Approximating the P-S reflection coefficient for small incidence angles

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ABSTRACT

We derive an approximation to the Zoeppritz equations for the converted-wave reflection coefficient, R_{PS}, given by:

$$R_{PS} = \frac{-2i_1\left(\alpha_2\beta_2\rho_2\Delta\rho + 2\rho_1\Delta\mu\right)}{\left(\rho_1\alpha_1 + \rho_2\alpha_2\right)\left(\rho_1\beta_1 + \rho_2\beta_2\right)} \tag{1}$$

where α, β, ρ, μ and i denote P-wave velocity, S-wave velocity, density, shear modulus, and P-wave angle of incidence, respectively; $\Delta\rho = \rho_2 - \rho_1$, $\Delta\mu = \mu_2 - \mu_1$, and subscripts 1 (upper) and 2 (lower) denote the two homogeneous isotropic media.

The approximation is made assuming small angles of incidence rather then small changes in the elastic parameters. The initial goal of this research is to derive a simple expression for the P-SV reflection coefficient. By doing that we hope to obtain simple relations that govern the change in polarity in the case of converted P-SV waves.

Using this simplified expression, we have investigated the conditions under which R_{PP} and R_{PS} have the same sign (the unusual situation) because this will mean that features that should be correlated on P-P and P-S sections will have opposite polarity. We obtained the following necessary conditions for this to occur: either $\Delta\rho > 0, \Delta\beta < 0$ and $\Delta\mu < 0$, or $\Delta\rho < 0, \Delta\beta > 0$ and $\Delta\mu > 0$.

We also reviewed other approximations to the Zoeppritz equations. From them we chose to code the expressions derived by Aki and Richards and by Wang. The accuracy of the new approximation was then tested by comparing its results with the results of these previous approximations for three interface models. For these three models, the new approximation proved to be the most accurate one out to beyond 15° incidence and, in some cases, beyond 30°.

INTRODUCTION

Up until the 1980s or so, preserving the true amplitude of seismic waves during data processing was not considered very important. Now AVO techniques are widely used as a direct indicator of oil and gas deposits when interpreting seismic data. Amplitude variation with offset (AVO) can be used to evaluate elastic rock properties from seismic data. This type of analysis is usually performed on P-wave reflections. Recent developments in ocean-bottom seismic (OBS) have made possible the acquisition of high-quality multicomponent data. The use of converted-wave (P-SV) data for AVO analysis can give better estimates of S-wave velocities and density contrasts (Jin et al., 2000).

Also related to AVO is the study of P-wave and S-wave polarity. A polarity standard for the recording of multicomponent data is yet to be officially adopted, but has been proposed by (Brown et al., 2000). Apart from this, a study is required to examine the conditions governing the amplitude and polarity relationships for P-P reflection compared with the P-SV reflection from the same interface.

The polarity problem comes into play when we try to correlate the P-P and P-SV seismic sections. Apparently, although there is normally a single sign relationship between the P-P reflection coefficient (R_{PP}) and the P-SV reflection coefficient (R_{PS}), that is, normally R_{PP}/R_{PS}, < 0 (Brown et al. 2000), this is not so for all possible combinations of rock parameters. In this ‘normal’ case, the P-P and P-SV events will have the same apparent polarity, that is, both peaks or both troughs on corresponding half-cycles. For some reflectors the P-P and P-SV reflections will have opposite polarity ($R_{PP}/R_{PS} > 0$). This unusual circumstance can make correlation of P-P and P-SV sections a bit tricky. Increased knowledge of when this can happen – for what combinations of rock parameters – will be helpful in multicomponent interpretation. Some examples of normal and opposite polarity are shown in Table 1 (Brown et al., 2000).

A system of equations developed separately by Knott and Zoeppritz is normally used in determining the reflection coefficients of the P and SV reflected and transmitted waves. They are more commonly used in the form derived by Zoeppritz (Aki and Richards, 1980). Several approximate solutions for these equations have been obtained, more common being those developed by Bortfeld (1961), Aki and Richards (1980) and Shuey (1985), by assuming relatively small changes in medium properties. One of our goals is to derive or approximate the mathematical conditions that determine amplitudes in general and polarity in particular for P-P and P-SV arrivals.

In this paper we analyze and test some of the known approximations to the Zoeppritz equations and also derive and test a new approximation. The objective is to find an expression that is not constrained by the usual restriction: small changes in medium parameters. In exchange we have to impose other restrictions, namely small angles of incidence. We focus almost entirely on the behaviour of the P-SV reflection coefficient at an interface when we vary the velocities and densities in the upper and lower media.

APPROXIMATIONS TO THE ZOEPPRITZ EQUATIONS

The approximations of Bortfeld (1961), Aki and Richards (1980) and Shuey (1985), though they differ somewhat in detail, are really equivalent to each other (Wang, 1999; Zhang, 1991). For instance, Shuey (1985) modifies the expression developed by Aki and Richards (1980) by replacing β and $\Delta\beta$ with $\Delta\sigma$ and σ, where σ is Poisson's ratio. The merit of this is to make his approximations more meaningful in the AVO context by grouping the different terms in a manner that is more appropriate to this type of analysis. A shortcoming of Shuey's approach is that he only develops the formula for the P-P reflection. Actually, both reflected P-P and converted P-S waves contain information that is valuable for AVO.

The two most recent approximations of the reflection coefficients that we know of were developed by Zhang (1991) and Wang (1999). Zhang developed eight new formulae for the reflection and transmission coefficients of compressional (P-P), shear (S-S) and converted (P-S and S-P) waves. All of these formulae are based on the previous approximations developed by Aki and Richards (1980). They are developed by expanding the Aki-Richards approximation in a power series of the sine of incidence angle. Thus, these formulae are limited from the very beginning to small changes in medium parameters.

Wang (1999), on the other hand, starts the whole approach with the exact equations for the reflection coefficients; these formulae are also found in Aki and Richards (1980). To keep the tradition Wang, of course, makes his own notation and chooses to use in the development of his formulae the P- and S-wave vertical slownesses. The mathematical expressions for the P- and S-wave reflection coefficients and the assumptions and limitations that are imposed are presented succinctly in Appendix A.

The main steps that Wang takes in developing the various approximations are as follows. He expands the denominator of the exact Zoeppritz equations using a Taylor series in p (ray parameter). Then, he truncates after the p^4 term, rewrites the expressions, and calls them the pseudoquartic approximations with respect to the ray parameter, p. By imposing more limitations and assumptions, the coefficients of p^2 and p^4 terms from the previous step are simplified and new pseudoquartic approximations are obtained. For the P-P case, this expressions is then truncated after the p^2 term and a quadratic approximation is obtained.

Wang’s whole paper is focused on different approximations for the P-P reflection and transmission coefficients. For the P-SV case, only the pseudoquartic formulae are shown and they are not further simplified to quadratic formulae. These are the formulae that we begin with in our investigations.

Wang also imposes his limitations without stating any physical or mathematical grounds. The only instance where he studies the effect of one of his assumptions is when he linearizes the quadratic expression for the P-P reflection coefficient.

Nevertheless, we feel Wang's (1999) approach is sound because he not only derives the approximations using a Taylor series expansion from the exact equations but also shows that, by neglecting the last term of his quadratic approximation, we end up with a formula that is linear in each of the three elastic parameters and equivalent to the approximations obtained by his predecessors. Still, as we show later, Wang’s pseudoquartic approximation for the P-SV reflection coefficient needs to be corrected or rederived.

ACCURACY OF THE APPROXIMATIONS

In the following, we use the notation shown in Figure 1.

