Rays in transversely isotropic media

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ABSTRACT

The theory of characteristics related to the solution of partial differential equations of the hyperbolic type is applied to the coupled $qP-qS_V$ wave propagation problem in a transversely isotropic (T.I.) medium. The characteristics or rays are the paths along which energy is transported from one point to another in any media type. The determination of ray paths in such a media type is often a preliminary step in addressing more complex problems in anisotropic wave propagation such as amplitude computations and the related polarization vectors, quantities that are significantly more difficult to obtain in any type of anisotropic medium when compared with the isotropic case.

Equations for tracing the progress of a ray through a homogeneous T.I. medium, which has applications in several areas of seismology, will be presented. The problem of reflection and refraction at an interface separating two T.I. media has been treated in an earlier report. Used with the results presented here, a method will be developed to explore two-point ray propagation in media where the rotationally invariant axis of the T.I. wave front is not aligned with the interfaces separating two media and some simple results shown.

INTRODUCTION

The partial differential equations arising in seismological related problems are generally linear second order hyperbolic. It is difficult, if not impossible, to consult a text on partial differential equations of the hyperbolic type, or systems thereof, and not be presented with at least an introduction to the method of characteristics in the solution of this equation type. It is a powerful method, which has been well developed from a theoretical point of view, and there are numerous classical practical applications. A simplistic overview of the basis of this method is that a Hamiltonian or eikonal equation that is a nonlinear partial differential equation, homogeneous in powers of order 2 in terms of the partial derivatives of the spatially dependent phase function may, given certain minimal criteria, be solved for the paths (rays) along which the energy propagates. This, together with an amplitude term (the solution of an energy transport equation), is usually assumed to comprise a solution of the original hyperbolic problem.

As the theoretical aspects of the most general form of this problem have been treated by a number of authors in diverse publications, only a brief summary of their results will be given here. The degenerate case of ellipsoidal transverse isotropy will be discussed as it contains most of the inherent concepts of wave propagation in a T.I. medium, and many solution quantities may be obtained in relatively simple closed form. As two-point ray tracing procedures are numerical in nature, a trial solution may be obtained using ellipsoidal theory and employed in the iterative process of the more complex general case. Both unconverted and converted ray propagation will be considered.

THEORETICAL OVERVIEW

In a generally inhomogeneous anisotropic medium in a Cartesian coordinate system, the equations of motion are given by

$$\frac{\partial}{\partial x_i}\left(c_{ijk\ell} \frac{\partial U_k}{\partial x_\ell} \right) = \rho \frac{\partial^2 U_j}{\partial t^2}; \tag{1}$$

U_j , $j = 1,3$ are the components of the of the particle displacement vector, $\mathbf{U}$; ρ is the density; c_{ijkl}, the anisotropic parameters; and t is time. The density and anisotropic parameters are generally spatially $\left(x_i\right)$ dependent. (Cerveny and Psencik, 1972; Cerveny, 1972, and Vlaar, 1968).

The geometrical optics, or Asymptotic Ray Theory (A.R.T.) solution of the above equation, has the general form

$$U_k\left(x_i, t\right) = \sum_{n=0}^{\infty} A_k^{(n)}\left(x_i\right) f_n\left[t - \tau\left(x_i\right)\right] \tag{2}$$

for some amplitude terms, $A_k^{(n)}\left(x_i\right)$, which are dependent only on the spatial coordinates. In the above, t refers to time. The aspect of the problem involving the energy transport along the rays will not be discussed here as it warrants a separate, fairly mathematically intensive treatment. The second expression in the n^{th} term in the asymptotic series in equation (2), $f_n\left(\zeta\right)$, is a generalized function containing a phase-dependence, $\tau\left(x_i\right)$. The wavefront moving through the anisotropic medium is assumed to be described by the equation

$$t = \tau\left(x_i\right). \tag{3}$$

The function $f_n\left(\zeta\right)$ is chosen with the property that

$$\frac{df_n\left(\zeta\right)}{d\zeta} = f_{n-1}\left(\zeta\right) \quad , \quad f_n\left(\zeta\right) \equiv 0.0 \, , \left\{\forall \; n : n < 0\right\}. \tag{4}$$

A function type that is commonly used in seismological applications of the elastodynamic problem, including the aforementioned phase function, $\tau\left(x_i\right)$, is the time harmonic expression,

$$f_n\left[t - \tau\left(x_i\right)\right] = \frac{\exp\left[i\omega\left(t - \tau\left(x_i\right)\right)\right]}{\left(i\omega\right)^n}. \tag{5}$$

