial z} \right) - \rho(z) \frac{\partial^2 \phi}{\partial t^2} = 0 \quad (1)$$

As a consequence of assuming the medium radially symmetric, there is no dependence on  $\theta$  in (1). Lamé's parameter,  $\mu(z)$ , is defined in terms of the shear-wave velocity,  $v_{SH}(z)$ , and density,  $\rho(z)$ , as  $\mu(z) = \rho(z) v_{SH}^2(z)$ . A point source disturbance is assumed at  $(r = r_0, z = 0)$ . The boundary condition containing the source specification at the free surface,  $z = 0$ , is

$$\left. \frac{\partial \phi}{\partial z} \right|_{z=0} = \frac{\delta(r - r_0)}{2\pi r \mu_0} g(t) \quad (2)$$

with  $\mu_0 = \mu(z = 0)$  and  $g(t)$  the time dependence of the source wavelet. The required initial time dependent conditions are

$$\phi \Big|_{t=0} = \left. \frac{\partial \phi}{\partial t} \right|_{t=0} = 0 \quad (3)$$

The above equations are sufficient to specify the problem<sup>1</sup> under the assumption that some specification of  $\phi$  or its relevant spatial derivative is specified at some introduced finite boundary. Any solution will be required to satisfy radiation conditions at infinite boundaries. It has been stated that there are no lateral inhomogeneities within the medium being considered so that the problem may, with little effort, be reduced to one of decreased spatial dimension by applying a finite integral transform to the radial coordinate.

The finite Hankel transform is defined as

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<sup>1</sup> The equation for the displacement of the *SH* arrival  $\mathbf{u} = u \mathbf{e}_\theta$ ,  $\mathbf{e}_\theta$  a unit vector perpendicular to the plane of motion is given as

$$\mu(z) \left\{ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial u}{\partial r} \right) + \frac{u}{r^2} \right\} + \frac{\partial}{\partial z} \left\{ \mu(z) \frac{\partial u}{\partial z} \right\} - \rho(z) \frac{\partial^2 u}{\partial t^2} = 0,$$

subject to the initial conditions  $u \Big|_{t=0} = \left. \frac{\partial u}{\partial t} \right|_{t=0} = 0$  and the free boundary condition, due

to a point torque source specified as  $\left. \frac{\partial u}{\partial z} \right|_{z=0} = -\frac{g(t)}{2\pi\mu_0} \frac{d}{dr} \left[ \frac{\delta(r)}{r} \right]$ . The potential, used here,

is introduced through the substitution of  $\mathbf{u} = u \mathbf{e}_\theta = \nabla \times (0, 0, \phi)$  in the elastodynamic equation (Aki and Richards, 1980).

$$R(k_n, z, t) = \int_0^a \phi(x, z, t) J_0(k_n r) r dr \quad (4)$$

(Sneddon, 1972 and Appendix A), where  $k_n$  are the zeroes of the Bessel function  $J_0$  obtained from the transcendental equation

$$J_0(k_n a) = 0. \quad (5)$$

Additionally, the  $k_n$  are the discrete horizontal wavenumbers. This aspect will be discussed in more detail in a later section. The value of  $\phi$  at  $r = a$  is chosen such that this boundary is perfectly reflecting, requiring that

$$\phi|_{r=a} = 0 \quad (6)$$

Additionally,  $a$  is placed a sufficient distance from  $r=0$  so that any of the reflections from it due to the source located at  $(0 \leq r_0 < a, z = 0)$  will not arrive at any of the “geophone” positions  $(0 \leq r_G < a, 0 < z_G < b)$  within the time window for which the synthetic seismogram is to be computed.