In order to become more acquainted with the previous approximations, after studying them, we tested their accuracy. Thus, we coded a few of them in MATLAB

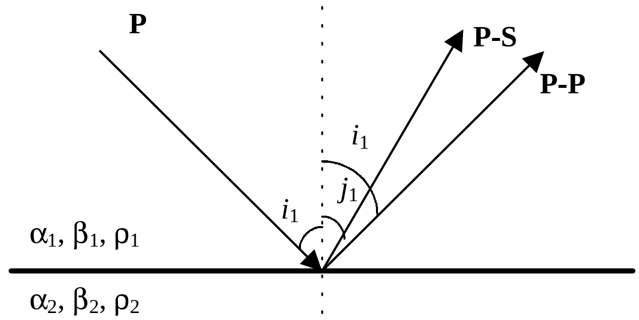

FIG. 1. The interface model and notation of this paper. P is the incident wave, P-P and P-S the reflected P wave and converted reflected S wave, respectively, *j* is S-wave angle of incidence, and other symbols are defined in the Abstract.

and tried them on three interface models that entail large changes in medium parameters. As emphasized before, some of the approximations are equivalent to each other, and thus we chose to code only the Aki-Richards approximation, Wang's pseudoquartic approximation – in two variants – and Wang's expressions for the exact Zoeppritz equations.

We also developed our own approximation that is only constrained to small angles of incidence. We tested its behaviour for various ranges of elastic parameters and angles of incidence.

The simple derivation of a new approximation

The exact formula for the P-S reflection coefficient is given by Aki and Richards (1980, p. 150) as:

$$R_{PS} = \frac{-2\dfrac{\cos i_1}{\alpha_1}\left(ab + cd\,\dfrac{\cos i_2}{\alpha_2}\dfrac{\cos j_2}{\beta_2}\right)p\alpha_1}{\beta_1 D}, \tag{1}$$

where they use the following notation:

$$a = \rho_2\left(1 - 2\beta_2^2 p^2\right) - \rho_1\left(1 - 2\beta_1^2 p^2\right), \tag{2}$$

$$b = \rho_2\left(1 - 2\beta_2^2 p^2\right) + 2\rho_1\beta_1^2 p^2, \tag{3}$$

$$c = \rho_1\left(1 - 2\beta_1^2 p^2\right) + 2\rho_2\beta_2^2 p^2, \tag{4}$$

$$d = 2\left(\rho_2\beta_2^2 - \rho_1\beta_1^2\right), \tag{5}$$

$$E = b\frac{\cos i_1}{\alpha_1} + c\frac{\cos i_2}{\alpha_2}, \tag{6}$$

$$F = b\frac{\cos j_1}{\beta_1} + c\frac{\cos j_2}{\beta_2}, \tag{7}$$

$$G = a - d\,\frac{\cos i_1}{\alpha_1}\frac{\cos j_2}{\beta_2}\,, \tag{8}$$

$$H = a - d\,\frac{\cos i_2}{\alpha_2}\frac{\cos j_1}{\beta_1}\,, \tag{9}$$

$$D = EF + GHp^2\,. \tag{10}$$

If we now limit our interest only to small angles of incidence, we can impose a few approximations that will help simplify equation (1). So, we now expand the sine and cosine terms from the above expressions using the Taylor series:

$$\cos u \approx 1 - \frac{u^2}{2!} + \frac{u^4}{4!} + ... \tag{11}$$

$$\sin u \approx \frac{u}{1!} - \frac{u^3}{3!} + \frac{u^5}{5!} - .... \tag{12}$$

Keeping only the first term seems a reasonable enough approximation for small angles of incidence, thus:

$$\left.\begin{array}{l}\cos u \approx 1\\ \sin u \approx u\end{array}\right\}\ \text{for}\ u = i_1, i_2, j_1, j_2\,. \tag{13}$$

Using the next higher order approximation would make the final expression for R_{PS} too complicated.

The sine of the angle of incidence is also found implicitly in the formula for p, and thus:

$$p = \frac{\sin i_1}{\alpha_1} \approx \frac{i_1}{\alpha_1}\,. \tag{14}$$

Since we limit our study to small angles of incidence and are only keeping the terms up to the first order in small quantities, it is also safe to assume that:

$$p^2 \approx 0\,. \tag{15}$$

If we now apply these approximations to equations (1) to (10), we have:

$$a \approx \rho_2 - \rho_1\,, \tag{16}$$

$$b \approx \rho_2\,, \tag{17}$$

$$c \approx \rho_1\,, \tag{18}$$

$$d = 2\left(\rho_2\beta_2^2 - \rho_1\beta_1^2\right), \tag{19}$$

$$E = \frac{b}{\alpha_1} + \frac{c}{\alpha_2} = \frac{\rho_2}{\alpha_1} + \frac{\rho_1}{\alpha_2}, \tag{20}$$

$$F = \frac{b}{\beta_1} + \frac{c}{\beta_2} = \frac{\rho_2}{\beta_1} + \frac{\rho_1}{\beta_2}, \tag{21}$$

$$G = a - \frac{d}{\alpha_1\beta_2} = (\rho_2 - \rho_1) - \frac{2\left(\rho_2\beta_2^2 - \rho_1\beta_1^2\right)}{\alpha_1\beta_2}, \tag{22}$$

$$H = a - \frac{d}{\alpha_2\beta_1} = (\rho_2 - \rho_1) - \frac{2\left(\rho_2\beta_2^2 - \rho_1\beta_1^2\right)}{\alpha_2\beta_1}, \tag{23}$$

$$D = EF + GHp^2 \approx EF = \left(\frac{b}{\alpha_1} + \frac{c}{\alpha_2}\right)\left(\frac{b}{\beta_1} + \frac{c}{\beta_2}\right). \tag{24}$$

By substituting the new expressions for the terms *a*, *b*, *c*, *d*, *E*, *F*, *G*, *H* and *D* into equation (1), we get:

$$R_{PS} = \frac{-2\dfrac{\cos i_1}{\alpha_1}\left(ab + cd\,\dfrac{\cos i_2}{\alpha_2}\,\dfrac{\cos j_2}{\beta_2}\right)p\alpha_1}{\beta_1 EF}, \tag{25}$$

$$R_{PS} = \frac{-2p\cos i_1\left(ab + cd\,\dfrac{\cos i_2}{\alpha_2}\,\dfrac{\cos j_2}{\beta_2}\right)}{\beta_1\left(\dfrac{\rho_2}{\alpha_1} + \dfrac{\rho_1}{\alpha_2}\right)\left(\dfrac{\rho_2}{\beta_1} + \dfrac{\rho_1}{\beta_2}\right)}, \tag{26}$$

$$R_{PS} = \frac{-2\dfrac{i_1}{\alpha_1}\left(\rho_2(\rho_2 - \rho_1) + \dfrac{2\rho_1\left(\rho_2\beta_2^2 - \rho_1\beta_1^2\right)}{\alpha_2\beta_2}\right)}{\dfrac{1}{\alpha_1\alpha_2\beta_2}\left[(\rho_1\alpha_1 + \rho_2\alpha_2)(\rho_1\beta_1 + \rho_2\beta_2)\right]}, \tag{27}$$

$$R_{PS} = \frac{-2\dfrac{i_1}{\alpha_1\alpha_2\beta_2}\left(\alpha_2\beta_2\rho_2(\rho_2 - \rho_1) + 2\rho_1\left(\rho_2\beta_2^2 - \rho_1\beta_1^2\right)\right)}{\dfrac{1}{\alpha_1\alpha_2\beta_2}\left[(\rho_1\alpha_1 + \rho_2\alpha_2)(\rho_1\beta_1 + \rho_2\beta_2)\right]}, \tag{28}$$

$$R_{PS} = \frac{-2i_1\left(\alpha_2\beta_2\rho_2(\rho_2 - \rho_1) + 2\rho_1\left(\rho_2\beta_2^2 - \rho_1\beta_1^2\right)\right)}{(\rho_1\alpha_1 + \rho_2\alpha_2)(\rho_1\beta_1 + \rho_2\beta_2)}. \tag{29}$$

or

$$R_{PS} = \frac{-2i_1\left(\alpha_2\beta_2\rho_2(\rho_2-\rho_1)+2\rho_1\Delta\mu\right)}{(\rho_1\alpha_1+\rho_2\alpha_2)(\rho_1\beta_1+\rho_2\beta_2)} \tag{30}$$

where:

$$\Delta\mu = \mu_2 - \mu_1 = \rho_2\beta_2^2 - \rho_1\beta_1^2 \,. \tag{31}$$

As stated in the Introduction, one of the goals of this research has been to develop a simple formula for the P-SV reflection coefficient. The new formula would be highly accurate for near-vertical incidence and would supposedly provide us with a straightforward relationship between the sign of R_{PP}/R_{PS} and the change in the elastic parameters of the medium over an interface.

