A generalized Gardner relation

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ABSTRACT

Gardner's relation is extended to include dependence on both shear and compressional velocity. Data for fitting are generated from work on Gardner-type relations by Wang (2000). It is found that a single relation of this form can reasonably represent several lithologies.

INTRODUCTION

Gardner's relation ($\rho = C\alpha^{1/4}$, ρ = density, α = compressional velocity) has been widely used as a means of providing approximate densities or compressional velocities, when one was available and the other not (Gardner et al., 1974). Other empirical forms for a density-velocity relation are possible, such as Birch's Law and Lindseth's relation. One of the advantages of Gardner's form however is in its application to AVO, as demonstrated by Smith and Gidlow (1987). In traditional AVO one uses information on P-P reflection coefficients as input to an inversion process that yields $\Delta\rho/\rho$, $\Delta\alpha/\alpha$, and $\Delta\beta/\beta$ as output (β = shear velocity, and $\Delta x/x$ is the difference x divided by the average of x across an interface). This procedure is strongly stabilized by reducing the number of output variables from three to two. One way to accomplish this for linear inversion is by substituting for the density contrast with a relation of the form

$$\frac{\Delta\rho}{\rho} = c_1 \frac{\Delta\alpha}{\alpha} + c_2 \,. \tag{1}$$

Integrating this yields

$$\ln \rho = c_1 \ln \alpha + c_2 \; \alpha + c_3 \tag{2}$$

or

$$\rho = c_3 \; \alpha^{c_1} e^{c_2 \, \alpha} \tag{3}$$

This is exactly the form of Gardner's relation, with c_1 = ¼ and c_2 = 0, and the substitution of Equation (1) is precisely what Smith and Gidlow proposed. Such a simple manipulation is not possible with other empirical density-velocity relations.

There are a variety of changes one could propose and still retain the essential form of Gardner's relation. Castagna et al. (1993) for instance proposed a set of lithology specific relations. Work by Dey and Stewart (1997), Potter and Stewart (1998), and Potter (1999) has used Blackfoot log data to develop ρ-β relations. More recently, Wang (2000) has used laboratory data to develop ρ-α and ρ-β relations for six different lithologies, for both gas- and water-saturation.

Dey (2001) noted the possibility of developing an empirical relation involving all three variables. We reason here that because of the greater flexibility in fitting permitted by use of an additional variable, this method may have the ability to represent multiple lithologies with accuracy, using only a single expression. Individual lithologies have well-defined relations between α and β (see, for instance, Mavko et al., Section 7.8), so it is hoped that fitting ρ to both α and β will allow the α-β relations to be incorporated implicitly into a generalized Gardner relation.

One use of such a relation would be to predict shear velocity logs from the more commonly obtained density and sonic logs. When lithology is know, α-β relations are generally used to predict β from α. However if lithology is not known, a single lithology-independent $\beta(\rho,\alpha)$ relation would be very useful. Another potential application, given the appropriate functional form, is an analogy to the Smith-Gidlow AVO method. In this paper we propose a function of the form:

$$\rho = C\alpha^A \beta^B \tag{4}$$

which yields the relation

$$\frac{d\rho}{\rho} = A\frac{d\alpha}{\alpha} + B\frac{d\beta}{\beta}\,. \tag{5}$$

The potential use of Equation (5) in AVO is explored elsewhere in this volume (Ursenbach and Stewart, 2001).