Substitution of equation (2) into equation (1) and considering only those equations which are related to the leading or zero order term in the asymptotic series, yields a third order polynomial whose solution provides three possible eikonal equations, homogeneous in powers of two in the slowness vector components, $p_i \quad (i=1,2,3), \left(p_i = \partial\tau\left(x_j\right)/\partial x_i\right)$. After the above substitution, the zero order equation in terms of $\mathbf{A}^{(0)}\left(x_i\right)$ is obtained as

$$\Gamma_{jk} A_k^{(0)} - A_j^{(0)} = 0\,, \tag{6}$$

where

$$\Gamma_{jk} = p_i p_\ell a_{ijk\ell} \quad , \quad a_{ijk\ell} = c_{ijk\ell}/\rho \tag{7}$$

and the $a_{ijk\ell}$ have the dimensions of velocity squared. Explicit expressions for the Γ_{jk} in the case of a transversely isotropic medium may be found in Cerveny and Psencik (1972) and for completeness are given in the Appendix. The existence of a nontrivial solution of equation (6) is contingent on $\mathbf{A}^{(0)}\left(x_i\right)$ not being identically equal to zero, thus requiring that the following condition be satisfied:

$$\det\left(\Gamma_{jk} - G\delta_{jk}\right) = 0\,. \tag{8}$$

The normalized eigenvectors obtained from introducing the solutions of equation (8) into equation (6) are the most often referred to in the geophysical literature as polarization vectors associated with one of the three distinct propagation modes. These polarization vectors should more accurately be prefixed with “zero order” as higher order terms in the asymptotic series introduce additional components or component additions (Cerveny and Ravindra, 1970 and Daley and Hron, 1977).

The three values that the eigenvalue or characteristic value, G, may have yield the eikonal equations corresponding to the quasi-compressional, qP, propagation mode and two quasi-shear modes, qS_V and qS_H. As $\mathbf{\Gamma}$ is positive definite, the eigenvalues are real and positive and in general, unique. This is referred to in numerical analysis as the irreducible case. In special cases a variable transformation will put the cubic equation in a form that may be solved to obtain three analytic expressions for the eigenvalues. Schoenberg and Helbig (1996) present the solutions for an orthorhombic media; however, for a generally anisotropic medium, numerical methods must be used:

$$G_\ell\left(x_i, p_i\right) = 1, \qquad \left(\ell = 1,2,3\right) \qquad \left(i = 1,2,3\right), \tag{9}$$

where p_i are were previously defined as

$$p_i = \frac{\partial\tau\left(x_j\right)}{\partial x_i}, \tag{10}$$

and are the components of the slowness vector,

$$\mathbf{p} = \left(p_1, p_2, p_3\right). \tag{11}$$

Each of the three modes of propagation modes has, in general, a unique eikonal equation and hence slowness surface. An exception to this is the isotropic case where the eikonals of the two shear modes are identical or degenerate.

The quantity $G_\ell\left(x_i, p_i\right)$ is a homogeneous function of order 2 in p_i. From Euler's theorem on homogeneous functions, the relations,

$$\sum_{i=1}^{3} p_i \frac{\partial G_\ell}{\partial p_i} = 2G_\ell, \tag{12}$$

are obtained. Additionally, the eikonal or, equivalently, the characteristic equations, sometimes also referred to as Hamiltonians, are nonlinear partial differential equations whose solutions define the characteristics or rays of the corresponding eikonal. The equations of the characteristics of these partial differential equations may be written as (Courant and Hilbert, 1962):

$$\frac{dx_i}{dt} = \frac{1}{2}\frac{\partial G_\ell}{\partial p_i}, \tag{13}$$

$$\frac{dp_i}{dt} = -\frac{1}{2}\frac{\partial G_j}{\partial x_i}. \tag{14}$$

If the initial conditions, $\mathbf{x} = x_i^0$ and $\mathbf{p} = p_i^0$ are known at some time, t_0, the above set of coupled ordinary differential equations may be solved for $\mathbf{x}\left(t\right)$ and $\mathbf{p}\left(t\right)$. This may be done numerically if the $c_{ijk\ell}$ are arbitrary functions of position, using methods such as a Runge-Kutta algorithm. An orthogonal coordinate system, comprised of the tangent to a point on the ray and the two vectors defining the tangent plane to the energy propagation surface at this point, are referred to as ray-centred coordinates and are useful in many ray tracing and amplitude determination applications.(Gassmann, 1964, Psencik, 1979 and Cerveny and Hron, 1980).

The order 3, rank 4 tensor, c_{ijkl}, with 21 generally independent members is usually replaced in seismological problems by the 6×6 symmetric matrix C_{mn} (Musgrave, 1970), whose elements have the dimensions of velocity squared, using the standard scheme:

$$c_{ij\,|\,k\ell} \quad \rightarrow \quad C_{mn}. \tag{15}$$

This simplification is made subject to the following replacements:

$$\begin{array}{llllll} [\; ij \rightarrow m & : & k\ell \rightarrow n \;] & & & \\ 11 \rightarrow 1 & & 22 \rightarrow 2 & & 33 \rightarrow 3 & \\ 23 \text{ or } 32 \rightarrow 4 & & 13 \text{ or } 31 \rightarrow 5 & & 12 \text{ or } 21 \rightarrow 6 & . \end{array} \tag{16}$$

TRANSVERSE ISOTROPY: ELLIPSOIDAL

As a transversely isotropic medium is rotationally invariant, chosen here to be with respect to the vertical, z, axis, the slowness vector may be reduced to a radial and vertical component so that, $\mathbf{p} = (p_1, p_3)$. It will be convenient here to adopt the notation changes, $(x_1, x_3) \to (x, z)$ and $(p_1, p_3) \to (p, q)$. The ellipsoidal anisotropic eikonal for qP wave propagation may be written in the form

$$G_{qP}(x, z, p, q) = A_{11} p^2 + A_{33} q^2 = 1 \tag{17}$$

The coefficients, A_{ii}, which have the dimensions of velocity squared, are related to the C_{mn} as $A_{mn} = C_{mn}/\rho$. Their relationship to the $a_{ijkl} = c_{ijkl}/\rho$ was given in the previous section.