If we analyze equation (28) obtained in the previous derivation, there are a few statements that we can make:

- When the sign of R_{PS} changes from negative to positive, it has to pass through zero.
- R_{PS} becomes zero when the numerator is zero, that is when

$$\alpha_2\beta_2\rho_2\Delta\rho + 2\rho_1\Delta\mu = 0 \,. \tag{32}$$

- We observe that, at least in this approximation, the change in sign is not influenced by the change in P-wave velocity (α_1 is not involved in the expression above).
- The sign changes when the following condition is met:

$$\frac{\Delta\mu}{\Delta\rho} = -\frac{\alpha_2\beta_2\alpha_2}{2\rho_1} \,. \tag{33}$$

Thus, the numerator from equation (28) can become zero only when the contrast in density ($\Delta\rho$) and the contrast in shear moduli ($\Delta\mu$) have opposite signs. From this, we can write the following two cases:

Case 1:

$$\begin{array}{l} \rho_2-\rho_1>0 \\ \rho_2\beta_2^2-\rho_1\beta_1^2<0 \end{array} \;\Rightarrow\; \begin{array}{l} \rho_2>\rho_1 \\ \rho_2\beta_2^2<\rho_1\beta_1^2 \end{array} \;\Rightarrow\; \begin{array}{l} \rho_2>\rho_1, \beta_2<\beta_1 \\ \rho_2\beta_2^2<\rho_1\beta_1^2 \end{array}$$

Case 2:

$$\begin{array}{l} \rho_2-\rho_1<0 \\ \rho_2\beta_2^2-\rho_1\beta_1^2>0 \end{array} \;\Rightarrow\; \begin{array}{l} \rho_2<\rho_1 \\ \rho_2\beta_2^2>\rho_1\beta_1^2 \end{array} \;\Rightarrow\; \begin{array}{l} \rho_2<\rho_1, \beta_2>\beta_1 \\ \rho_2\beta_2^2>\rho_1\beta_1^2 \end{array}$$

If we use the notation of Aki and Richards (1980), we can express the two conditions as follows:

Case 1:

$$\Delta\rho > 0, \Delta\beta < 0 \quad \Rightarrow \quad \frac{\Delta\rho}{\rho} > 0, \frac{\Delta\beta}{\beta} < 0$$

$$\Delta\mu < 0 \qquad\qquad \frac{\Delta\mu}{\mu} < 0$$

Case 2:

$$\Delta\rho < 0, \Delta\beta > 0 \quad \Rightarrow \quad \frac{\Delta\rho}{\rho} < 0, \frac{\Delta\beta}{\beta} > 0$$

$$\Delta\mu > 0 \qquad\qquad \frac{\Delta\mu}{\mu} > 0$$

where $\mu = (\mu_1 + \mu_2)/2$, $\beta = (\beta_1 + \beta_2)/2$, $\rho = (\rho_1 + \rho_2)/2$.

So far, these are the simplest conditions that we can find for the change in the polarity of the reflected P-SV waves at small offsets. Further analysis of the conditions governing the changes in sign of R_{PP} and R_{PS} will be the subject of our future research. For now we focus on studying the accuracy of this approximation.

TESTING THE APPROXIMATIONS

In order to test the approximations, we used three models of geologic media, the same as in Brown et al. (2000). Each of them consists of a plane interface between two layers characterized by quite different elastic parameters. The models were called:

The 'normal' situation: α_1 = 2000 m/s; α_2 = 3500 m/s,
β_1 = 800 m/s; β_2 = 1800 m/s,
ρ_1 = 1900 kg/m^3; ρ_2 = 2400 kg/m^3

Clastic over salt: α_1 = 3600 m/s; α_2 = 4500 m/s,
β_1 = 2400 m/s; β_2 = 2500 m/s,
ρ_1 = 2600 kg/m^3; ρ_2 = 2100 kg/m^3

Shale over gas sand: α_1 = 2150 m/s; α_2 = 1750 m/s,
β_1 = 860 m/s; β_2 = 1250 m/s,
ρ_1 = 2200 kg/m^3; ρ_2 = 1950 kg/m^3.

The second and third models involve ‘parameter reversals’, that is, the three rock parameters do not all change in the same direction across the interface. This type of situation is often encountered in the subsurface covered by multicomponent seismic surveys. For example, in blocked well logs from the Blackfoot field, Margrave et al. (2001) show eight interfaces, three of which have ‘parameter reversals’: in each case β changing in the opposite direction to that of α and ρ. So, although the majority of actual geologic cases may be ‘normal’, with $R_{PP}/R_{PS} < 0$, there is a real possibility that $R_{PP}/R_{PS} > 0$, especially if somewhat 'anomalous' lithologies are present, like salt, gas sand, or coal. Knowledge of any such parameter reversals will forewarn one to expect reversals of polarity in correlating events from P-P to P-S sections, even after care has been taken to produce only normal-polarity sections (Brown et al., 2000).

We used the MATLAB software to write the code for the four different approximations, and also for the exact R_{PS} expression. These exact values of R_{PS}, together with the results of the four approximations, are presented in Table 1. We have analyzed the accuracy of the approximations for incidence angles of 0, 5,10, 20 and 30°.

Model 1: The 'normal' situation					
α_1=2000 α_2=3500	(m/s)	β_1=800 β_2=1800	(m/s)	ρ_1=1900 ρ_2=2400	(kg/m^3)
i_1	$R_{ZOEPPRITZ}$	R_{AKI}	$R_{WANG\,1}$	$R_{WANG\,2}$	R_{APPROX}
0	0	0	0	0	0
5	-0.0789	-0.1129	-0.0789	-0.1297	-0.0796
10	-0.1533	-0.2166	-0.1533	-0.2532	-0.1592
20	-0.2684	-0.3608	-0.2681	-0.4560	-0.3183
30	-0.2642	-0.3569	-0.2521	-0.5505	-0.4775
Model 2: Clastic over salt					
α_1=3600 α_2=4500	(m/s)	β_1=2400 β_2=2500	(m/s)	ρ_1=2600 ρ_2=2100	(kg/m^3)
i_1	$R_{ZOEPPRITZ}$	R_{AKI}	$R_{WANG\,1}$	$R_{WANG\,2}$	R_{APPROX}
0	0	0	0	0	0
5	0.0172	0.0181	0.0172	0.0071	0.0173
10	0.0340	0.0358	0.0340	0.0136	0.0346
20	0.0647	0.0674	0.0647	0.0230	0.0692
30	0.0891	0.0914	0.0891	0.0244	0.1039
Model 3: Shale over gas sand					
α_1=2150 α_2=1750	(m/s)	β_1=860 β_2=1500	(m/s)	ρ_1=2200 ρ_2=1950	(kg/m^3)
i_1	$R_{ZOEPPRITZ}$	R_{AKI}	$R_{WANG\,1}$	$R_{WANG\,2}$	R_{APPROX}
0	0	0	0	0	0
5	-0.0255	-0.0215	-0.0255	-0.0358	-0.0256
10	-0.0499	-0.0418	-0.0499	-0.0703	-0.0513
20	-0.0918	-0.0743	-0.0918	-0.1311	-0.1026
30	-0.1190	-0.0897	-0.1190	-0.1742	-0.1539

Table 1. Examples of R_{PS} versus angle of incidence (i_1) in degrees calculated with the exact Zoeppritz formulae and four other approximate expressions for three different interface models. $R_{ZOEPPRITZ}$ is the result obtained with the actual Zoeppritz equations while R_{AKI}, R_{WANG1}, R_{WANG2}, and R_{APPROX} are results of the approximations developed by Aki and Richards (1980), Wang (1999) (two approximations), and by us, respectively.

THE ACCURACY OF THE APPROXIMATIONS

In order to better illustrate the results shown in Table 1 and also to make it easier to compare the four approximations, we also plotted the results for different ranges of velocities, densities and angles of incidence (Figures 2 to 6). Emphasis is put on the behaviour at small angles of incidence (5°). We used the same domain of variation for the rock parameters of both media (i.e. α_1 and α_2, β_1 and β_2, ρ_1 and ρ_2).

