# Optimal Zoeppritz approximations

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## ABSTRACT

New approximations are developed for $R_{PP}$ and $R_{PS}$ that are optimal in the sense that they preserve as much accuracy as possible with as much simplicity as possible. In particular, they are expressed in a form that is pseudo-linear after the form of the Aki-Richards approximation. Thus they would be very simple to apply in an essentially exact non-linear inversion. Accurate analogues of the two-term Shuey equation are also given for both $R_{PP}$ and $R_{PS}$.


## INTRODUCTION

The Zoeppritz equations describe the partitioning of energy among reflections and transmissions of compressional and shear waves at an interface (Aki and Richards, 1980). The coefficients for the various modes are dependent on the six parameters describing density ($\rho_1$, $\rho_2$) and velocity ($\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$) of upper and lower earth layers, as well as the angle of the incident wave with respect to the normal. The model upon which the Zoeppritz equations are based assumes plane waves impinging on a planar, non-slip boundary between two isotropic and elastic semi-infinite half-spaces. Because of these assumptions, the Zoeppritz coefficients are not always accurate enough to describe real seismic responses and their amplitude variation with offset (AVO), but they have nonetheless provided much assistance in practical exploration geophysics.

The same may be said of the Aki-Richards approximation (Aki and Richards, 1980). This approach recognizes that only four of the six earth parameters are independent, and may be represented, for instance, by $\beta/\alpha$, $\Delta\rho/\rho$, $\Delta\alpha/\alpha$, and $\Delta\beta/\beta$, where $x = ½ [x_1 + x_2]$, $\Delta x = (x_2 - x_1)$, $x = \alpha, \beta, \rho$. In the Aki-Richards approximation, the exact Zoeppritz coefficients are linearized with respect to the three contrast variables, $\Delta\rho/\rho$, $\Delta\alpha/\alpha$, and $\Delta\beta/\beta$, and this results in sufficiently simple expressions that amplitude variations with offset can be interpreted in terms of quantities relevant to exploration seismology. Of course their usefulness is challenged for interfaces with large contrasts, but still their role has been, and continues to be, a significant one.

One of the appeals of the Aki-Richards approximation has been its ease of application to inversion problems. In this context we ask whether there would be a role for an approximation which is nearly as straightforward to apply as that of Aki-Richards, but which is also nearly as accurate as the exact results of Zoeppritz. One obvious application would be to improve inversion results. However, one of the more far-reaching benefits may be because of the need to remove some of the assumptions of the Zoeppritz equations. As their inadequacy in the presence of anisotropy, absorption, non-planar interfaces, etc. becomes better understood, it is important to separate various contributions to error. Accuracy with simplicity will be at a premium. Thus retaining the framework of the Aki-Richards approximation while minimizing its error would be one valuable contribution to further development of AVO techniques.

In what follows we present an argument for what form an optimal Zoeppritz approximation should take for two key coefficients, the compressional and shear wave reflectivities, $R_{PP}$ and $R_{PS}$. We then develop explicit expressions and discuss their accuracy and application.

## RATIONALE OF METHOD

It is instructive to consider how the Zoeppritz coefficients actually behave under various approximations. Consider Figure 1 in which the exact value of $R_{PP}$ for a set of typical earth parameters is compared to the value of its Taylor expansion the parameter $\Delta\alpha/\alpha$, when the truncation is at the linear, quadratic, or cubic level in $\Delta\alpha/\alpha$. (No approximations are made in $\Delta\beta/\beta$ or $\Delta\rho/\rho$ at this point). It is clear that the higher the degree of terms that are present, the closer the approximation comes to the exact value. However it is equally clear that, near the critical angle, there is not any rapid convergence to an accurate result.