The anisotropic parameters, A_{11} and A_{33}, may be arbitrary functions of position; $A_{ii} = A_{ii}(x, z), (i = 1, 3)$. Assuming that the initial values (x_0, z_0) and (p_0, q_0) at t_0 are known, a solution may be determined for equations (8) and (9) which, for this case, may be written as:

$$\frac{dx}{dt} = p\ A_{11} \tag{18}$$

$$\frac{dz}{dt} = q\ A_{33} \tag{19}$$

$$\frac{dp}{dt} = \frac{1}{2}\left[p^2 \frac{\partial A_{11}}{\partial x} + q^2 \frac{\partial A_{33}}{\partial x} \right] \tag{20}$$

$$\frac{dq}{dt} = \frac{1}{2}\left[p^2 \frac{\partial A_{11}}{\partial z} + q^2 \frac{\partial A_{33}}{\partial z} \right]. \tag{21}$$

The above system of ordinary differential equations may be solved numerically to yield the characteristic paths or rays along which energy is transported provided they also satisfy the eikonal equation from which they were derived. Clearly, equations (18) and (19) satisfy Gauss's theorem. The quantities, dx/dt and dz/dt, are the Cartesian components of the ray velocity vector, $\mathbf{v_r}(\alpha)$, α being the angle the ray angle relative to some coordinate, usually the vertical, while dp/dt and dq/dt are the time derivatives of the components of the slowness vector and implicitly contain information related to the anisotropic equivalent of Snell's Law for media with lateral and/or vertical variations of the anisotropic parameters, A_{ii}. The system of equations for the qS_V ellipsoidal case is similar to the above with the exception that A_{11} and A_{33} are both replaced by A_{55}, implying a spherical energy propagation surface.

In addition to the above set of equations, it has been stated that the eikonal equation must be satisfied along the total length of the rays. As a consequence, one of the above four differential equations may be eliminated and replaced by the eikonal. A more conservative approach is to numerically solve the system of ordinary differential equations and use the eikonal as a check on the accuracy of the method used.

Consider now the above problem simplified further, in that the coefficients A_{11} and A_{33} are independent of the spatial coordinates: i.e., they are constant in the medium being considered. For this case, $dp/dt = dq/dt = 0$ along the whole ray resulting in the rays being forced to follow straight line trajectories.

The magnitude of the ray velocity, $v_r(\alpha)$, is obtained from equations (18) and (19) and is given by:

$$v_r(\alpha) = \left\{ \left[\frac{dx}{dt} \right]^2 + \left[\frac{dz}{dt} \right]^2 \right\}^{1/2} = \frac{ds}{dt} \quad . \tag{22}$$

The quantity s is the length of the ray segment and the angle, α, which the ray makes with the vertical (z) axis, is defined in terms of the associated slowness or phase (wavefront normal) angle, θ, as

$$\tan\alpha = \frac{dx/dt}{dz/dt} = \frac{A_{11}}{A_{33}} \tan\theta . \tag{23}$$

The horizontal and vertical components of slowness, (p,q) may be expressed in terms of the phase angle, θ, and the phase angle dependent phase (wavefront normal) velocity as

$$p = \frac{\sin\theta}{V_N(\theta)} \quad , \quad q = \frac{\cos\theta}{V_N(\theta)} \tag{24}$$

where the normal velocity, $V_N(\theta)$, is obtained from the eikonal equation (9) using the above two relationships for the qP mode propagation, as

$$V_N(\theta) = \left[A_{11}\sin^2\theta + A_{33}\cos^2\theta \right]^{1/2} \tag{25}$$

and for the qS_V mode

$$V_N(\theta) = \sqrt{A_{55}} \tag{26}$$

Utilizing (22) and (23), the following expression for the qP ray velocity may be obtained

$$\frac{1}{v_r^2(\alpha)} = \frac{\sin^2\alpha}{A_{11}} + \frac{\cos^2\alpha}{A_{33}} \tag{27}$$

indicating that the ray surface, the surface which transports the energy through the medium, is an ellipsoid of revolution about the vertical axis. The equations defining the magnitude of the ray velocity and the ray angle allow for the full specification of the ray velocity vector, $\mathbf{v_r}$. In the qS_V case, the energy propagation surface is spherical; but it should be noted that the polarization vector for this propagation mode is not in general perpendicular to the ray as in a comparable wavefront type in an isotropic medium.

At an interface between media in welded contact, the anisotropic equivalent of Snell's Law states that the horizontal component of the slowness vector, p, is the same for all wave types, both reflected and refracted, resulting from the incidence of a given wavefront type. The vertical component of slowness, q, may then be obtained from the eikonal equation.