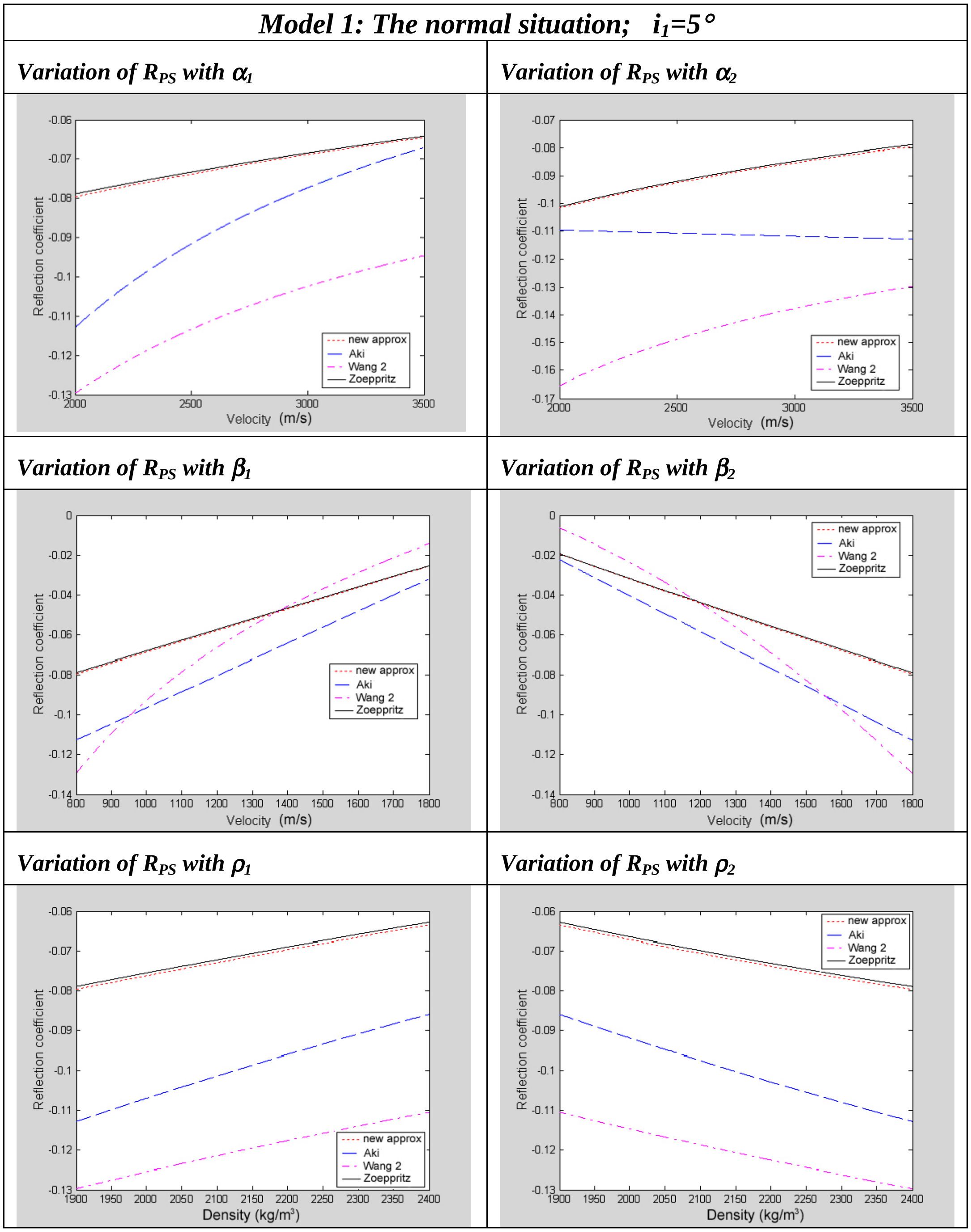

FIG. 2. Variation of the reflection coefficient with the change in velocity and density for an incidence angle of 5°; the plotted curves correspond to the exact values of R_{PS} and to three of its approximations – Aki and Richards (1980), Wang (1999) and our new approximation. The interface model is 'model 1' from Table 1.

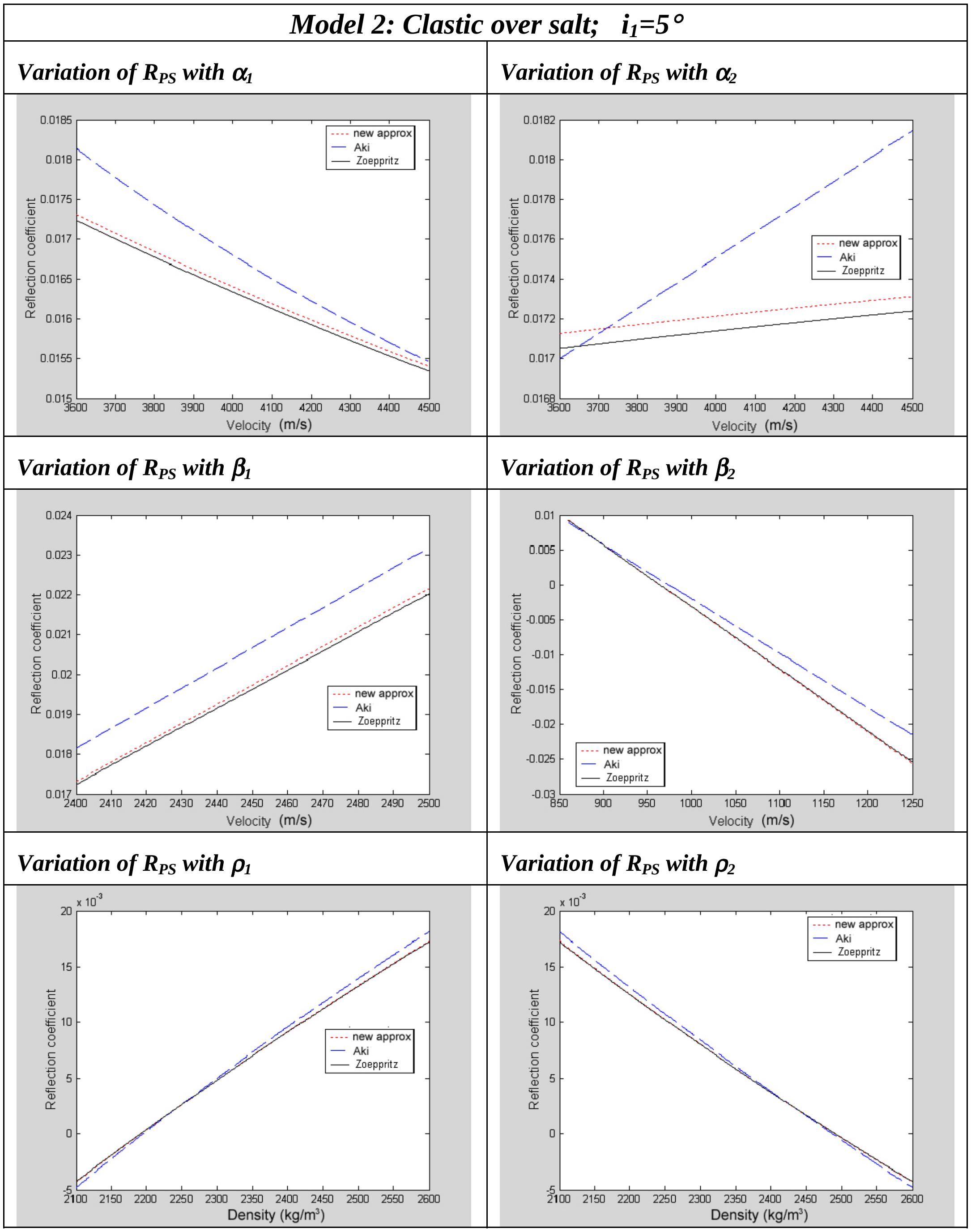

FIG. 3. Variation of the reflection coefficient with the change in velocity and density for an incidence angle of 5°; the plotted curves correspond to the exact values of R_{PS} and to two of its approximations – Aki and Richards (1980) and our new approximation. The interface model is 'model 2' from Table 1. In the last three plots, the curves obtained using our approximation and the exact Zoeppritz equation are very close and cannot be distinguished from each other.