METHOD & RESULTS

We develop an empirical expression of the form of Equation (4) using Wang's empirical lithology-specific expressions as input. Gardner's relation is intended to describe water-saturated rocks, so we use the water saturation version of Wang's equations. First, for each lithology, we generate a set of densities in a range normal for that lithology. The ranges actually employed are shown in Table I:

Table I. Ranges of densities values employed for generating input data for each lithology

Lithology	Dolomite	Limestone	Sandstone	Shaley Sandstone	Unconsol. Sandstone	Shale
Density Range (g/cm^3)	2.45-2.85	1.85-2.75	2.10-2.60	2.05-2.55	2.05-2.30	2.30-2.70

From these density values we employ Wang's equations [Equations (6) and (7) from his paper] to generate values of α and β. These are of the form $\alpha = c\,\rho^d$, which is simply a rearrangement of Gardner's relation, and similarly $\beta = c\,\rho^d$. We then fit the resulting (ρ,α,β) triples from all lithologies to the expression

$$\ln \rho = \ln C + A \ln \alpha + B \ln \beta \tag{6}$$

to obtain A, B, and ln$C$. For units of g/cm^3 and km/s, this yields $A = -0.091$, $B = 0.319$, and $C = 2.10$.

These are not our final suggested results however. A test was carried out to see how well α and β predicted by Wang's equations satisfy extant α-β relations (Mavko et al., 1998, Sec. 7.8). In fact they deviate somewhat, as illustrated for dolomite in the figure below:

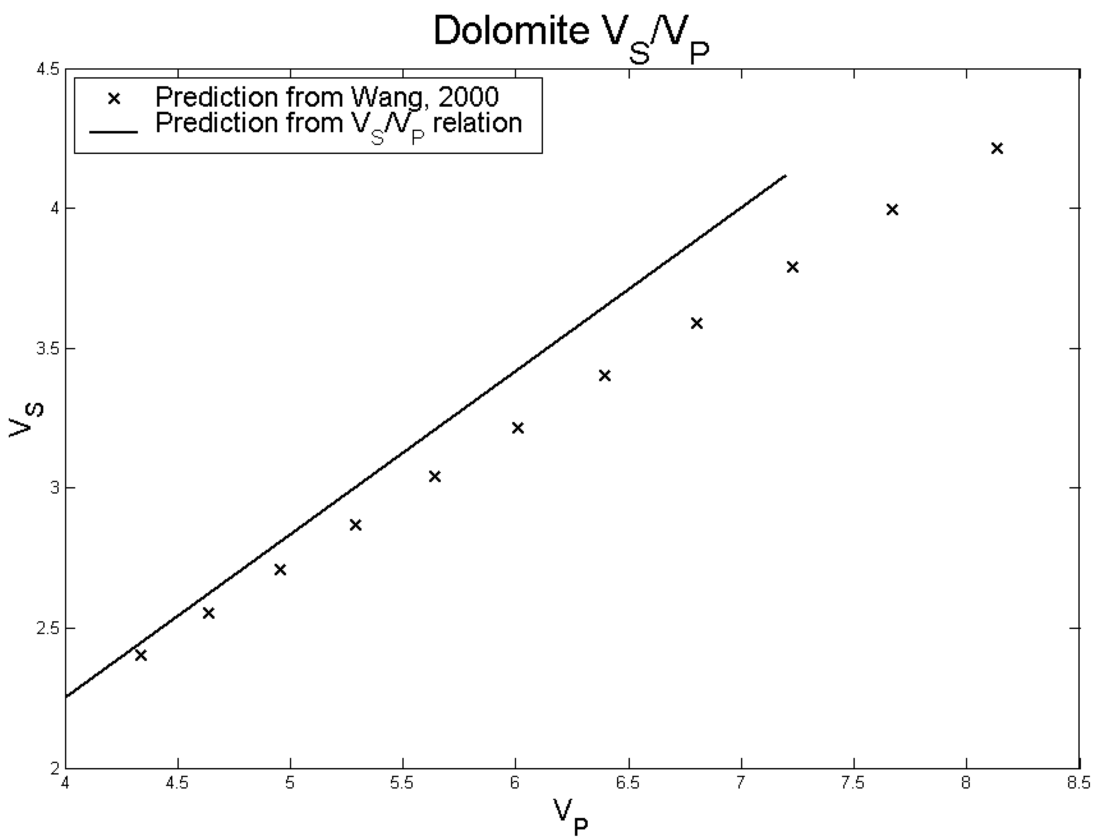

FIG. 1. Values of β/α predicted by Wang's equations compared to α-β relation from Mavko et al., Sec. 7.8.