The problem with the above is that the change in ray angle, α, and the corresponding value of the ray velocity, $v_r(\alpha)$, for the reflected or transmitted wave is usually required. For the case of ellipdoidal anisotropy, Snell's Law at a boundary may be written in terms of ray quantities, rather than wavefront normal (phase) or slowness, as it may be shown using (22) that the following holds for an ellipsoidally anisotropic medium:

$$p = \frac{v_r(\alpha)\sin\alpha}{A_{11}}, \tag{28.a}$$

for qP propagation; and for the qS_V mode,

$$p = \frac{\sin\alpha}{\sqrt{A_{55}}}. \tag{28.b}$$

This is the only type of anisotropy where a relationship of this type may be derived in such a simple form.

It is also not difficult to show that the normal at some point, $\mathbf{p} = (p,q)$, on the slowness surface defines the corresponding ray direction. The inverse of this is also valid: the normal to the ray surface at a point corresponding to a specific ray defines the slowness vector angle.

TRANSVERSE ISOTROPY: COUPLED P-S_V MOTION

The eikonal equations for qP and qS_V wave propagation in a general transversely isotropic medium may be obtained by a slight modification of those presented by Gassmann (1964) as:

$$G_{qP}(p,q,x,z) = A_{11}p^2 + A_{33}q^2 + \frac{A_\alpha}{2}\left\{(1+4\varepsilon_D)^{1/2} - 1\right\} = 1 \tag{29.a}$$

$$G_{qS_V}(p,q,x,z) = A_{55}(p^2+q^2) - \frac{A_\alpha}{2}\left\{(1+4\varepsilon_D)^{1/2} - 1\right\} = 1 \tag{29.b}$$

Quantities in the above, which require definition, are:

$$A_{\alpha} = \left(A_{11} - A_{55}\right) p^2 + \left(A_{33} - A_{55}\right) q^2 \tag{30}$$

$$\varepsilon_D = \frac{A_D p^2 q^2}{A_{\alpha}^2} \tag{31}$$

$$A_D = \left(A_{13} + A_{55}\right)^2 - \left(A_{11} - A_{55}\right)\left(A_{33} - A_{55}\right). \tag{32}$$

If equations (8) and (9) are applied to the eikonal equations defined above, the following results are obtained for the horizontal and vertical components of the ray velocity where "$+$" refers to qP and "$-$" to qS_V ray propagation:

$$\frac{dx}{dt} = \frac{p}{2}\left\{\left(A_{11} + A_{55}\right) \pm \left(\frac{A_{\alpha}^2}{4} + A_D p^2 q^2\right)^{-1/2} \left[\frac{A_{\alpha}\left(A_{11} - A_{55}\right)}{2} + A_D q^2\right]\right\} \tag{33}$$

$$\frac{dz}{dt} = \frac{q}{2}\left\{\left(A_{33} + A_{55}\right) \pm \left(\frac{A_{\alpha}^2}{4} + A_D p^2 q^2\right)^{-1/2} \left[\frac{A_{\alpha}\left(A_{33} - A_{55}\right)}{2} + A_D p^2\right]\right\} \quad . \tag{34}$$

The expressions for dp/dt and dq/dt have been omitted as it will be assumed that the anisotropic parameters, A_{ij}, are spatially independent so that both dp/dt and dq/dt are equal to zero.

As in the ellipsoidal case, the ray velocity is given by:

$$v_r\left(\alpha\right) = \left\{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dz}{dt}\right]^2\right\}^{1/2}; \tag{35}$$

or equivalently,

$$\mathbf{v_r} = \left(\frac{dx}{dt}, \frac{dz}{dt}\right), \tag{36}$$

and the ray angle with respect to the vertical spatial axis is obtained from equations (31) and (32) as:

$$\tan\alpha = \frac{dx/dt}{dz/dt} = \frac{dx}{dz}. \tag{37}$$

To summarize to this point, given the anisotropic parameters, A_{ij}, the eikonal equation and spatial location of the origin at some initial time, $t = t_0$, a ray, either qP or qS_V may be traced in the medium of interest. Further, the ray path at any time may be specified together with its velocity and the angle it makes with the vertical axis.

If the ray encounters an interface where it may either be reflected or transmitted, the anisotropic equivalent of Snell's Law must be introduced into the computation. This topic has been dealt with in an earlier report (Daley, 2001) and the reader is referred there for a more comprehensive treatment. Formulae for the generalized form of Snell's Law in transversely isotropic media, where the anisotropic axes need not be aligned with the model axes, are given there. For continuity, a brief discussion of specific topics from the above paper will be included in a subsequent section.