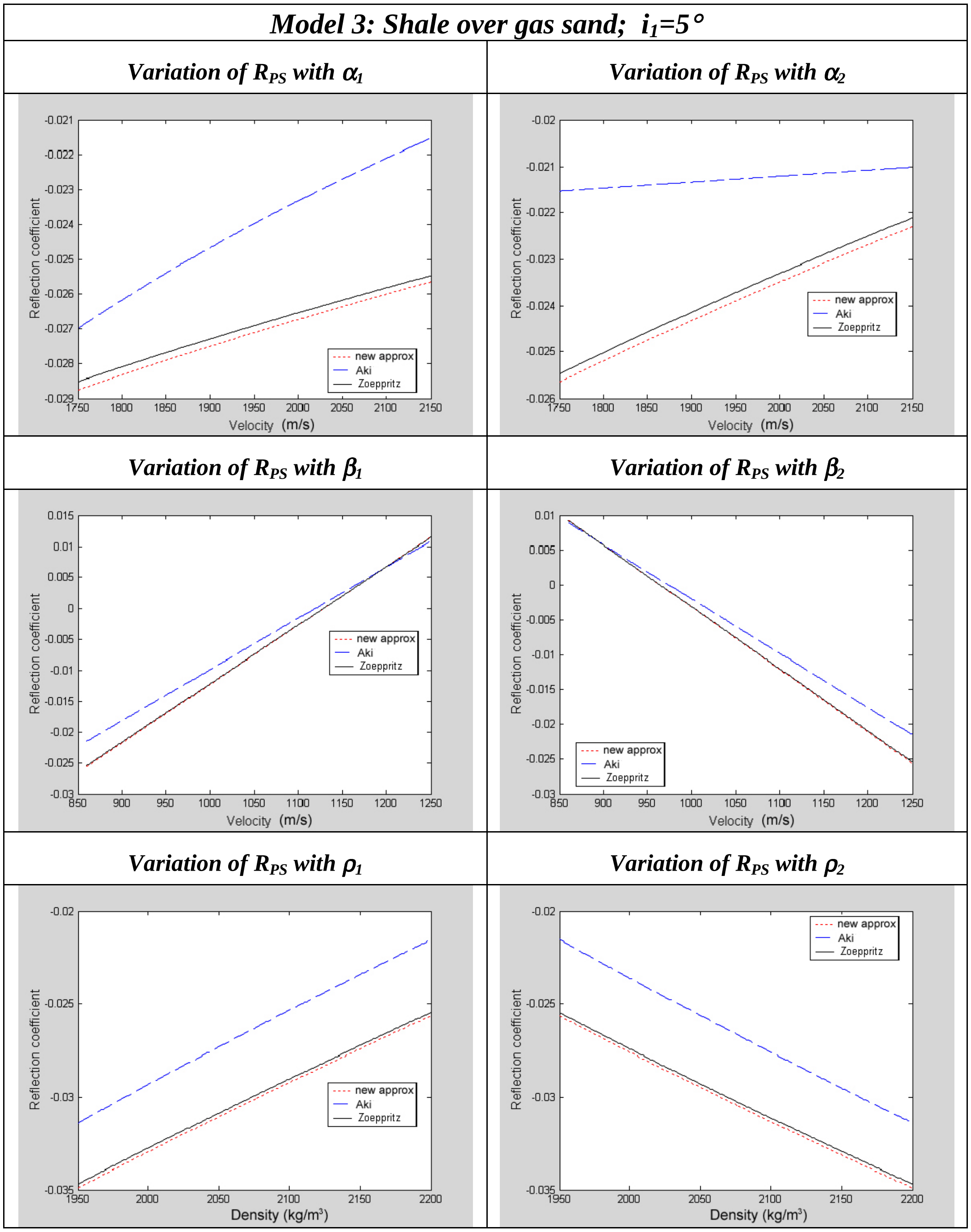

FIG. 4. Variation of the reflection coefficient with the change in velocity and density for an incidence angle of 5°; the plotted curves correspond to the exact values of R_{PS} and to two of its approximations – Aki and Richards (1980) and our new approximation. The interface model is 'model 3' from Table 1. In the two middle plots, the curves obtained using our approximation and the exact Zoeppritz equation are very close cannot be distinguished from each other.

Because the second pseudoquartic approximation developed by Wang (1999) for R_{PS} turned out to be very poor, we show it only in Figures 2 and 6. The first pseudoquartic approximation is very accurate and we show it only in Figure 6, where it can be distinguished from the exact curve. In this figure the reflection coefficients are plotted against the angle of incidence.

After examining Figures 2, 3 and 4, we can draw the following conclusions:

- Our expression approximates the exact P-SV reflection coefficient at $i_1 = 5°$ very well for all ranges of velocities and densities tested.
- As we expect, the approximation of Aki and Richards (1980) gets better when the differences between the rock parameters of the two media become smaller.
- The second pseudoquartic approximation developed by Wang (1999) gives very poor results (Figures 2) and we are entitled to say that the expression provided by him is either a very poor approximation or is erroneous.
- The plots that result from varying the shear-wave velocities and the densities in the upper and lower media show a very obvious symmetry. In other words, the change in shear velocity, $\Delta\beta$, or density, $\Delta\rho$, is more important than the way in which this change is manifested (increase or decrease across the interface).
- The plots resulting from varying the P-wave velocities in the upper and lower media are not symmetric. This can be because α_1 is also hidden inside the p (ray-parameter) terms of the Zoeppritz equation and influences in a different manner the behaviour of the R_{PS} curves.

Figure 5 shows the variation of R_{PS} with α_1 and ρ_1 for incidence angles of 10°, 20° and 30°. By analyzing these plots, we can conclude that, for the Aki-Richards formula, the accuracy of the approximation is not perturbed by changes in the angle of incidence, i_1. On the other hand, our new approximation loses its accuracy quickly as i_1 increases. It becomes relatively inaccurate for angles greater than 25° or 30°.

Figures 6 and 7 also support this affirmation. Figure 6 shows the more familiar variation of R_{PS} with the angle of incidence for all three models from Table 1 and for all approximations studied. It is clear that, at least up to the P-P critical angle (if there is one), the first pseudoquartic formula provides the best approximation.

These plots also show the inaccuracy of the second pseudoquartic approximation.

Figure 7 presents error charts that were computed for our approximation and the Aki-Richards (1980) approximation. The error was computed as the absolute value of the difference between the approximate and exact values of R_{PS}:

$$E = \left| R_{exact} - R_{approx} \right| . \tag{34}$$

The 'model 2' plot in Figure 7 shows that, in this case the new approximation can only be confidently utilized for incidence angles that are smaller than 15° whereas, in models 1 and 3, it is good up to 25 or 30°. This is because in model 2 the changes in elastic parameters across the interface are smaller and, in this case, the Aki-Richards approximation is more accurate.