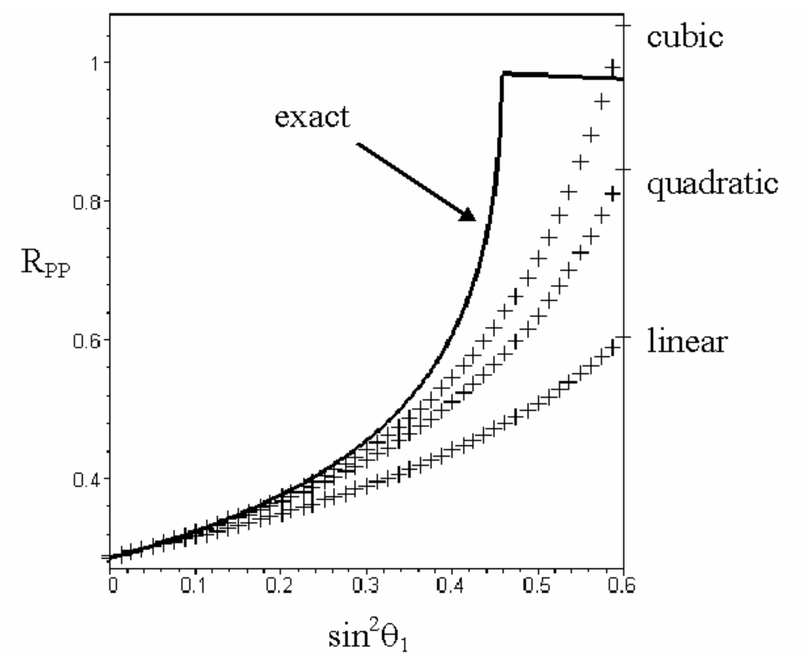


Figure 1: The value of $R_{PP}$ for various levels of approximation in $\Delta\alpha/\alpha$. No approximations have been made in $\Delta\beta/\beta$ or $\Delta\rho/\rho$. The specific earth parameters that have been used are $\Delta\alpha/\alpha$ = .3857, $\Delta\beta/\beta$ = -.1857, $\Delta\rho/\rho$ = .19524, and $\beta/\alpha$ = 0.4. $\theta_1$ is the angle of incidence in the upper layer.

In contrast, when the Taylor expansion is applied to $\Delta\beta/\beta$ instead of to $\Delta\alpha/\alpha$, even the linear expansion is accurate up to the critical point, as shown in Figure 2. In Figure 3 we see that for $\Delta\rho/\rho$ a linear expansion is good even past the critical point.

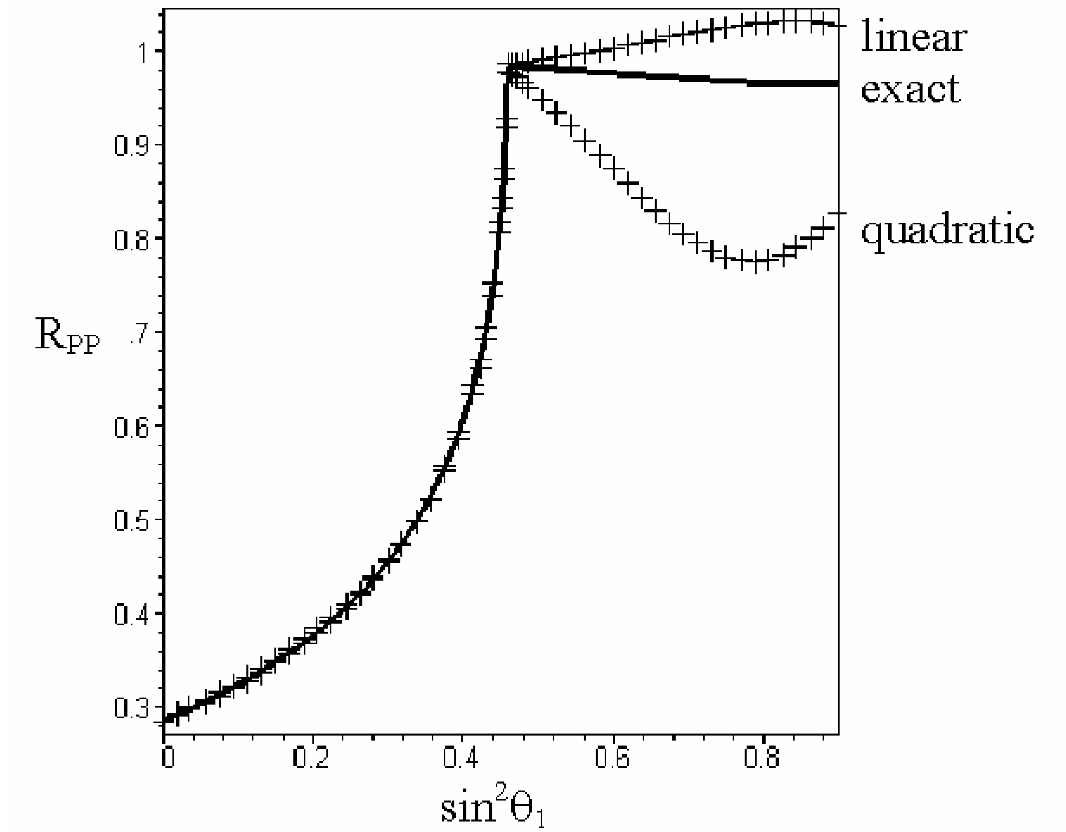


Figure 2: The value of $R_{PP}$ for various levels of approximation in $\Delta\beta/\beta$.