The inverse transform of (4) is the infinite series

$$\phi(x, z, t) = \frac{2}{a^2} \sum_{n=1}^{\infty} \frac{R(k_n, z, t) J_0(k_n r)}{[J_0'(k_n a)]^2}. \quad (7)$$

Introducing the finite Hankel transform defined above, the details of which may be found in Appendix A, results in the following problem of reduced spatial dimensionality in terms of  $R(k_n, z, t)$

$$\frac{\partial}{\partial z} \left( \mu(z) \frac{\partial R}{\partial z} \right) - k_n^2 \mu(z) R - \rho(z) \frac{\partial^2 R}{\partial t^2} = 0 \quad (8)$$

with the boundary conditions in terms of the new dependent quantity  $R(k_n, z, t)$  now being

$$\frac{\partial R}{\partial z} \Big|_{z=0} = \frac{g(t)}{4\pi\mu_0} \quad \text{and} \quad R|_{r=a} = 0 \quad (9)$$

assuming  $r_0 = 0$ <sup>1</sup> with initial conditions

<sup>1</sup> For an integral involving the Dirac delta function the following property holds:

$$\int_c^d h(\xi) \delta(\xi - c) d\xi = \frac{h(c)}{2} \quad \text{and} \quad \int_c^d h(\xi) \delta(\xi - d) d\xi = \frac{h(d)}{2}, \quad c \text{ and } d \text{ being finite values.}$$

$$R|_{t=0} = \frac{\partial R}{\partial t} \Big|_{t=0} = 0. \quad (10)$$

As the solution must be confined to some finite depth, say  $0 \leq z \leq b$ , a further condition on  $\phi$  and hence  $R$  must be imposed at  $z = b$ . This boundary will also be made perfectly reflecting so that  $\phi|_{z=b} = R|_{z=b} = 0$ . If the number of equally-spaced depth points at an interval of  $\Delta z$  between  $z = 0$  and  $z = b$  is  $M$ ,  $R$  must be zero at depth  $z = M \Delta z$ . The value chosen for  $b$  should take into consideration what is contained in the remainder of this section.

The introduction of this perfectly reflecting boundary presents a bit more of a problem with spurious reflected arrivals than does that at  $r = a$ . This is due to the fact that all one-dimensional arrays in the  $z$  direction may have to be made unnecessarily large. An absorbing boundary condition as proposed by Clayton and Enquist (1977) may be applied. However, it has been determined by numerical experimentation and consulting literature on this subject that the introduction of temporal attenuation leads to better results in this problem type. For this reason, an extra term,  $-\xi(z) \frac{\partial \phi}{\partial t} \left[ -\xi(z) \frac{\partial R}{\partial t} \right]$  is fairly arbitrarily added to equation (1)[(8)]. The quantity  $\xi(z)$  is set equal to zero except in a small zone  $b_{att} < z < b$  where  $b - b_{att} \approx$  half of a wavelength. A discussion of wavelengths is contained in the next section. The addition of this term slightly modifies the equations of the finite difference analogues, (12) – (15). As the introduction of the attenuation term results in a quantity which need only be computed once for a given set of synthetic traces, it does not add greatly to the CPU time and space allocation, while removing reflections from the boundary at  $z = b$ . A standard for choosing  $\xi(z)$  is in the region  $(b_{att}, b)$  is to let  $\xi(z) = \rho(z)/f_0$  and to be cautious require  $b - b_{att} \approx$  one wavelength. The quantity  $f_0$  is defined in the next section. It should be noted at this point that the region of interest in depth ( $z$ ) is now  $0 \leq z \leq z_{att}$

As a consequence of the above discussion (8) has the altered form

$$\frac{\partial}{\partial z} \left( \mu(z) \frac{\partial R}{\partial z} \right) - k_n^2 \mu(z) R - \xi(z) \frac{\partial R}{\partial t} - \rho(z) \frac{\partial^2 R}{\partial t^2} = 0 \quad (11)$$

with the added boundary condition

$$R|_{z=b} = 0 \quad (12)$$

Using explicit finite difference analogues to approximate the derivatives in (11), accurate to  $\mathbf{O}((\Delta z)^2, (\Delta t)^2)$  in the spatial coordinate,  $z$ , and time,  $t$ , the following expression results

$$R_i^{j+1} = -K_0 R_i^{j-1} - K_1 R_i^j + K_2 R_{i+1}^j + K_3 R_{i-1}^j \quad (13)$$

where the subscript “ $i$ ” refers to the spatial coordinate ( $z$ ) and the superscript “ $j$ ” to time ( $t$ ), the coefficients ( $K_l, l = 0, 4$ ) are given by