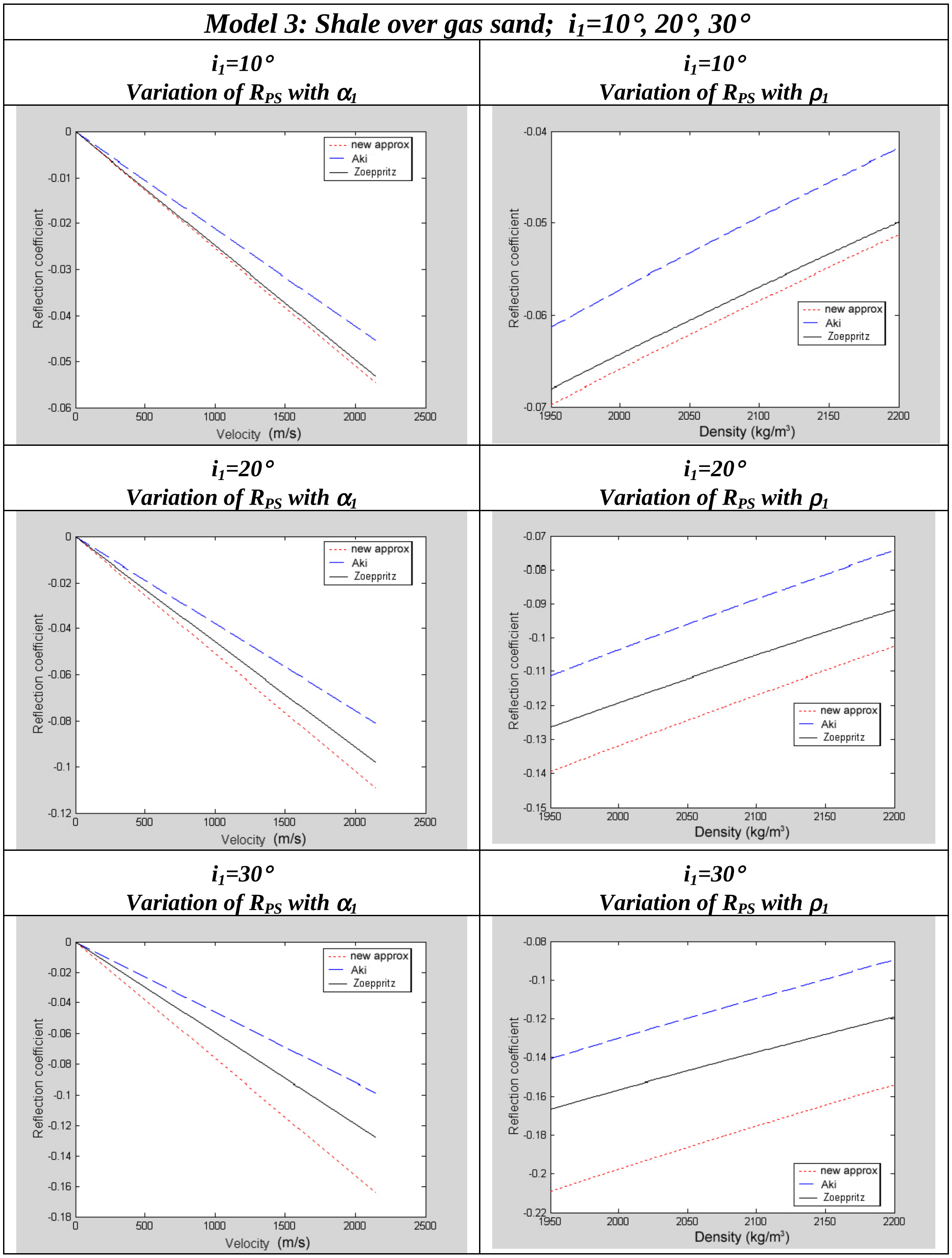

FIG. 5. Variation of the reflection coefficient with the change in α_1 and ρ_1 for incidence angles of 10°, 20°, 30°; the plotted curves correspond to the exact values of R_{PS} and to two of its approximations – Aki and Richards (1980) and our new approximation. The interface model is 'model 3 – shale over gas sand' from Table 1.

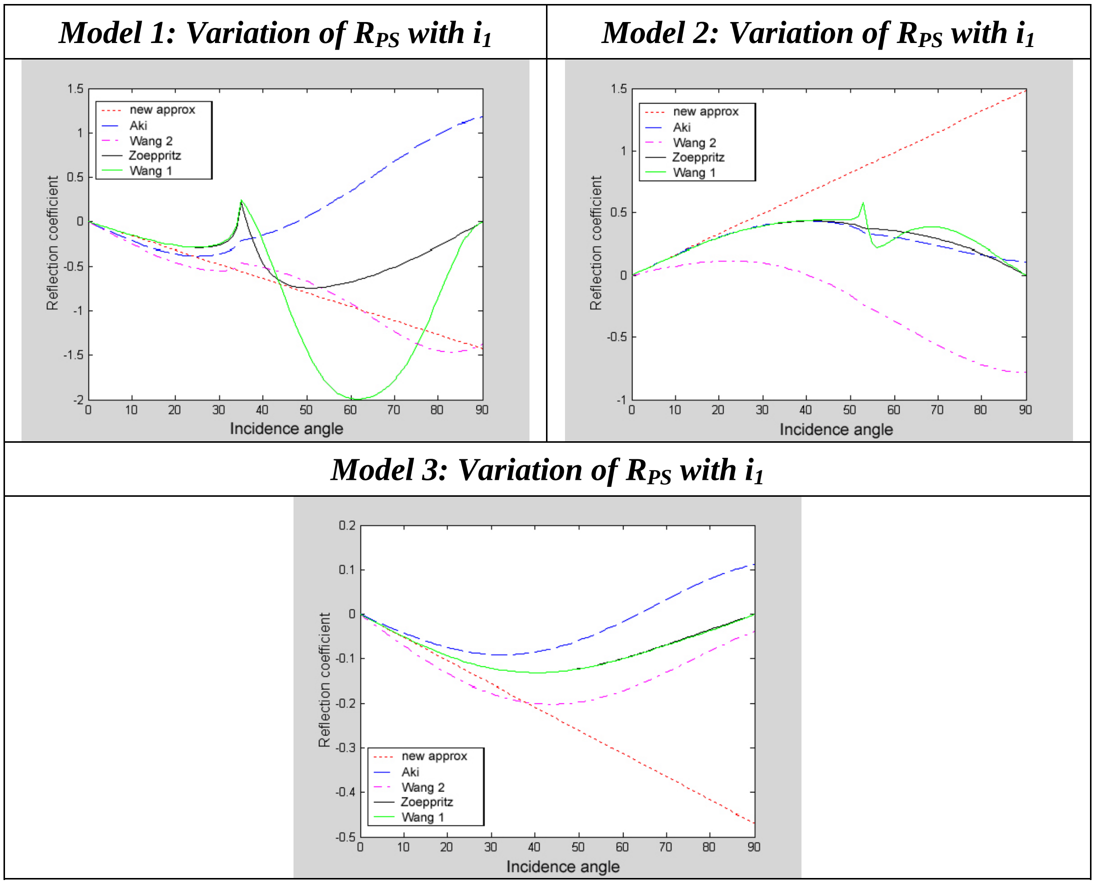

FIG. 6. Variation of the reflection coefficient with the incidence angle; the plotted curves correspond to the exact values of R_{PS} and to four of its approximations – Aki and Richards (1980), Wang (1999) – two approximations and our new approximation. The graphs were plotted for all three models from Table 1.

Still, if we stick to small angles of incidence, the accuracies of the two approximations are comparable, even for relatively small changes in medium parameters, as shown for 'model 4' (Table 2).

Table 2. Examples of P-SV reflection coefficients versus angle of incidence, calculated for small changes in elastic parameters with the exact Zoeppritz formulae, Aki-Richards approximation, and our new approximation.

Model 4: Small changes in elastic parameters			
α_1=2150 α_2=2160 (m/s)	β_1=800 β_2=810 (m/s)	ρ_1=2200 ρ_2=2210 (kg/m^3)	
i_1	$R_{ZOEPPRITZ}$	R_{AKI}	R_{APPROX}
5°	-0.0011	-0.0012	-0.0012