There is generally much less data scatter about a best-fit α-β relation than about a best-fit ρ-α or ρ-β relation. Thus it seems important to constrain the fitting to satisfy the α-β relations. To accomplish this we weighted the data by creating additional points. The first way this was done was as follows. For each (ρ,α,β) in the data set, we also included (ρ, aβ+b, cα+d), where (a,b,c,d) are appropriate constants obtained from the α-β relations. After including these points in the fit we obtained $A = 0.103$, $B = 0.146$, and $C = 1.83$ using 1000 density points for each lithology. The correlation coefficient had a value of 0.83. The sum of A and B is 0.249, very close to the ¼ exponent of Gardner's relation. It is reasonable that they are of similar size, but likely a coincidence that they are so nearly identical. C is also of similar magnitude to the corresponding constant in Gardner's relation, which is 1.741 (for units of g/cm^3 and km/s).

A visual representation of the fit is presented in Figure 2 below. Density is plotted against shear and compressional velocity in the following way. The horizontal and vertical axes represent α and β respectively. For each data triple, the density is plotted at the appropriate location as a short line, whose angle with the horizontal represents the magnitude of the density. A horizontal line represents the minimum density in Table I, and a vertical line represents the maximum density. The black lines represent input triples, and the lighter lines over top represent the density predicted by Equation (4) after fitting. For each (α,β) pair, the input and output ρ values cross over at their midpoints. For a very good fit the lighter line eclipses the black line. The displayed results employed 5 density points for each lithology, so the fitting results differ slightly from those given above. One observes a few results that are noticeably less satisfactory than others. These correspond to low densities in the limestone lithology. In general though, this appears to be a reasonable method for fitting data from several lithologies