$$K_0 = \left( 1 - \frac{\Delta t \xi_i}{2\rho_i} \right) \tilde{K}_0 \quad (14)$$

$$K_1 = \left[ 2 - (a_{i+1} + a_i) \frac{(\Delta t)^2}{\rho_i (\Delta z)^2} - \frac{k_n^2 (\Delta t)^2 a_{i+1}}{\rho_i (\Delta z)^2} \right] \tilde{K}_0 \quad (15)$$

$$K_2 = \frac{(\Delta t)^2 a_{i+1}}{\rho_i (\Delta z)^2} \tilde{K}_0 \quad (16)$$

$$K_3 = \frac{(\Delta t)^2 a_{i-1}}{\rho_i (\Delta z)^2} \tilde{K}_0 \quad (17)$$

with

$$\tilde{K}_0 = \frac{1}{\left( 1 + \frac{\Delta t \xi_i}{2\rho_i} \right)} \quad (18)$$

and the quantities  $a_i$  are defined as

$$a_i = \frac{2\mu_i \mu_{i-1}}{\mu_i + \mu_{i-1}} \quad (19)$$

The incremental steps in the time,  $t$ , and depth,  $z$ , directions are  $\Delta t$  and  $\Delta z$ , respectively,  $\rho_i$  is the density at the  $i$ -th depth grid point and the determination of the quantities,  $a_i$ , is discussed in more detail in Appendix B.

To introduce the source term at  $z = 0$  it will be assumed that the first three grid points in  $z_i$  ( $i = 0, 1, 2$ ) are homogeneous. Using the free surface boundary condition, equation (9), the explicit finite difference analogue for  $i = 0$  is

$$R_0^{j+1} = -R_0^{j-1} + \left[ 2 - \frac{2v_0^2 (\Delta t)^2}{(\Delta z)^2} - k_n^2 v_0^2 (\Delta t)^2 \right] R_0^j + 2v_0^2 \left[ \frac{(\Delta t)^2}{(\Delta z)^2} \right] R_1^j - \frac{g(t) v_0^2 (\Delta t)^2}{2\pi\mu_0 (\Delta z)} \quad (20)$$

The set of equations (13) through (20), together with the initial and boundary conditions, provide, in theory, all that is required to develop a hybrid finite difference program for the  $SH$  potential wave equation in a vertically inhomogeneous isotropic

medium. The inverse transform  $R \rightarrow \phi$  is an infinite series, the truncation of which will be discussed in a subsequent section.

### TIME, DISTANCE AND VELOCITY

When dealing with problems involving finite difference methods in the solution, it is often more instructive to use periods ( $T$ ) and wavelengths ( $WL$ ) as units defining time, distance, and velocity than the more standard  $km$  or  $m$  and *seconds*. A wavelength is related to the predominant frequency of the source wavelet,  $f_0$ , and some velocity within the medium, usually, the velocity at the source location. Thus if  $v_s$  is the velocity at the source the related wavelength, which may be taken as a reference wavelength for zones within the medium of different velocities, is defined as

$$\lambda_s = \frac{v_s}{f_0} \quad (21)$$

Other useful measures are the maximum and minimum wavelengths within the medium being considered written in terms of the maximum and minimum velocities,  $v_M$  and  $v_m$ , encountered in the medium and the predominant frequency as

$$\lambda_M = \frac{v_M}{f_0} \quad (22)$$

and

$$\lambda_m = \frac{v_m}{f_0} \quad (23)$$

A period may be expressed in terms of the predominant frequency  $f_0$  as

$$T = \frac{1}{f_0} \quad (24)$$

from which it follows that velocities are normalized to have the dimensions of  $WL/T$  (wavelengths/period).

The reason for this renormalization of these quantities is that sampling in a spatial dimension is required in *points/wavelength* ( $pts/WL$ ). In standard, explicit finite-difference programs, the minimal requirement for keeping grid dispersion within acceptable limits, for finite-difference schemes of  $O((\Delta x_i)^2, (\Delta t)^2)$  accuracy, is of the order of 10  $pts/WL$  relative to  $\lambda_m$ . Through numerical experimentation it has been found, with the exception of very special cases, that 40  $pts/WL$  is sufficient for an accuracy of about 2-3% when known exact solutions of specific problems are

compared with this numerical method of computing synthetic seismograms using the method discussed here. At 100 *pts/WL* this becomes less than 1%.