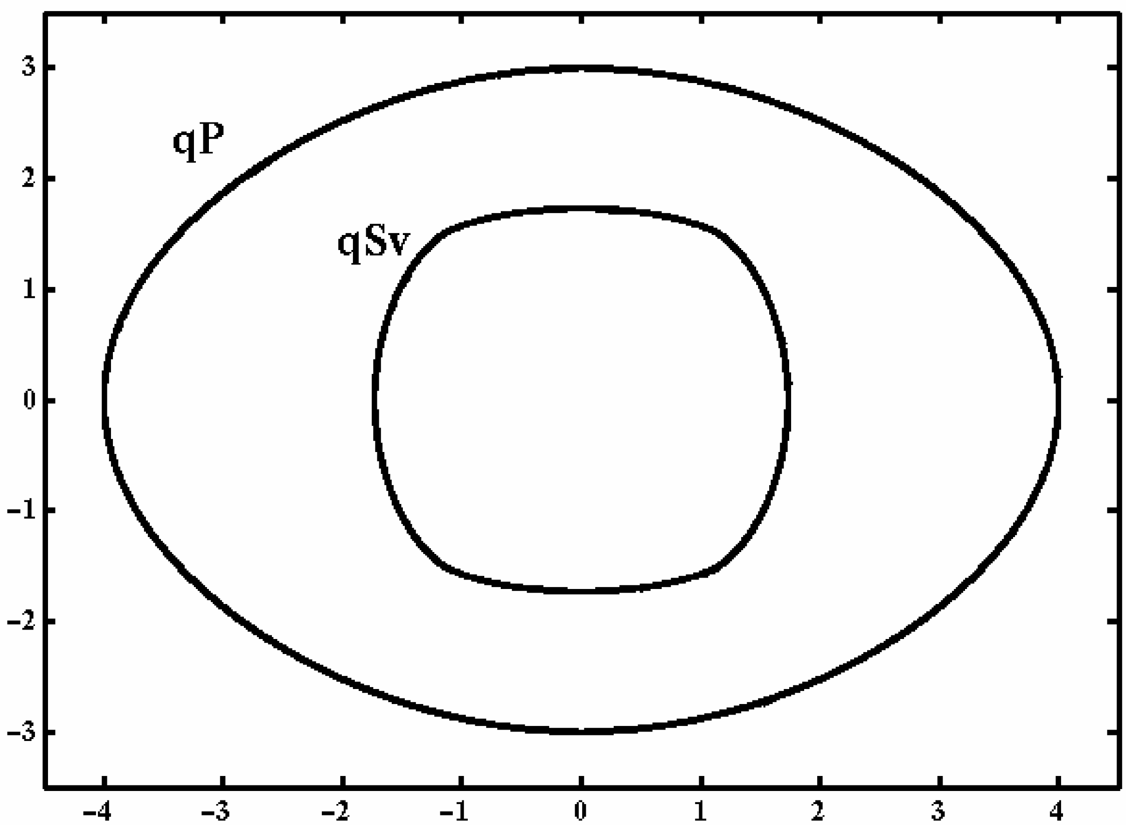

FIG. 1: The unrotated qP and qS_V ray surfaces for the layer model being considered. The axis coordinates are an indication of velocity.

RAY TRACING IN A ROTATED T.I. MEDIUM

In a plane parallel-layered model, composed of T.I. media in which the axis of anisotropy in each layer is not required to be aligned with the model axis, two-point ray tracing generally involves solving a number of nonlinear equations for Snell's Law embedded in a nonlinear equation for the two point ray tracing equation. For a plane N-layered medium, this equation may be written in algorithm form as:

$$\begin{aligned} r - \sum_{j=1}^{N} n_j^{(qP\downarrow)} h_j \tan\alpha_j^{(qP\downarrow)} - \sum_{j=1}^{N} n_j^{(qP\uparrow)} h_j \tan\alpha_j^{(qP\uparrow)} - \\ \sum_{j=1}^{N} n_j^{(qS_V\downarrow)} h_j \tan\alpha_j^{(qS_V\downarrow)} - \sum_{j=1}^{N} n_j^{(qS_V\uparrow)} h_j \tan\alpha_j^{(qS_V\uparrow)} = 0 \end{aligned} . \tag{38}$$

In the above, the following quantities require definition:

$r-$ horizontal distance between source and receiver;

h_j- thickness of the j^{th} layer;

$n_j^{(qP)}-$ number of qP ray segments travelling downwards $(\downarrow)$ or upwards $(\uparrow)$ in the j^{th} layer;

$n_j^{(qS_V)}-$ number of qS_V ray segments travelling downwards $(\downarrow)$ or upwards $(\uparrow)$ in the j^{th} layer;

$\alpha_j^{(qP)}-$ the acute angle a downward $(\downarrow)$ or upward $(\uparrow)$ propagating qP ray segment in the j^{th} layer makes with the vertical model axis;

$\alpha_j^{qS_V}-$ the acute angle a downward $(\downarrow)$ or upward $(\uparrow)$ propagating qS_V ray segment in the j^{th} layer makes with the vertical model axis.

Equation (38) simplifies to the following two equations in the case of one layer over a halfspace for incident and reflected qP ray segments, and an incident qP ray and a reflected qS_V ray.

$$r-h_1\tan\alpha_1^{(qP\downarrow)}-h_1\tan\alpha_1^{(qP\uparrow)}=0. \tag{39}$$

$$r-h_1\tan\alpha_1^{(qP\downarrow)}-h_1\tan\beta_1^{(qS_V\uparrow)}=0 \tag{40}$$

Not much is gained by considering a complex media composed of many layers and ray segments as all of the theory presented is used in these simple cases, and the results may be displayed in a manner that may be more indicative of what the effects are of rotating the axis of anisotropy with respect to the model axis on the reflected $qP-qP$ and $qP-qS_V$ arrivals.