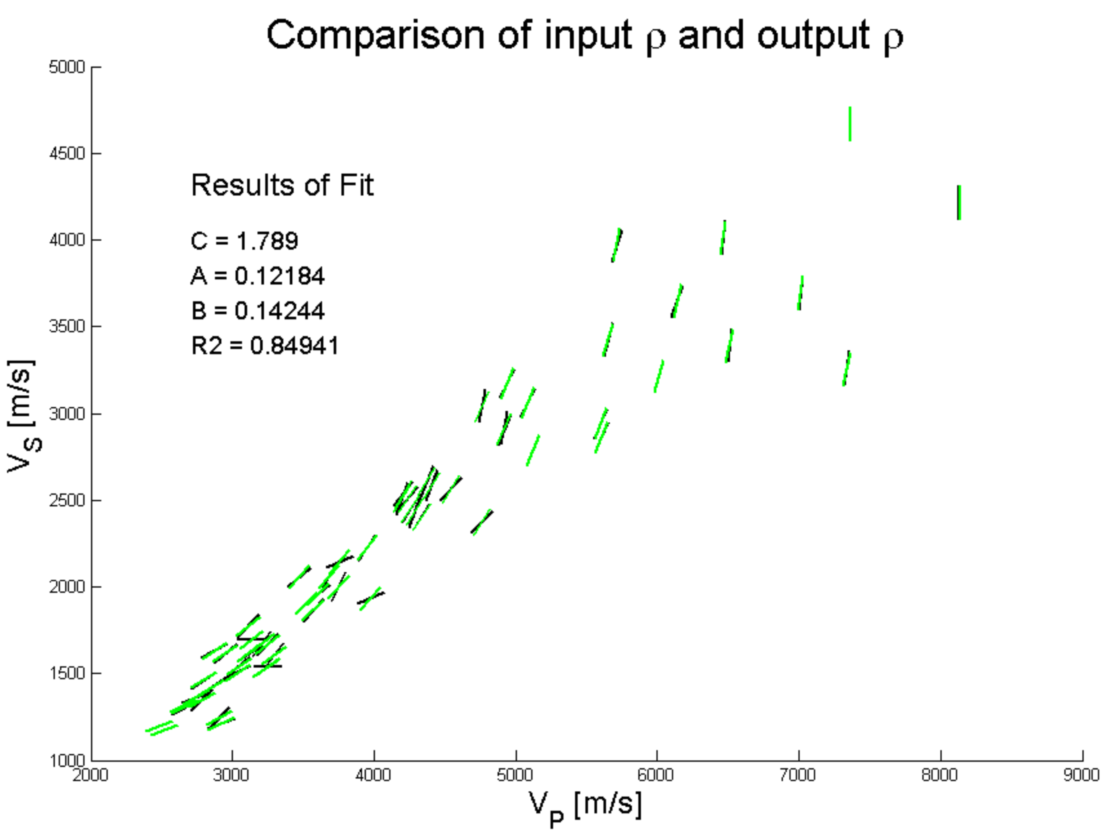

FIG. 2. Comparison of fitted density to input density. The density for a given (V_P=α, V_S=β) pair is given by the angle of the line with respect to the horizontal (horizontal = minimum density in range, vertical = maximum density). The black lines represent the original input densities, from which α and β were predicted by Wang's Equations (6) and (7). This data was then fit to

Equation (4), and the values of ρ predicted by the fit of Equation (4) for the same (α, β) pairs are given by the lighter colored line on top of the black line.

A second method for including extra points was to replace each (ρ,α,β) with (ρ,α,aα+b) and (ρ,cβ+d,β). Fitting this data yielded $A = 0.0799$, $B = 0.164$, and $C = 1.87$, again using 1000 density points for each lithology. The correlation coefficient again had a value of 0.83, and plotting the result gave a result similar to that in Figure 2. The sum of *A* and *B* is 0.244, again close to ¼.

Comparison with Gardner's Relation

To test the value of this result we compare it to three other relations. The first is the original Gardner's Relation, the second is Gardner's relation with coefficients refit to the present data, and the third is a Gardner-type relation depending only on V_S and not on V_P. The three expressions therefore are

$$\rho = 1.741\, V_P^{\,0.25} \quad \text{(Gardner's relation)} \tag{7}$$

$$\rho = Ap\, V_P^{\,Bp} \quad \text{(Gardner's relation refit)} \tag{8}$$

$$\rho = As\, V_P^{\,Bs} \quad \text{(Gardner-like } V_S \text{ relation)} \tag{9}$$

A least-squares method is used to determine *Ap*, *Bp*, *As,* and *Bs* from the logarithmic version of Equations (8) and (9). The fitting coefficients obtained are $Ap = 1.61$, $Bp = 0.278$, $As = 1.97$, and $Bs = 0.235$. For each of Equations (4), (7), (8), and (9), we then determine an goodness-of-fit function as

$$\text{Goodness-of-fit} = \sqrt{\,[\,(1/N)\,\Sigma|\rho_{\text{input}} - \rho_{\text{output}}|\,]} \tag{10}$$

where N is the number of points and Σ indicates a sum over all points. The goodness values calculated are 0.119 g/cm^3 for Equation (4), 0.154 g/cm^3 for Equation (7), 0.122 g/cm^3 for Equation (8), 0.120 g/cm^3 for Equation (9). Thus Equation (4) is slightly superior to Equations (8) and Equation (9), indicating a strong correlation between V_P and V_S.

CONCLUSIONS

Gardner's relation, $\rho = 1.741\,\alpha^{1/4}$ (for ρ in g/cm^3 and α in km/s), is generalized herein to $\rho = C\,\alpha^A\,\beta^B$, where *C* is similar in size to 1.741 and $A + B \approx$ ¼. This result is expected to be reasonably accurate for several water-saturated lithologies. Limestone appears to be the least well described. This result would be useful both for predicting shear velocities and for substituting velocity contrasts for density contrasts in AVO inversion approximations.

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