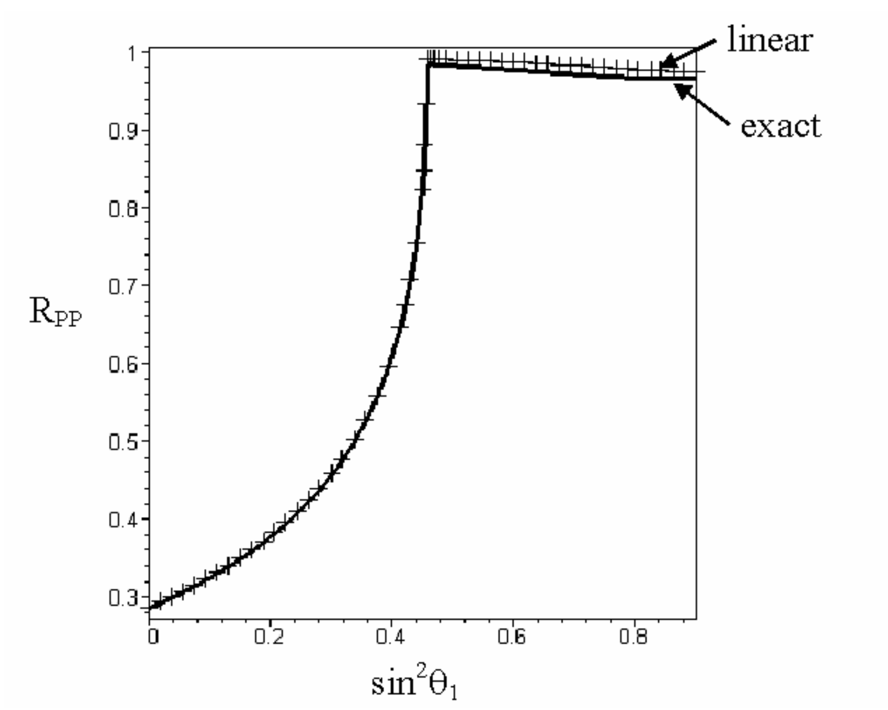


Figure 3: The value of $R_{PP}$ for a linear approximation in $\Delta\rho/\rho$.

The trends observed in Figures 1-3 have been verified for a wide range of earth parameters. It is reasonable that $R_{PP}$ is more sensitive to approximations in $\Delta\alpha/\alpha$ since, for typical geological interfaces, the first critical point is controlled by $\alpha_2/\alpha_1 = (2+\Delta\alpha/\alpha)/(2-\Delta\alpha/\alpha)$. It therefore seems reasonable to propose that an optimal approximation for $R_{PP}$, at least for sub-critical angles and $\alpha_2 > \beta_1$, consists of linearizing with respect to $\Delta\beta/\beta$ and $\Delta\rho/\rho$ only. We will refer to this as a pseudo-linear approximation, primarily because of the form in which we will present it in a later section. In Figure 4 we have compared this approximation against the exact $R_{PP}$ and against the Aki-Richards approximation.