### STABILITY CRITERION

As an explicit finite difference approach to this problem has been taken, there is the possibility of error amplification if  $\Delta z$  and  $\Delta t$  are chosen arbitrarily. A stability criterion must be established. The von Neumann stability conditions may be applied here as a result of the separability of variables in the problem. With this method a harmonic decomposition of error is done at the grid points at a specific time level. Following the procedure which may be found in Mitchell (1969, pp. 209-210), the inequality

$$v_M^2 \left( \frac{\Delta t}{\Delta z} \right)^2 + \frac{k_n^2}{4} v_M^2 (\Delta t)^2 < 1 \quad (25)$$

is obtained,  $v_M$  being the maximum velocity encountered on the velocity grid, which in the case being considered is the maximum velocity along the velocity-depth profile. This condition must be satisfied if the problem is to be considered stable. Since it has been assumed that  $\Delta z$  has been chosen and the roots  $k_n$  and velocity  $v_M$  are known, the inequality (21) essentially serves as the determining condition for the time step  $\Delta t$ . It is usual to compute this maximum value for  $\Delta t$  and then choose its actual value used in the program to be a suitable “rounded off” value somewhat less than the actual computed value of  $\Delta t$ .

### TRUNCATION OF THE INFINITE SERIES SUMMATION

The inverse transform of the problem discussed in a previous section is taken over all the positive roots of the equation

$$J_0(k_n a) = 0, \quad j = 1, \dots, \infty \quad (26)$$

As the infinite series must be replaced by a finite series, the question is, how many terms are necessary to adequately approximate the infinite series in the inverse summation? This question has been briefly addressed in the paper of Martynov and Mikhailenko (1984). The analysis presented here is more detailed and leads to fairly practical guidelines for obtaining a value for the minimum number of terms needed for an accurate, yet practical, approximation of the infinite series.

The number of terms in the series summation is related to the smoothness and duration of the source pulse,  $g(t)$ , in the time domain as well as the distance  $a$  to the perfectly reflecting boundary in the radial,  $r$ , direction. The time dependence of the Gabor wavelet used as the source pulse here is

$$g(t) = \sin[2\pi f_0(t - \tau)] \exp\left[-(2\pi f_0(t - \tau)/\gamma)^2\right] \quad -\tau < t < \tau \quad (27)$$

here  $2\tau = \lambda / f_0$  is the approximate duration of the pulse, indicating that

$g(t) \cong 0.0$  for  $|t| > \tau$ . The predominant frequency in Hz of the pulse is  $f_0$ , and  $\gamma$  is a dimensionless damping factor which controls the size of the side lobes and thus the width of the pulse in the time domain.

The two quantities specifying the shape of the pulse in the time domain,  $f_0$  and  $\gamma$ , have a direct effect on the width of the spectrum  $F(\omega)$  in the frequency domain.  $F(\omega)$  is the Fourier time transform of  $g(t)$ , which may apart from some multiplicative constant be written analytically as

$$F(\omega) = \frac{\pi^{1/2}\gamma}{i\omega_0} \exp\left\{-\frac{\gamma^2}{4}\left[1 + (\omega/\omega_0)^2\right]\right\} \sinh\left(\frac{\omega\gamma^2}{2\omega_0}\right) \quad (28)$$

with the circular frequency  $\omega$  being defined by  $\omega = 2\pi f$ .