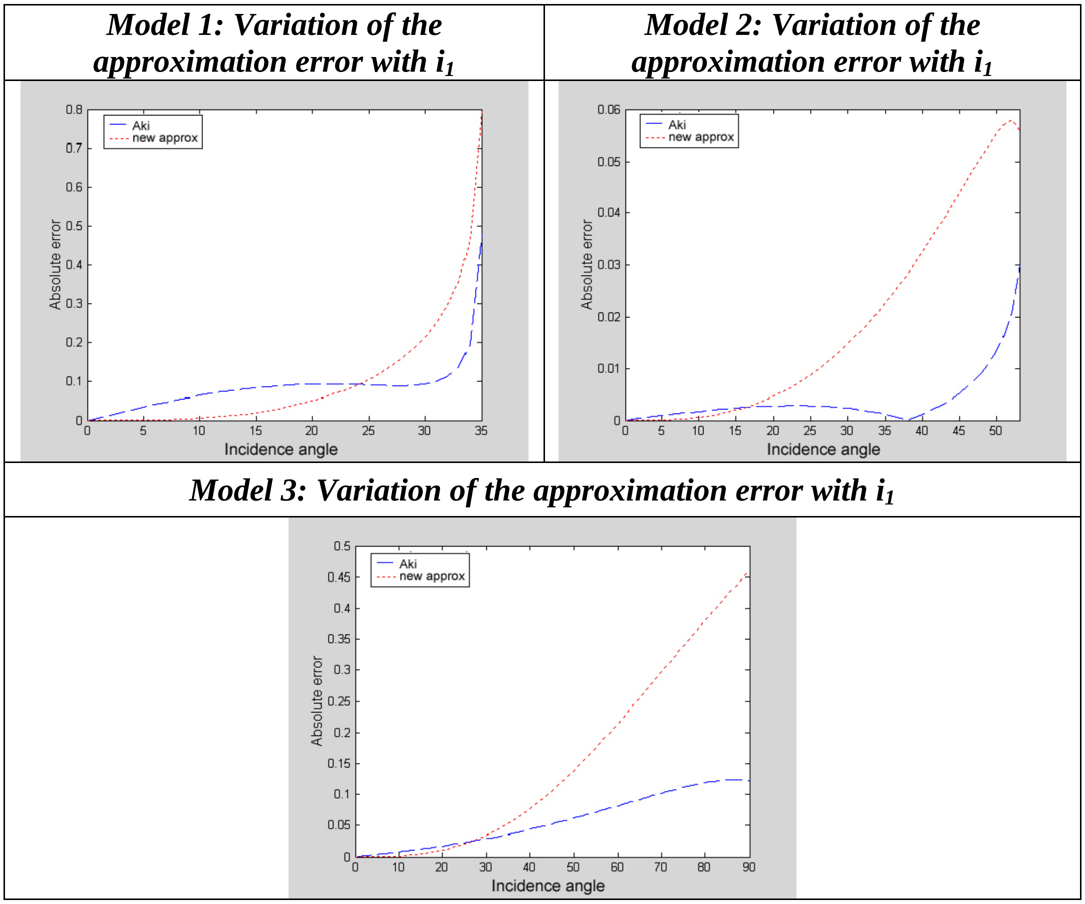

FIG. 7. The absolute error in the estimation of R_{PS}; the plotted curves correspond to the error in the PS reflection coefficients calculated with two approximations – Aki and Richards (1980) and our new approximation. The error curves were plotted for all three models from Table 1.

CONCLUSIONS AND FUTURE WORK

The testing done so far proves that the new expression developed for R_{PS} is a very accurate approximation for small angles of incidence (near-zero offsets). The plots of the P-SV reflection coefficients presented in Figures 2 to 6 and the error plots from Figure 7 show that the new approximation works well under the imposed restrictions.

We can impose the same restrictions on the exact expression for the P-P reflection coefficient. The resulting approximation has the same mathematical form as the well-known zero-offset P-P reflection coefficient, that is:

$$R_{PP} = \frac{\rho_2\alpha_2 - \rho_1\alpha_1}{\rho_2\alpha_2 + \rho_1\alpha_1}. \tag{35}$$

A few subsequent observations that are worth pursuing further. One is the fact that the change in P-wave velocity across an interface does not influence the change in the sign of R_{PS} at small offsets. One might have expected this because the shear-wave velocity also does not appear in the zero-offset formula for R_{PP}.

Also important is the observation that Wang's (1999) second pseudoquartic approximation gives very poor results for R_{PS}. Rederiving it or finding an error in its derivation may prove to be useful in developing further simplified quadratic or linear approximations for R_{PS} (by utilizing the same method that was used by Wang for the R_{PP} approximations).

A linear expression for R_{PS} – the equivalent of Shuey's R_{PP} formula – may be valuable for converted-wave AVO.

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APPENDIX A - APPROXIMATIONS TO THE ZOEPPRITZ EQUATIONS

The Aki-Richards approximation for the P-S reflection coefficient

Assumptions

- The approximation is made for the reflection and transmission at a plane horizontal boundary between two media.
- The two half-spaces must have similar properties (small changes in medium parameters at the interface).
- As a result, the majority of the incident energy will be transmitted. Thus, the transmission coefficients such as T_{PP} or T_{SS} are of order one, whereas the reflection coefficients are small.

$$\Delta\alpha = \alpha_2 - \alpha_1, \qquad \alpha = (\alpha_2 - \alpha_1)/2,$$
$$\Delta\beta = \beta_2 - \beta_1, \qquad \beta = (\beta_2 + \beta_1)/2,$$
$$\Delta\rho = \rho_2 - \rho_1, \qquad \rho = (\rho_2 + \rho_1)/2,$$

$$\frac{\Delta\alpha}{\alpha}, \frac{\Delta\beta}{\beta}, \frac{\Delta\rho}{\rho} << 1,$$

$$\Delta i = i_2 - i_1 \approx \tan i \left(\frac{\Delta\alpha}{\alpha}\right),$$

$$\Delta j = j_2 - j_1 \approx \tan j \left(\frac{\Delta\beta}{\beta}\right).$$

$$R_{PS} = \frac{-p\alpha}{2\cos j}\left[\left(1 - 2\beta^2 p^2 + 2\beta^2 \frac{\cos i}{\alpha}\frac{\cos j}{\beta}\right)\frac{\Delta\rho}{\rho} - \left(4\beta^2 p^2 - 4\beta^2 \frac{\cos i}{\alpha}\frac{\cos j}{\beta}\right)\frac{\Delta\beta}{\beta}\right].$$

There is a tendency for the coefficient of $\frac{\Delta\beta}{\beta}$ to be larger than the coefficients of the other two velocity and density ratios (only $\frac{\Delta\rho}{\rho}$ is present in the R_{PS} formula). According to Aki and Richards (1980), this means that changes in shear velocity are more efficient in scattering elastic waves.

Shuey's approximation for the P-P reflection coefficient

Assumptions

- The formula used to obtain the new approximation is the approximation developed by Aki and Richards (1980).
- The approximations used are the same as in Aki and Richards (1980) (stated above).

The formula for R_{PP}

$$R_{PP} = R_0 + \left[A_0 R_0 + \frac{\Delta\sigma}{(1-\sigma)^2}\right]\sin^2 i + \frac{1}{2}\frac{\Delta\alpha}{\alpha}\left(\tan^2 i - \sin^2 i\right).$$

where: R_0 is the normal-incidence reflection coefficient,
$\sigma = (\sigma_1 + \sigma_2)/2$; σ_1 and σ_2 are Poisson's ratios in the two media,
$\Delta\sigma = (\sigma_1 - \sigma_2)$.

Wang's approximation for the P-P and P-S reflection coefficients

The starting formula for R_{PP}	
• The formula used to obtain the new approximation is the exact formula for the P-P/P-S reflection coefficient – an adaptation of the formula found in Aki and Richards (1980).	
	$R_{PP}(p) = \dfrac{E + Fp^2 + Gp^4 - Dp^6}{A + Bp^2 + Cp^4 + Dp^6}.$
where:	$A = (p_2 q_{\alpha 1} + p_1 q_{\alpha 2})(p_2 q_{\beta 1} + p_1 q_{\beta 2}),$ $B = -4\Delta\mu(p_2 q_{\alpha 1} q_{\beta 1} - p_1 q_{\alpha 2} q_{\beta 1}) + (\Delta\rho)^2 + 4(\Delta\mu)^2 q_{\alpha 1} q_{\alpha 2} q_{\beta 1} q_{\beta 2},$ $C = 4(\Delta\mu)^2 (q_{\alpha 1} q_{\beta 1} + q_{\alpha 2} q_{\beta 2}) - 4\Delta\mu\Delta\rho,$ $D = 4(\Delta\mu)^2,$ $E = (\rho_2 q_{\alpha 1} - \rho_1 q_{\alpha 2})(\rho_2 q_{\beta 1} + \rho_1 q_{\beta 2}),$ $F = -4\Delta\mu(\rho_2 q_{\alpha 1} q_{\beta 1} + p_1 q_{\alpha 2} q_{\beta 2}) - (\Delta\rho)^2 + 4(\Delta\mu)^2 q_{\alpha 1} q_{\alpha 2} q_{\beta 1} q_{\beta 2},$ $G = 4(\Delta\mu)^2 (q_{\alpha 1} q_{\beta 1} - q_{\alpha 2} q_{\beta 2}) + 4\Delta\mu\Delta\rho,$ $H = 2(\rho_2 q_{\beta 1} - \rho_1 q_{\beta 2})\rho_1 q_{\alpha 2}(\alpha_1/\alpha_2),$ $I = -4\Delta\mu(q_{\beta 1} - q_{\beta 2})\rho_1 q_{\alpha 1}(\alpha_1/\alpha_2).$
Additional notation:	q is the vertical slowness, $\sigma = (\sigma_1 + \sigma_2)/2$; σ_1 and σ_2 are Poisson's ratios in the two media, $\Delta\mu = \rho_2 \beta_2^2 - \rho_1 \beta_1^2$ is the contrast in shear moduli.