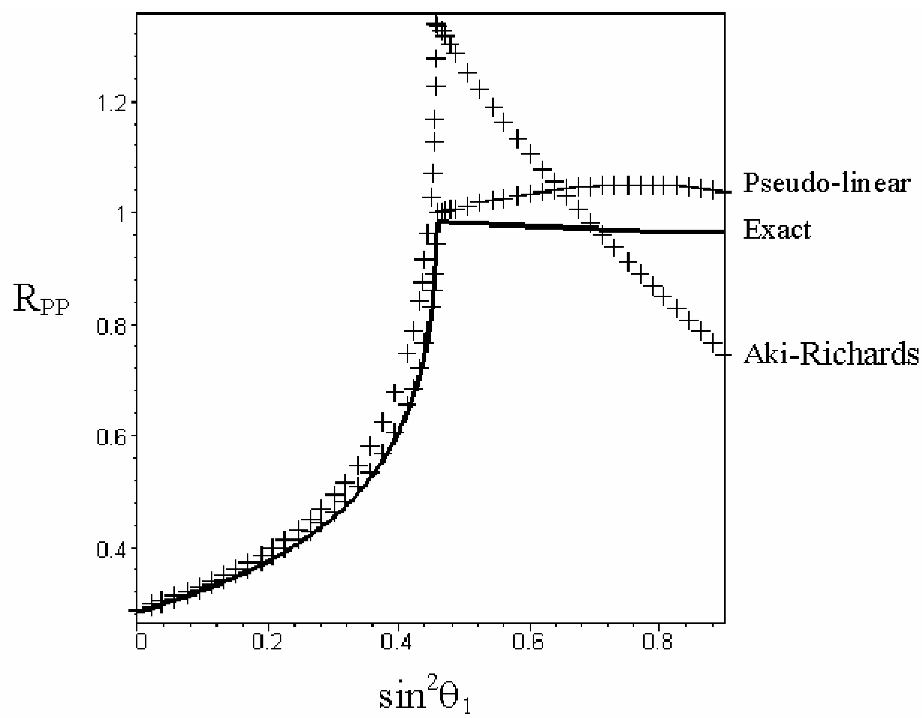


Figure 4: Comparison of $R_{PP}$ expressions. The pseudo-linear approximation, proposed above, is very similar to that of the linear approximation in Figure 2. The principal weakness of the Aki-Richards approximation appears to be in its estimation of the critical-point value.

We now turn to consideration of the converted wave reflectivity, $R_{PS}$. Plots analogous to Figures 1-3 tell a very similar story for the behavior of approximations with respect to $\Delta\alpha/\alpha$ and $\Delta\rho/\rho$. However, while a quadratic truncation with respect to $\Delta\beta/\beta$ is quite accurate, the linear truncation deviates slightly but visibly from the exact value. We propose that an optimal approximation for $R_{PS}$ is to expand linearly in $\Delta\rho/\rho$ and quadratically in $\Delta\beta/\beta$, i.e., retaining terms proportional to $(\Delta\beta/\beta)^2$ and $(\Delta\beta/\beta)(\Delta\rho/\rho)$. In principle we could also include the 3$^{rd}$ order term, $(\Delta\beta/\beta)^2(\Delta\rho/\rho)$, but we choose not to, as we have found to be less important than the other two, and it adds much more complexity. In Figure 5 we present both the proposed approximation and one without the quadratic terms.