As cylindrical symmetry has been assumed, the spatial frequency is characterized by the horizontal wavenumber,  $k$ , which is related to the angular frequency,  $\omega$ , of a harmonic spectral component through the relation

$$k = \omega p \quad (29)$$

Here,  $p$  is the horizontal component of the slowness vector, or ray parameter. The Alekseev-Mikhailenko method is essentially a wave-number method, in that only discrete wave-numbers are considered, so that  $k$  is not a continuous independent variable, but is related to discrete values of  $\omega$  (Korn and Muller, 1983). This may also be inferred from the fact that the inverse summation series is taken over a discrete set of roots of the transcendental equation  $J_0(k_n a) = 0$ . Thus, only those values of  $\omega$  at which the spectrum of the source pulse is nonzero need to be considered in the numerical computation. The infinite inverse series summation may be approximated by a finite number of terms, which are determined by estimating an upper bound on  $\omega$ , say  $\omega_M$ , beyond which the spectrum of  $F(\omega)$  is essentially zero. After determining  $\omega_M$ , the quantity  $k_M$  may be obtained from the relation

$$k_M = \omega_M p \quad (30)$$

For the source pulse being considered, the maximum value of  $F(\omega)$  occurs at  $\omega = \omega_0 = 2\pi f_0$ . The width of  $F(\omega)$  in the frequency domain, that is, the range where  $|F(\omega)|$  is not small when compared with  $|F(\omega_0)|$ , depends on  $\gamma$ . The larger



the value of  $\gamma$ , the narrower the spectrum in the frequency domain and the longer the pulse duration in the time domain.

In estimating  $k_M$  as  $k_M = \omega_M p$ , the value assigned to  $p$  should be the maximum possible value. The range of  $p$  is usually restricted to those values related to geometrical rays associated with body waves. As  $p = \sin \theta / v$ ,  $\theta$  and  $v$  being the acute angle measured from the geometrical ray to the vertical and the velocity of propagation along the ray, respectively. The maximum value that  $p$  can attain results from setting  $\theta = \pi/2$ , while  $v$  is chosen to be  $v_m$ , the minimum shear wave velocity encountered in the medium. With  $k_n = \frac{\xi_n}{a}$ , where the  $\xi_n$  are obtained from the transcendental equation (5) (A.7 for  $n \gg 1$ ), the following holds

$$k_M = \frac{\omega_M}{v_m} = \frac{\xi_M}{a} \quad (31)$$

or

$$\xi_M = \frac{a\omega_M}{v_m} \quad (32)$$

This relationship indicates that the number of terms in the inverse summation series varies linearly with  $a$ . As the spectrum width in the frequency domain decreases as a result of an increasing value of  $\gamma$ , less terms are required to approximate the infinite series. For large values of  $n$ , or equivalently, large values of the roots  $\xi_n$ , the asymptotic expression for the roots of the transcendental equation  $J_0(\xi_n) = 0$  is  $\xi_n \approx n\pi$  (Appendix A). As a consequence,

$$\xi_M \approx N\pi \quad (33)$$

so that with  $\omega_M = 2\pi f_M$

$$N \approx \frac{2f_M a}{v_m} \quad (34)$$

where  $N$  is the predicted number of terms in the inverse series to accurately approximate the infinite summation.

## CONCLUSIONS

An introduction of the use of finite integral transforms to reduce the problem of a hyperbolic equation in several spatial dimensions and time to that of one in a single spatial dimension and time has been presented. Although the example considered was somewhat simplistic, the motivation for this report was to provide the basic theory of the method and some insight into the numerical techniques involved. Assuming a

band-limited source wavelet, it was shown that the infinite inverse summation may, in a numerical sense, be adequately approximated by a finite series with reasonable accuracy. As mentioned in the text, a fairly general inhomogeneous dependence of the elastic parameters on the spatial coordinates may be considered using this method (Mikhailenko, 1984 and 1985). However, the pursuit of those extensions here may have been counterproductive as one can become overwhelmed with the mathematics and the objective was to present a tutorial for persons wanting a basic understanding of the method.

## APPENDIX A: FINITE HANKEL TRANSFORMS

Denoting the finite Hankel transform of a function  $f(x)$  as  $\mathbf{H}[f(x)]$  the theorem which follows is sufficient for the introduction of the finite Hankel transform into a hyperbolic equation for which a solution is sought in the text will be stated (Sneddon, 1972). It is first necessary to present the following specification of Dirichlet's conditions.

**Definition:** A function  $f(x)$  is said to satisfy Dirichlet's conditions on the interval  $0 \leq x \leq a$  if it is bounded on the interval, and the interval  $0 \leq x \leq a$  can be partitioned into a finite number of subintervals inside each of which the function  $f(x)$  is continuous and monotonic; either increasing or decreasing.