Both of the ray angles that are required to be determined in equations (39) and (40), $\alpha^{(qP)}$ and $\alpha^{(qS_V)}$ are functions of the incident phase angle $\theta^{(qP)}$ or equivalently a function of the horizontal slowness, p. The solution of the nonlinear two-point ray-tracing equation may be formulated in either of these variables. For a many-layered plane interface model, the parameterization is usually done in terms of $\theta^{(qP)}$ or p in the layer where the fastest propagating ray segment exists. This layer may change with angle so that a certain amount of extra computer code must be written to accommodate this.

In a medium where the axes of anisotropy in each plane layer are aligned with the interfaces, the $(\downarrow)$ and $(\uparrow)$ quantities are equal and hence additive for a given mode propagation, either qS_V or qP.

For a structure with curved or dipping interfaces, the above procedure must be modified and a ray code specified, with each ray segment in the ray code being treated sequentially. When considering curved-layer models the possibility of multiple travel time branches for a given ray must also be considered.

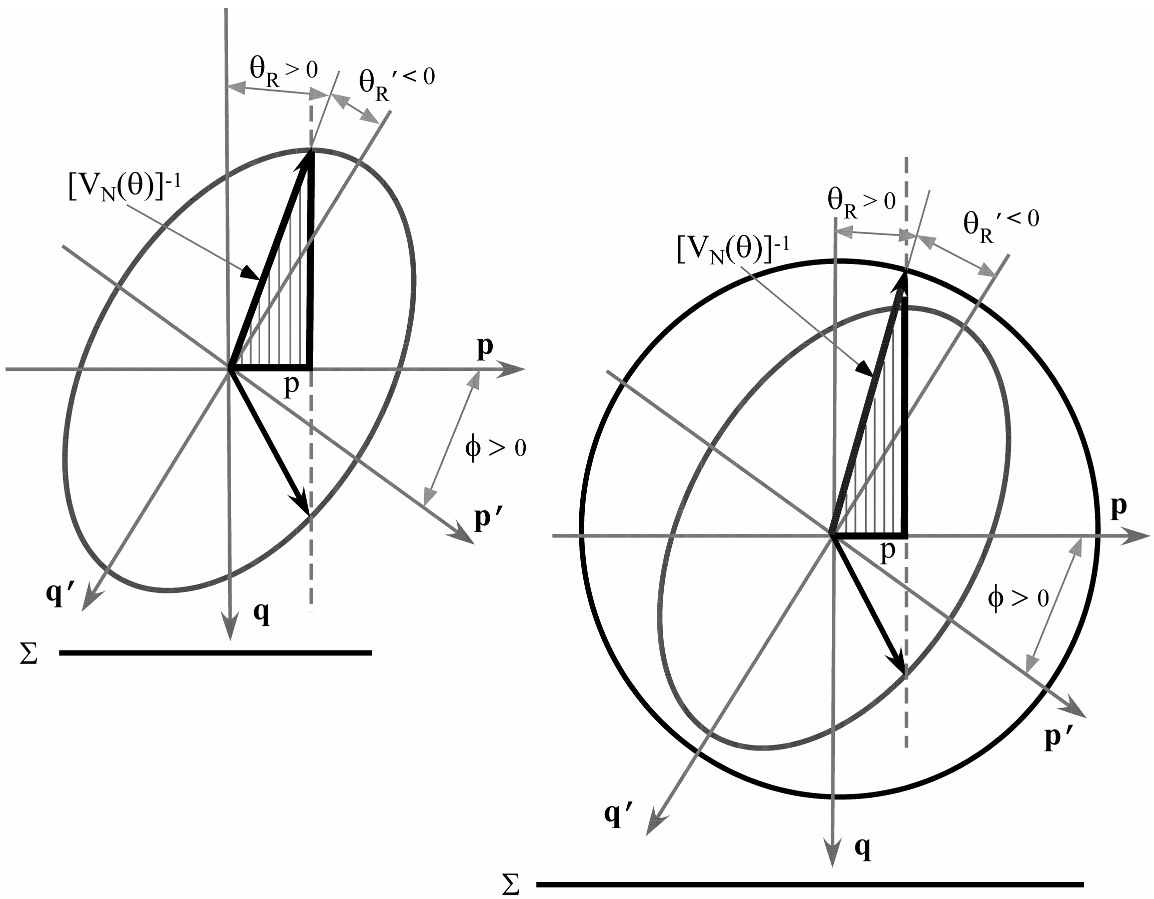

FIG. 2: Analogy of Snell's Law for rotated slowness surfaces in a transversely isotropic media. The reflected angle of either the $qP-qP$ or $qP-qS_V$ reflected arrival is obtained from the highlighted right-angle triangle nonlinear problem in each of the schematics. (See Daley (2001) for a more detailed discussion.)