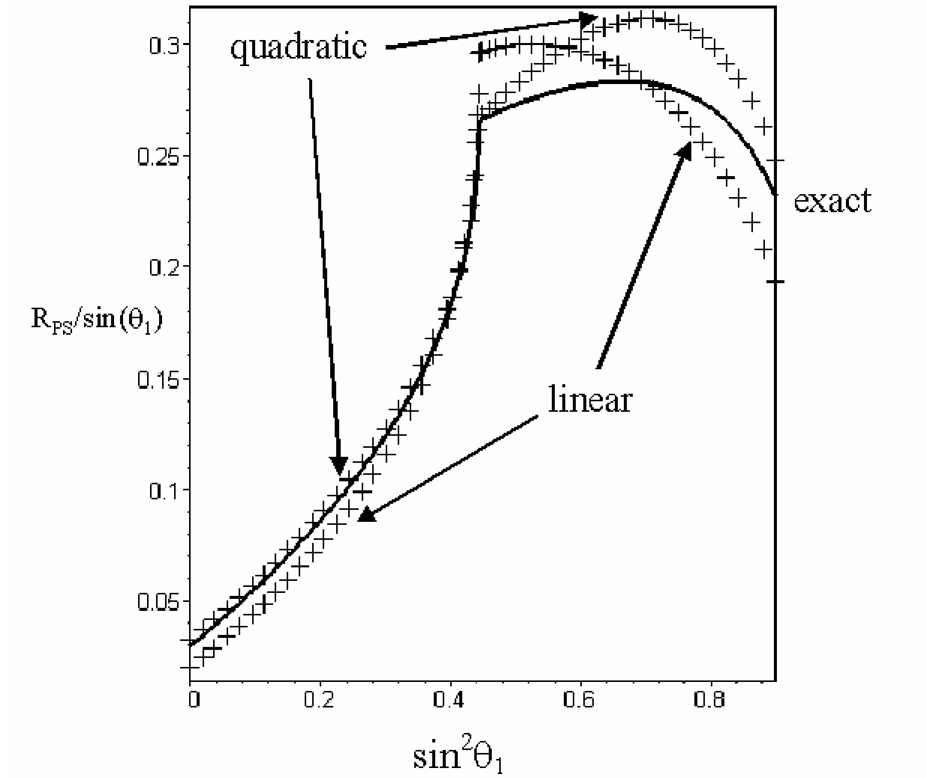


Figure 5: Approximations for $R_{PS}$ compared to the exact result. The "linear" line is linear in $\Delta\beta/\beta$ and $\Delta\rho/\rho$. The line marked "quadratic" has additional terms in $(\Delta\beta/\beta)^2$ and $(\Delta\beta/\beta)(\Delta\rho/\rho)$.

## RESULTS

The formal approximations of $R_{PP}$ or $R_{PS}$ with respect to different variables were straightforwardly obtained using a symbolic mathematics program (*Maple* 7, 2002) in order to generate the plots in Figures 1-5. However such formal results often extended over several pages and considerably more effort was required to distill this information into a compact and comprehensible form. Our goal in this process was to find expressions as close as possible in form to the enduring Aki-Richards approximation. To illustrate this we present, below, both the standard Aki-Richards reflectivities, $R_{PP}{}^{A\text{-}R}$ and $R_{PS}{}^{A\text{-}R}$, as well as the results that we were able to obtain for the proposed pseudo-linear approximations, $R_{PP}{}^{P\text{-}L}$ and $R_{PS}{}^{P\text{-}L}$. Note that $R_{PP}{}^{P\text{-}L}$ is somewhat simpler that $R_{PS}{}^{P\text{-}L}$ because of the latter's quadratic terms, but that both follow the Aki-Richards structure:

$$R_{PP}^{A-R} = \frac{1}{2\cos^2\theta}\frac{\Delta\alpha}{\alpha} - 2\left(\frac{\beta}{\alpha}\right)^2 \sin^2\theta \frac{\Delta\mu}{\mu} + \frac{1}{2}\frac{\Delta\rho}{\rho} \qquad (1)$$

$$R_{PP}^{P-L} = \left[\frac{4\cos\theta_1\cos\theta_2}{Q^2}\right]\left\{\frac{1}{2\cos\theta_1\cos\theta_2}\frac{\Delta\alpha}{\alpha} - 2\left(\frac{\beta}{\alpha}\right)^2 \sin\theta_1\sin\theta_2\frac{\Delta\mu}{\mu} + \frac{1}{2}\left[1-\left(\frac{\Delta\alpha}{2\alpha}\right)^2\right]\frac{\Delta\rho}{\rho}\right\} \qquad (2)$$

(Here arrows have been used to emphasize the analogous linear structure of the two equations.)

$$R_{PS}^{A-R} = -\frac{\sin\theta}{2\cos\bar{\varphi}}\left\{\frac{\Delta\rho}{\rho} + 2\left(\frac{\beta}{\alpha}\cos\theta\cos\bar{\varphi} - \left(\frac{\beta}{\alpha}\right)^2 \sin^2\theta\right)\frac{\Delta\mu}{\mu}\right\} \qquad (3)$$

$$\begin{aligned} R_{PS}^{P-L} = &-\frac{\sin\theta_1}{2\cos\varphi}\,\frac{2\cos\theta_1\left[1+\dfrac{\Delta\alpha}{2\alpha}\right]}{Q}\times \\ &\left\{\left[1+\frac{\cos^2\varphi}{2P}\frac{\Delta\beta}{\beta}\left(C_\rho + \frac{2(\beta/\alpha)^2\sin^2\theta_1}{[1-\Delta\alpha/(2\alpha)]^2}C_{\rho\mu}\right)\right]\frac{\Delta\rho}{\rho}\right. \\ &\left.+2\left(\frac{\beta}{\alpha}\frac{\cos\theta_2\cos\varphi}{[1+\Delta\alpha/(2\alpha)]} - \left(\frac{\beta}{\alpha}\right)^2\frac{\sin^2\theta_1}{[1-\Delta\alpha/(2\alpha)]^2}\left[1+\frac{\cos^2\varphi}{2P}\frac{\Delta\beta}{\beta}\left(C_\mu - C_{\rho\mu}\right)\right]\right)\frac{\Delta\mu}{\mu}\right\} \end{aligned} \qquad (4)$$