**Theorem:** If  $f(x)$  satisfies Dirichlet's conditions on the interval  $0 \leq x \leq a$  and if its finite Hankel transform in that range is defined to be

$$\mathbf{H}[f(x)] = \hat{f}(\xi_n) = \int_0^a x f(x) J_\mu(\xi_n x) dx \quad (\text{A.1})$$

where  $\xi_n$  is the root of the transcendental equation

$$J_\mu(\xi_n a) = 0 \quad (\text{A.2})$$

then at any point of  $0 \leq x \leq a$  at which the function  $f(x)$  is continuous

$$f(x) = \frac{2}{a^2} \sum_n \hat{f}(\xi_n) \frac{J_\mu(\xi_n x)}{[J'_\mu(\xi_n a)]^2} \quad (\text{A.3})$$

where the sum is taken over all the positive roots of equation (A.2) and the prime indicates differentiation with respect to the argument.

The particular case of interest here is when  $\mu = 0$  for which the following relation is valid (Abramowitz and Stegun, 1972)

$$J'_0(x) = -J_1(x) \quad (\text{A.4})$$

so that the inverse series, (A.3), reduces to

$$f(x) = \frac{2}{a^2} \sum_n \hat{f}(\xi_n) \frac{J_0(\xi_n x)}{[J_1(\xi_n a)]^2} \quad (\text{A.5})$$

Computation of the roots of  $J_\mu(\xi_n a) = 0$  presents less of a problem than might be apparent as they need be tabulated only once and the first 20 of these may be found in Abramowitz and Stegun (1972), accurate to 14 floating point digits and formulae exist for computing the next 20 to the same accuracy. Any roots of an order larger than this may be obtained with acceptable accuracy using the asymptotic formula

$$\xi_n \approx \beta - \frac{\nu-1}{8\beta} - \frac{4(\nu-1)(7\nu-31)}{3(8\beta)^3} - \dots \quad (\text{A.6})$$

(Abramowitz and Stegun, 1972), where  $\nu = 4\mu^2$  and  $\beta = \left(n + \frac{\mu}{2} - \frac{1}{4}\right)\pi$ . For the case of  $\mu = 0$ , (A.6) becomes

$$\xi_n \approx \beta + \frac{1}{8\beta} + \frac{124}{3(8\beta)^3} + \dots \quad (\text{A.7})$$

with  $\beta = \left(n - \frac{1}{4}\right)\pi$ .

It may be instructive to proceed step by step through the application of the forward finite Hankel transform on the first term of equation (1). Noting that  $\mu(z)$  is independent of  $r$  it will be omitted from the derivation

$$\int_0^a \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) \right] J_0(k_n r) r dr = \left[ J_0(k_n r) r \frac{\partial \phi}{\partial r} \right]_0^a - k_n \int_0^a \left\{ \frac{d J_0(k_n r)}{dr} \right\} \left\{ r \frac{\partial \phi}{\partial r} \right\} r dr \quad (\text{A.8})$$

The first term on the left hand side of (A.8) is zero as  $J_0(k_n a) = 0$ . Continuing has

$$\begin{aligned} -k_n \int_0^a \left\{ \frac{d J_0(k_n r)}{dr} \right\} \left\{ r \frac{\partial \phi}{\partial r} \right\} r dr = \\ -k_n \left[ r \frac{d J_0(k_n r)}{dr} \phi \right]_0^a + k_n^2 \int_0^a \left[ \frac{1}{r} \frac{d}{dr} \left\{ r \frac{d J_0(k_n r)}{dr} \right\} \right] \phi r dr \quad (\text{A.9}) \end{aligned}$$

As in (A.8) the first term of the left hand side of (A.9) is zero as  $\phi|_{r=a}$  has been assumed to be zero. Utilizing the differential equation defining  $J_0$  results in

$$\int_0^a \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) \right] J_0(k_n r) r dr = -k_n^2 \int_0^a \phi(r, z, t) J_0(k_n r) r dr \quad (\text{A.10})$$

such that the final form of the finite integral transform application (equation (4)) is

$$\int_0^a \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) \right] J_0(k_n r) r dr = -k_n^2 R(k_n, z, t) \quad (\text{A.11})$$