NUMERICAL RESULTS

A model consisting of a T.I. plane-layer over a halfspace with the rotationally invariant axis not aligned with the model axes defined by the plane interface will be used to demonstrate two point ray tracing techniques. The anisotropic parameters defining the layer are $A_{11}=16.0\,km^2/s^2$, $A_{33}=9.0\,km^2/s^2$, $A_{55}=3.0\,km^2/s^2$ and $A_{13}=4.6158\,km^2/s^2$, and the rotationally invariant axis is at an angle of $\phi=30^\circ$ with respect to the vertical model axis measured positive clockwise. The layer thickness is $2.0\,km$. The unrotated qP and qS_V ray surfaces are displayed in Figure 1 and it may be seen from this figure that there are no triplication points on either surface. This phenomenon would greatly increase the complexity of the computer code and is presently being evaluated as to whether it should be pursued. In what follows, all incident and reflected angles are the acute angles measured with respect to the vertical axis, either in the model or rotated (primed) systems.

At an interface where there is reflection, transmission, or mode conversion of rays, the horizontal component of slowness is required to remain constant. The reference plane is the tangent to the interface at the point where any of the above takes place. This is a generalized version of Snell's Law and is shown schematically for a situation similar to what will be discussed here in Figure 2. This generalized Snell's Law has the form,

$$pV_N\left(\theta_r'\right)-\sin\theta_r=0\,; \qquad (36)$$

or in terms of only θ_r as:

$$pV_N\left(\theta_r-\phi\right)-\sin\theta_r=0\,. \qquad (37)$$

The definitions of θ_r, θ_r', and ϕ may be inferred from Figure 2. The value of p is the horizontal slowness in the model system and the phase (wavefront normal) velocities for both the qP and qS_V in the primed system may be obtained from substituting the expressions for the slowness vector components, p and q, from equation (19) into the eikonal equations given by equations (24a) and (24b).

As mentioned the two-point ray-tracing technique will be demonstrated using the previously specified single-layer model and a common shot configuration. The source will be assumed to be located at zero offset and surface receivers will be located from $-2.5\,km$ to $2.5\,km$ at $0.05\,km$ intervals. The $qP-qP$ ray diagrams and traveltime curve are shown in Figure 3 while the $qP-qS_V$ case is displayed in Figure 4.

The method of solution of the outer nonlinear equation involving the two-point ray tracing is dependent on the type of problem being considered. For the common source case treated here, it is required that two rays be determined such that one arrives at a surface position less than the minimum offset being considered and one at a point greater than the maximum offset. Once a specific set of quantities involving these two rays are known, all of the intermediate rays comprising the common shot array may be determined within a prescribed tolerance in usually less than 6 iterations if the method of false position (*regula falsi*) is used. Other techniques may be indicated for different applications. The inner tier of nonlinear equations involving Snell's Law computation has been set constant as Newton's Method, not because of speed, but rather because it runs consistently without problems.

CONCLUSIONS

A brief discussion in which a summary of the theory required for ray tracing in a transversely isotropic medium, with arbitrary orientation of the axis of rotational invariance within the medium, has been presented. The special case of ellipsoidal anisotropy was used to address some specific topics without introducing an excessive amount of mathematical complexity. Using a simple one-layered model, results were presented which gave at least some indication of what can be done with this technique, as the methods may be incorporated in both surface acquisition geometries as well as in vertical seismic profiling (VSP) applications.