where

$$P = 1-(\beta/\alpha)^2\sin^2\theta_1/[1-\Delta\alpha/(2\alpha)]^2$$

$$Q = \left(1+\frac{\Delta\alpha}{2\alpha}\right)\cos\theta_1 + \left(1-\frac{\Delta\alpha}{2\alpha}\right)\cos\theta_2$$

$$S = -2\{\cos\theta_1\cos\varphi[1-\Delta\alpha/(2\alpha)] + (\beta/\alpha)\sin^2\theta_1\}$$

$$C_\rho = 1+\frac{4\dfrac{\beta}{\alpha}\cos\theta_2}{\left(1-\dfrac{\Delta\alpha}{2\alpha}\right)Q}\left[S - 4\frac{\beta}{\alpha}\sin^2\theta_1 - \frac{4\left(\dfrac{\beta}{\alpha}\right)^2 S\sin^2\theta_1}{\left(1-\dfrac{\Delta\alpha}{2\alpha}\right)^2 P}\right]$$

$$C_\mu = 1+\frac{\dfrac{\beta}{\alpha}\cos\theta_2}{P}\left[\frac{\cos\varphi}{\left(1+\dfrac{\Delta\alpha}{2\alpha}\right)} + \frac{8S}{\left(1-\dfrac{\Delta\alpha}{2\alpha}\right)Q}\right]$$

$$C_{\rho\mu} = \frac{8S\sin^2\theta_1\cos\theta_2}{[1-\Delta\alpha/(2\alpha)]^3 PQ}\left(\frac{\beta}{\alpha}\right)^3$$

and

$$\frac{\Delta\mu}{\mu} = \frac{\Delta(\rho\beta^2)}{\rho\beta^2} = 2\frac{\Delta\beta}{\beta} + \frac{\Delta\rho}{\rho} + O\left[\left(\frac{\Delta\beta}{\beta}\right)^n\left(\frac{\Delta\rho}{\rho}\right)^{3-n}\right]_{n=1,2,3} \approx 2\frac{\Delta\beta}{\beta} + \frac{\Delta\rho}{\rho}$$

$\theta_1$ is the angle of P-wave incidence and reflection in the first layer. $\theta_2$ is the P-wave transmission angle in the second layer. The Aki-Richards expressions employ the average $\theta = (\theta_1 + \theta_2) / 2$. Similarly, the angle $\overline{\varphi}$ is the average of the angles of S-wave reflection, $\varphi_1$, and transmission, $\varphi_2$. In the pseudo-linear expressions the angle $\varphi$ is defined slightly differently, namely, by the relation $\sin\varphi = (\sin\varphi_1 + \sin\varphi_2) / 2$. The quantities $\sin\varphi$ and $\sin\overline{\varphi}$ differ by $O[(\sin\varphi_1)^3(\Delta\beta/\beta)^2]$, so the difference is small, but still of the same order as the pseudo-linear approximation for $R_{PS}$.

It is clear how the pseudo-linear form of the above equations can be used in AVO inversions. The pseudo-linear expressions are linear in the parameters $\Delta\alpha/\alpha$, $\Delta\rho/\rho$ and $\Delta\mu/\mu$, as indicated by the arrows, and these play a role analogous to their counterparts in the Aki-Richards approximation. If all other occurrences of $\Delta\alpha/\alpha$, $\Delta\beta/\beta$, and $\Delta\rho/\rho$ are initially set to zero, then all of the quantities in square bracketed will simplify to unity, and $\Delta\alpha/\alpha$, $\Delta\rho/\rho$ and $\Delta\mu/\mu\Delta$ (or $\alpha/\alpha$, $\Delta\beta/\beta$, and $\Delta\rho/\rho$) may be obtained through a simple linear inversion. These values may then be substituted in for the values originally set to zero, and thus by iteration we may easily obtain essentially the same results as with the full Zoeppritz equation. An alternate use of these equations is discussed in the next section.