## APPENDIX B: FINITE DIFFERENCE ANALOGUES

A finite difference approximation, with applications in an explicit difference schemes accurate to  $\mathbf{O}((\Delta\eta)^2)$  for the expression

$$\frac{\partial}{\partial \eta} \left[ \mu(\eta) \frac{\partial \phi(\eta, \xi_i)}{\partial \eta} \right] \quad (\text{B.1})$$

utilizing the Lax-Wendroff scheme (Ames, 1969) is

$$\begin{aligned} \frac{\partial}{\partial \eta} \left[ \mu(\eta) \frac{\partial \phi(\eta, \xi_i)}{\partial \eta} \right] &\approx \frac{1}{(\Delta\eta)^2} \left[ \mu_{n+1/2} \{ \phi_{n+1} - \phi_n \} - \mu_{n-1/2} \{ \phi_n - \phi_{n-1} \} \right] \\ &\approx \frac{1}{(\Delta\eta)^2} \left[ \mu_{n+1/2} \phi_{n+1} - \{ \mu_{n+1/2} + \mu_{n-1/2} \} \phi_n + \mu_{n-1/2} \phi_{n-1} \right] \end{aligned} \quad (\text{B.2})$$

where the subscripts involving  $n$  are related to the variable  $\eta$  under the assumption that the other independent variables,  $\xi_i$ , are held constant. The definition of  $\mu_{n+1/2}$  is given as

$$\mu_{n+1/2} = \frac{(\mu_{n+1} + \mu_n)}{2} \quad (\text{B.3})$$

and that for  $\mu_{n-1/2}$  follows.

An alternative expression for (B.1) may be written in the following form

$$\frac{\partial}{\partial \eta} \left[ \left\{ \frac{1}{\mu(\eta)} \right\}^{-1} \frac{\partial \phi(\eta, \xi_i)}{\partial \eta} \right] \quad (\text{B.4})$$

for which the equivalent analogue for explicit finite difference formulations is

$$\frac{\partial}{\partial \eta} \left[ \left\{ \frac{1}{\mu(\eta)} \right\}^{-1} \frac{\partial \phi(\eta, \xi_i)}{\partial \eta} \right] = \frac{1}{(\Delta \eta)} \left[ \left\{ \frac{1}{\mu_{n+1/2}} \right\}^{-1} \{\phi_{n+1} - \phi_n\} - \left\{ \frac{1}{\mu_{n-1/2}} \right\}^{-1} \{\phi_n - \phi_{n-1}\} \right] \quad (\text{B.5})$$

so that the finite difference approximation of (B.4) for use in an explicit scheme has the form

$$\frac{\partial}{\partial \eta} \left[ \mu(\eta) \frac{\partial \phi(\eta, \xi_i)}{\partial \eta} \right] \approx \frac{1}{(\Delta \eta)^2} [a_{n+1} \phi_{n+1} - (a_{n+1} + a_n) \phi_n + a_n \phi_{n-1}] \quad (\text{B.6})$$

with

$$a_{n+1} = \left\{ \frac{1}{\mu_{n+1/2}} \right\}^{-1} = \left\{ \frac{1}{2\mu_{n+1}} + \frac{1}{2\mu_n} \right\}^{-1} = \frac{2\mu_n \mu_{n+1}}{\mu_n + \mu_{n+1}} \quad (\text{B.7})$$

and  $a_n = \left\{ 1/\mu_{n-1/2} \right\}^{-1}$  being similar.

The quantity  $a_n$  may take on the more mathematically formal definition

$$a_n = \left[ \Delta \eta \int_{\eta_{n-1}}^{\eta_n} \frac{d\eta}{\mu(\eta)} \right]^{-1} = \frac{2\mu_n \mu_{n-1}}{\mu_n + \mu_{n-1}} \quad (\text{B.8})$$

Equation (B.6) is preferred over (B.2) as it has the effect of *smoothing* the implicit quantity  $\partial \mu(\eta)/\partial \eta$  which is useful when a jump discontinuity in  $\mu_n$  is encountered.

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