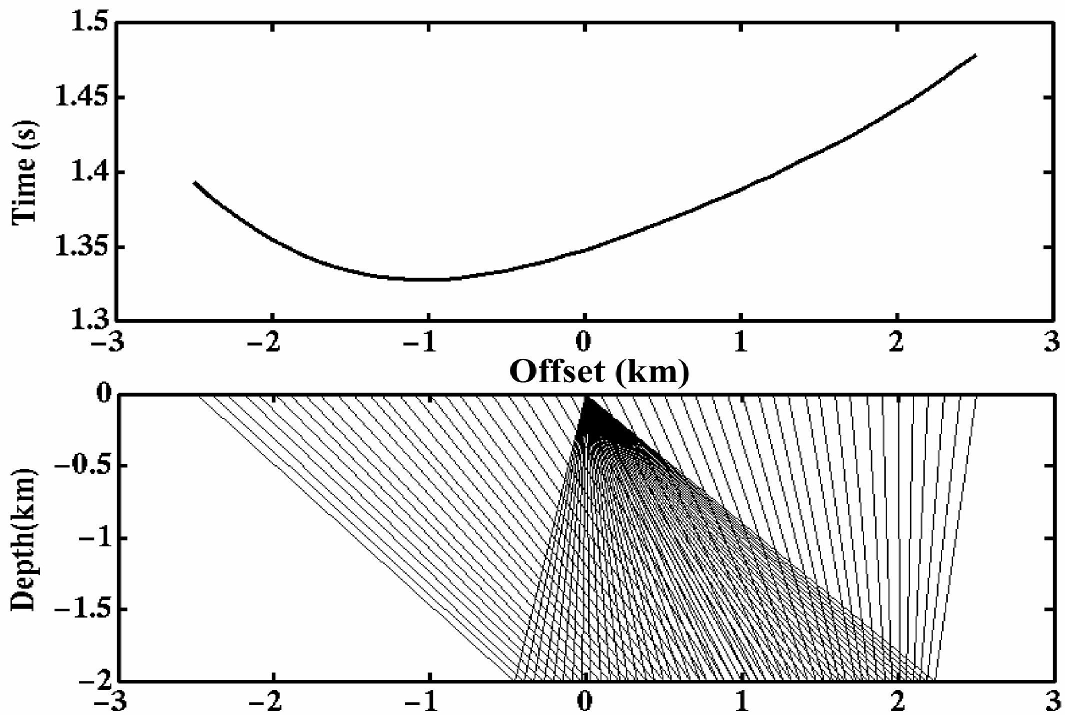

FIG. 3: The $qP-qP$ traveltime curve and two-point ray tracing plot of the rays used to introduce the method. Both the incident and reflected ray segments are qP.

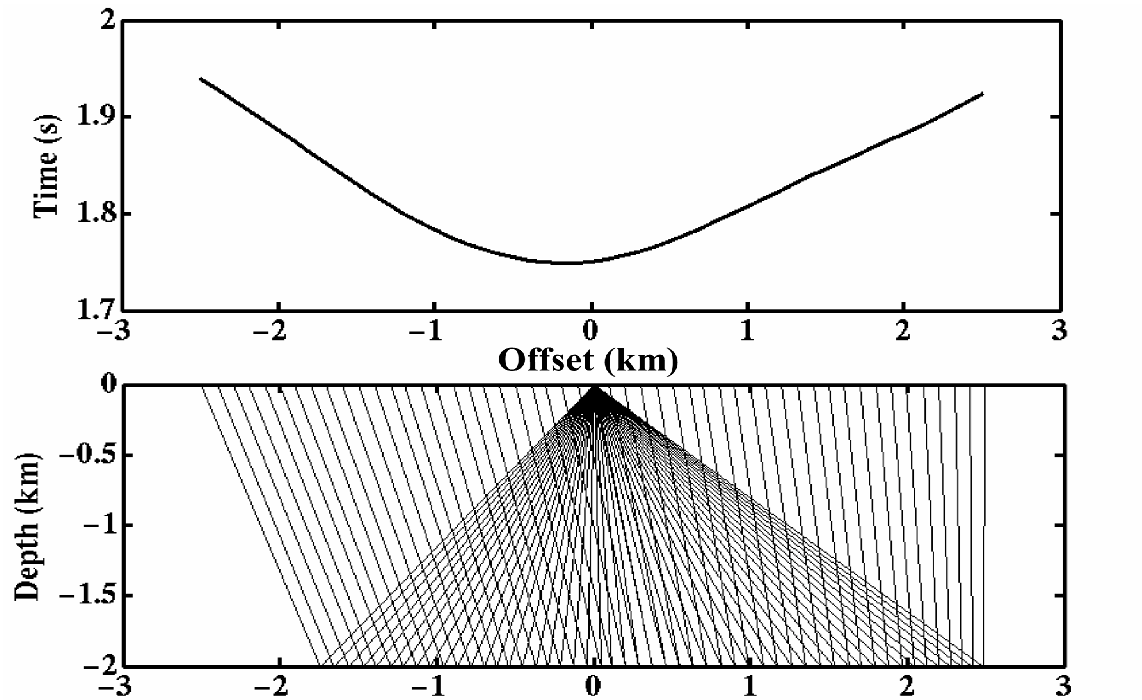

FIG. 4: The traveltime curve and two-point ray plot of the rays converted at the interface from qP to qS_V. The ray travels from the boundary back to the surface as a shear wave.

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APPENDIX: THE MATRIX Γ AND EIGENVALUE PROBLEM

The eigenvalues associated with a transversely isotropic media and the corresponding eigenvectors, known as polarization vectors, which determine the direction of particle motion, are obtained from the relation:

$$\left(\Gamma_{jk} - G\delta_{jk}\right) A_k^{(0)}\left(x_i\right) = 0\,, \tag{A.1}$$

where

$$\Gamma_{11} = A_{11}p_1^2 + A_{55}p_3^2 \tag{A.2}$$

$$\Gamma_{22} = A_{44}p_1^2 + A_{66}p_3^2 \tag{A.3}$$

$$\Gamma_{33} = A_{55}p_1^2 + A_{33}p_3^2 \tag{A.4}$$

$$\Gamma_{13} = \Gamma_{31} = \left(A_{13} + A_{55}\right) p_1 p_3 \tag{A.5}$$

$$\Gamma_{12} = \Gamma_{21} = \Gamma_{23} = \Gamma_{32} = 0\,. \tag{A.6}$$

Unless $\mathbf{A}^{(0)}\left(x_i\right) \equiv 0$, then the following condition must hold:

$$\det\left(\Gamma_{jk} - G\delta_{jk}\right) = 0\,, \tag{A.7}$$

that yields 3 eigenvalues corresponding to the qP, qS_V, and qS_H wave types. The corresponding eigenvectors, or polarization vectors indicate the direction of particle displacement. It should be noted that the wavefront normal, the ray vector and the polarization vectors are not, as in the isotropic case in the same direction, being generally in distinct directions.