## EXAMPLE

Ramos and Castagna (2001) recently put forward some expressions for use with converted wave AVO. In particular they describe a new "high-contrast" approximation for $R_{PS}$ that they compare with a "low-contrast" approximation obtained from the Aki-Richards approximation. Both are of the form $R_{PS} = A \sin\theta + B \sin^3\theta$, analogous to Shuey's two-term equation, which can be written as

$$R_{PP} = \frac{1}{2}\left(\frac{\Delta\alpha}{\alpha} + \frac{\Delta\rho}{\rho}\right) + \left(\frac{1}{2}\frac{\Delta\alpha}{\alpha} - 2\left(\frac{\beta}{\alpha}\right)^2 \frac{\Delta\mu}{\mu}\right)\sin^2\theta$$

Their low-contrast approximation has the benefit of being simple enough that information on rock properties can be readily extracted, once $A$ and $B$ have been fit to seismic data. Unfortunately the results are not always accurate. Their high contrast approximation is much more accurate, but there is no obvious way to extract properties from it.

The equations we have derived above are generally of the same accuracy of the high-contrast approximation of Ramos and Castagna. To illustrate this, we plot $R_{PS}^{P\text{-}L}$ and $R_{PS}^{P\text{-}L}$ truncated at $\sin^3\theta$ along with the exact value (Figure 6). As expected, the $R_{PS}^{P\text{-}L}$ approximation follows the exact value closely. In addition, the cubic truncation is accurate up to at least 20°. Comparison with Figure3b of Ramos and Castagna (2001) shows that the cubic truncation is of the same accuracy as their high-contrast approximation.

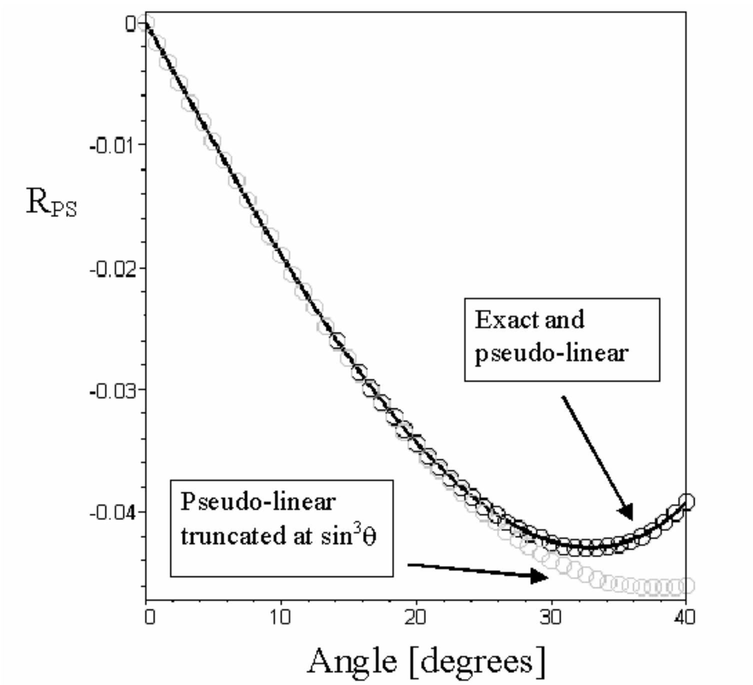


Figure 6: $R_{PS}^{P\text{-}L}$ and $R_{PS}^{P\text{-}L}$ truncated at $\sin^3\theta$ compared with the exact value. Compare to Figure 3b of Ramos and Castagna (2001).

In addition to its accuracy though, the pseudo-linear approximation is of a form such that rock properties can be extracted nearly as readily as with Ramos and Castagna's low-contrast approximation. We have derived expressions for the two-term $\sin\theta$ expansions of both $R_{PP}^{P\text{-}L}$ and $R_{PS}^{P\text{-}L}$. For again write Shuey's equation, and then the pseudo-linear analogue for comparison:

$$R_{PP} \approx \frac{1}{2}\left(\frac{\Delta\alpha}{\alpha}+\frac{\Delta\rho}{\rho}\right)+\left(\frac{1}{2}\frac{\Delta\alpha}{\alpha}-2\left(\frac{\beta}{\alpha}\right)^2\frac{\Delta\mu}{\mu}\right)\sin^2\theta \qquad (5)$$

$$R_{PP}^{P-L(Shuey)} = \frac{1}{2}\left(\frac{\Delta\alpha}{\alpha}+\frac{\Delta\rho}{\rho}\left[1-\left(\frac{\Delta\alpha}{\alpha}\right)^2\right]\right)+\left(\frac{1}{2}\frac{\Delta\alpha}{\alpha}\left[1+2\frac{\Delta\alpha}{\alpha}\frac{\Delta\rho}{\rho}\right]-2\left(\frac{\beta}{\alpha}\right)^2\frac{\Delta\mu}{\mu}\right)\sin\theta_1\sin\theta_2. \qquad (6)$$

We have again used arrows to emphasize the similarity of the two expressions. Indeed, if all the contrasts in square brackets are set to zero (highlighted in yellow), then the pseudo-linear expression becomes equal to the Shuey equation.