<table>
<tr><th colspan="2">The pseudoquartic approximation for R_{PP}</th></tr>
<tr><td>Assumptions:</td><td>

$$(A+Bp^2+Cp^4+Dp^6)^{-1} \approx \frac{1}{A}-\frac{B}{A^2}p^2-\left(\frac{C}{A^2}-\frac{B^2}{A^3}\right)p^4.$$

$$R_{PP}(p) \approx \frac{E}{A}+\left(\frac{F}{A}-\frac{BE}{A^2}\right)p^2-\left(\frac{G}{A}-\frac{BF}{A^2}-\frac{CE}{A^2}+\frac{B^2E}{A^3}\right)p^4.$$

$$\left.\begin{matrix}\left(\dfrac{\Delta\rho}{\rho}\right)^2 \approx 0 \\ \dfrac{\rho_2 q_{\alpha 1} q_{\beta 1}-\rho_1 q_{\alpha 2} q_{\beta 2}}{(\rho_2 q_{\alpha 1}+\rho_1 q_{\alpha 2})(\rho_2 q_{\beta 1}+\rho_1 q_{\beta 2})}\end{matrix}\right\} \Rightarrow$$

$$\frac{F}{A}-\frac{BE}{A^2} \approx -2\frac{\Delta\mu}{\rho}+(1-R_f)\left(\frac{\Delta\mu}{\rho}\right)^2 q_\alpha \rho_\beta.$$

</td></tr>
<tr><td></td><td>

$$R_{PP} \approx R_f-\left[2\frac{\Delta\mu}{\rho}-(1-R_f)q_\alpha q_\beta\left(\frac{\Delta\mu}{\rho}\right)^2\right]p^2$$

$$-\left[2R_f\left(\frac{\Delta\mu}{\rho}\right)^2-2q_\alpha q_\beta\left(\frac{\Delta\mu}{\rho}\right)^3+(1-R_f)q_\alpha^2 q_\beta^2\left(\frac{\Delta\mu}{\rho}\right)^4\right]p^4.$$

</td></tr>
<tr><td>where:</td><td>

$$R_f=\frac{(\rho_2 q_{\alpha 1}-\rho_1 q_{\alpha 2})}{(\rho_2 q_{\alpha 1}+\rho_1 q_{\alpha 2})}.$$

- The other notation is the same as that of Aki and Richards, presented earlier.

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</table>

<table>
<tr><th colspan="2">The quadratic approximation for R_{PP}</th></tr>
<tr><td>Assumptions:</td><td>

- The pseudoquartic formula developed above is truncated at the p^2 term and the following assumptions are made:

$$\tan\left(\frac{\Delta i}{2}\right) \approx \frac{1}{2}\frac{\Delta\alpha}{\alpha}\tan i,$$

$$R_f \approx \frac{1}{2}\left(\frac{\Delta\rho}{\rho}+\sec^2 i\frac{\Delta\alpha}{\alpha}\right),$$

$$\frac{q_{\alpha 1}\alpha_1}{q_{\alpha 2}\alpha_2} \approx 1+\alpha\Delta\alpha p^2,$$

- $\Delta\mu=\beta^2\Delta\rho+2\rho\beta\Delta\beta+\frac{1}{4}\Delta\rho(\Delta\beta)^2.$

</td></tr>
<tr><td></td><td>

$$R_{PP} \approx \left[\frac{1}{2}-2\left(\frac{\beta}{\alpha}\right)^2\sin^2 i\right]\frac{\Delta\rho}{\rho}+\frac{1}{2}\sec^2 i\frac{\Delta\alpha}{\alpha}-4\left(\frac{\beta}{\alpha}\right)^2\sin^2 i\frac{\Delta\beta}{\beta}$$

$$+\left(\frac{\beta}{\alpha}\right)^3\cos i\sin^2 i\left(\frac{\Delta\rho}{\rho}+2\frac{\Delta\beta}{\beta}\right)^2.$$

</td></tr>
</table>

The linearized approximation for R_{PP}	
Assumptions:	▪ If we ignore the terms that contain $\left(\frac{\Delta\rho}{\rho}+2\frac{\Delta\beta}{\beta}\right)^2$ in the quadratic approximation formula, we obtain an expression that is linear in each of the three elastic contrast terms $\left(\frac{\Delta\alpha}{\alpha},\frac{\Delta\beta}{\beta},\frac{\Delta\rho}{\rho}\right)$. ▪ This expression is equivalent to the ones developed by Bortfeld (1961), Aki and Richards (1980), and Shuey (1985).
	$R_{PP} \approx \left[\frac{1}{2}-2\left(\frac{\beta}{\alpha}\right)^2 \sin^2 i\right]\frac{\Delta\rho}{\rho}+\frac{1}{2}\sec^2 i\frac{\Delta\alpha}{\alpha}-4\left(\frac{\beta}{\alpha}\right)^2\sin^2 i\frac{\Delta\beta}{\beta}$.

The starting formula for R_{PS}	
Assumption:	$R_{PS}(p)=\left(2q_{\alpha 1}\frac{\alpha_1}{\beta_1}p\right)\frac{J+Kp^2-Dp^4}{A+Bp^2+Cp^4+Dp^6}$.
where:	*A*, *B*, *C and D* are the same as in the R_{PP} formula above $J=-\rho_2\Delta\rho-2\rho_1 q_{\alpha 2}q_{\beta 2}\Delta\mu$, $K=2(\rho_2+\Delta\rho)\Delta\mu-4q_{\alpha 2}q_{\beta 2}(\Delta\mu)^2$, $L=\rho_1\Delta\rho-2\rho_1 q_{\alpha 2}q_{\beta 1}\Delta\mu$, $M=-2\rho_1\Delta\mu$.
The pseudoquartic approximation for R_{PS}	
After a Taylor series expansion of the denominator, we have:	
	$R_{PS}(p)\approx\left(2q_{\alpha 1}\frac{\alpha_1}{\beta_1}p\right)\left[\frac{J}{A}+\left(\frac{K}{A}-\frac{BJ}{A^2}\right)p^2 -\left(\frac{D}{A}+\frac{BK}{A^2}+\frac{CJ}{A^2}-\frac{B^2J}{A^3}\right)p^4\right]$.
	$R_{PS}=-\frac{\alpha_1}{\beta_1}\frac{\Delta\mu}{\rho}q_\alpha p\left\{1-\left[\frac{1}{q_\alpha q_\beta}-2\frac{\Delta\mu}{\rho}+q_\alpha q_\beta\left(\frac{\Delta\mu}{\rho}\right)^2\right]p^2 +\left[\frac{2}{q_\alpha q_\beta}\frac{\Delta\mu}{\rho}-\left(\frac{\Delta\mu}{\rho}\right)^2-2q_\alpha q_\beta\left(\frac{\Delta\mu}{\rho}\right)^3+q_\alpha^2 q_\beta^2\left(\frac{\Delta\mu}{\rho}\right)^4\right]p^4\right\}$.
where:	$R_f=\frac{(\rho_2 q_{\alpha 1}-\rho_1 q_{\alpha 2})}{(\rho_2 q_{\alpha 1}+\rho_1 q_{\alpha 2})}$.