The converted-wave expression of course is more complicated, including as it does, higher-order terms. We first give the expression derived from the Aki-Richards approximation, as given, for instance, by Ramos & Castagna (2001), and then the pseudo-linear analogue:

$$\frac{R_{PS}}{\sin\theta} \approx -\left(\frac{1}{2}\frac{\Delta\rho}{\rho} + \frac{\beta}{\alpha}\frac{\Delta\mu}{\mu}\right) + \left(\frac{1}{2}\frac{\beta}{\alpha}\frac{\Delta\mu}{\mu} + \frac{\beta^2}{\alpha^2}\frac{\Delta\mu}{\mu} - \frac{1}{4}\frac{\beta^2}{\alpha^2}\frac{\Delta\rho}{\rho}\right)\sin^2\theta \tag{7}$$

$$\begin{aligned}\frac{R_{PS}^{P-L(Shuey)}}{\sin\theta_1} = &-\left(\frac{1}{2}\frac{\Delta\rho}{\rho}\left[1+\frac{\Delta\alpha}{2\alpha}\right]\left[1+\frac{\Delta\beta}{2\beta}\left(1-4\frac{\beta}{\alpha}\right)\right] + \frac{\beta}{\alpha}\frac{\Delta\mu}{\mu}\right) + \\ &\left\{\left[1+\frac{(\Delta\alpha/\alpha)^2+(\beta/\alpha)^2(\Delta\beta/\beta)}{1-(\Delta\alpha/2\alpha)^2}\right]\frac{1}{2}\frac{\beta}{\alpha}\frac{\Delta\mu}{\mu} + \right. \\ &\left[\frac{1+(1-8\beta/\alpha)\Delta\beta/2\beta}{1-\Delta\alpha/2\alpha}\right]\frac{\beta^2}{\alpha^2}\frac{\Delta\mu}{\mu} + \\ &\left[\frac{1+(1+32\beta/\alpha)\Delta\beta/2\beta}{1-\Delta\alpha/2\alpha} + \frac{1+\Delta\beta/2\beta}{(\beta/\alpha)^2}\frac{\Delta\alpha}{\alpha} - \right. \\ &\left.\left. 12\frac{\Delta\beta}{\beta} + 2\frac{1-\Delta\alpha/\alpha+(5/4)(\Delta\alpha/\alpha)^2}{(\beta/\alpha)(1-\Delta\alpha/2\alpha)}\frac{\Delta\beta}{\beta}\right]\left(-\frac{1}{4}\frac{\beta^2}{\alpha^2}\frac{\Delta\rho}{\rho}\right)\right\}\sin\theta_1\sin\theta_2\end{aligned} \tag{8}$$

Once again, the linear structure of the pseudo-linear result is evident.

Carcuz (2001) has suggested an AVO method given such two-term expressions for $R_{PP}$ and $R_{PS}$. Given two slopes and two intercepts, obtained from both compressional and converted wave data, then one has four equations in four variables – the three relative contrasts and the ratio $\beta/\alpha$. Thus one could in principle solve even for the $V_P$-$V_S$ ratio, which is normally required as input. Even with simple relations such as Equations (5) and (7), these are non-linear relations, but Carcuz indicates that one can readily obtain a unique and stable solution. A solution of Equations (5) and (7) could logically be used as the initial guess for a solution of Equations (6) and (8), whose error should always be far less than that of the data itself.

## CONCLUSIONS

Based on the observation that, for $\alpha_2 > \beta_1$, $R_{PP}$ and $R_{PS}$ are much more sensitive to P-wave velocity contrasts than to S-wave or density contrasts, we have developed new approximations for these two key Zoeppritz coefficients. These approximations are optimal in the sense that they are nearly as accurate as the exact expressions for parameters typical of geological interfaces, but they are simple in the sense of having a linear structure, even though they are not truly linear. This means that they may be employed in a fairly simple fashion to obtain quite accurate results, if the assumptions of the original Zoeppritz model are valid. They will also serve as a useful reference in working to relax the basic assumptions of the fundamental Zoeppritz equations.

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