Using a transition band in the weighted least-squares design of wavefield extrapolators

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ABSTRACT

Wavefield extrapolation methods are powerful in handling lateral velocity variations. However, the stability of the wavefield extrapolators is a major issue with these methods. The stability problem arises due to the presence of discontinuities at boundaries separating the wavelike and evanescent regions. Least squares methods can be used to design wavefield extrapolators that practically remain stable in a recursive scheme by minimizing the squared error between the desired and actual transforms or "the L_2 error".

Least squares methods can be classified into three major categories: unweighted least squares followed by a windowing function applied in space-frequency domain, weighted least squares using a smooth transition function connecting the wavefield and evanescent regions, and weighted least squares using a transition band (zero weight) for the transition region.

Using a transition function like a spline in the least squares approximation has been shown to be capable of designing practical stable operators. This paper shows another extrapolation method that uses weighted least squares with a transition band to design a wavefield extrapolator. This approach changes the error criterion in a particular way in order to remove or reduce the overshoot. That can be done by removing a region from the optimization. That region is called a transition band. Preliminary results for the Marmousi dataset show that this method can be used to design practical stable operators.

INTRODUCTION

Extrapolation techniques for a laterally variable velocity field are usually formulated in the space-frequency domain (Berkhout, 1981; Holberg, 1988; Hale, 1991) as a dip-limited approximation to the inverse Fourier transform of the phase shift operator. There are different ways to design spatial convolution operators for recursive wavefield extrapolation. Some methods are based on least squares, Taylor series, or both. Holberg (1988) used nonlinear least squares to design wavefield extrapolators. Hale (1991) introduced a method to calculate a stable explicit extrapolator based on the Taylor expansion of the exact constant-velocity, phase-shift operator in the frequency-wavenumber domain and the use of novel basis functions. Soubaras (1996) used the Remes-exchange algorithm to design wavefield extrapolators that have equiripple behavior. Thorbecke et al. (2004) have introduced a weighted least squares method, which is not perfectly stable but has a controlled instability. Recently, Margrave et al. (2005) introduced another method for designing spatial operators called the FOCI method. "FOCI" is an acronym for forward operator and conjugate inverse. Most of these methods have controlled instability. This means that the designed extrapolator is not perfectly stable but is practically stable in a marching scheme.

A wavefield extrapolator can be designed by the regular least squares error design, and then a smooth windowing function can be applied to truncate the final operator in the space-frequency domain, but the result will no longer be optimal. Different weighted least squares methods can also be used in filter design. The most straightforward approach is simply to change the desired response so that there is no discontinuity and, therefore, the Gibbs phenomenon is removed or reduced and more weight is put on the region of interest (Parks and Burrus, 1987). This method of weighted least squares was used by Thorbecke et al. (2004) to design a wavefield extrapolator where a spline function was used for the transition region in the desired transform that connects the wavelike and the evanescent regions. As a result, the desired transform does not have discontinuities.

In this report, we introduce another method for designing wavefield extrapolators that is also based on weighted least squares. This approach changes the error criterion in such a way as to remove or reduce the overshoot at the evanescent boundary. This is achieved by removing a region from the optimization. This region is called the transition band or "don't care region" as it is called in the literature of finite impulse response (FIR) filters. Extending this approach to wavefield extrapolator design can reduce the overshoot dramatically. We will show comparisons of this approach with Thorbecke's approach by looking at amplitude and phase spectra as well as some prestack and poststack examples.

THEORY OF WAVEFIELD EXTRAPOLATION METHODS

We begin with a 2D wavefield, $\hat{W}\left(k_x, k(x), \Delta z\right)$, which has already been Fourier transformed over the temporal and spatial coordinates. From the formula of the generalized phase shift plus interpolation (GPSPI) for 2D (Margrave and Ferguson, 1999)

$$\psi\left(x, z+\Delta z, \omega\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\psi}\left(k_x, z, \omega\right)\hat{W}\left(k_x, k(x), \Delta z\right)e^{-ik_x x}dk_x\,, \tag{1}$$

where

$$\hat{W}\left(k_x, k(x), \Delta z\right) = \exp\left(i\Delta z\sqrt{k(x)^2 - k_x^2}\right), \tag{2}$$

$$k(x) = \frac{\omega}{v(x)}\,, \tag{3}$$

ψ is the pressure wavefield after taking its Fourier transformation over the temporal coordinate, "^" means the forward Fourier transform over the transverse coordinate, $\hat{W}$ is a pseudodifferential operator, z is depth, Δz is the depth increment, x is the transverse coordinate, ω is the temporal frequency, and k_x is the transverse wavenumber.

Equation (1) is the limiting form of the phase shift plus interpolation (PSPI) (Gazdag and Squazerro, 1984) where

$$\hat{\psi}\left(k_x, z, \omega\right) = \int_{-\infty}^{\infty} \psi\left(x', z, \omega\right) e^{ik_x x'} dx'. \tag{4}$$

Inserting equation (4) into equation (1) gives

$$\psi\left(x, z+\Delta z, \omega\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{W}\left(k_x, k\left(x\right), \Delta z\right) \left[\int_{-\infty}^{\infty} \psi\left(x', z, \omega\right) e^{ik_x x'} dx' \right] e^{-ik_x x} dk_x, \tag{5}$$

where x' and x are the transverse coordinates at input and output, respectively. Rearranging the integrals in the above equation gives

$$\psi\left(x, z+\Delta z, \omega\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi\left(x', z, \omega\right) \left[\int_{-\infty}^{\infty} \hat{W}\left(k_x, k\left(x\right), \Delta z\right) e^{-ik_x\left(x-x'\right)} dk_x \right] dx'. \tag{6}$$

Rewriting equation (6) gives

$$\psi\left(x, z+\Delta z, \omega\right) = \int_{-\infty}^{\infty} \psi\left(x', z, \omega\right) W\left(x-x', k\left(x\right), \Delta z\right) dx'. \tag{7}$$

From equation (7), the wavefield extrapolation can be done in the $\omega - x$ domain as a spatial convolution where

$$W\left(x-x', k\left(x\right), \Delta z\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{W}\left(k_x, k\left(x\right), \Delta z\right) e^{-ik_x\left(x-x'\right)} dk_x. \tag{8}$$

The convolution in equation (7) becomes non-stationary when the velocity varies with *x*, and stationary when the velocity is constant. By using a non-stationary convolution operator, $W\left(x-x', x, \omega\right)$, lateral velocity variations can be handled where for each output point, a different operator can be used.

Addressing the stability problem

The problem with $W\left(x-x', k\left(x\right), \Delta z\right)$ is that it is not compactly supported (i.e. it is infinite in the spatial extent). The purpose of the different extrapolation methods is to find a stable compactly supported approximation to $W\left(x-x', k\left(x\right), \Delta z\right)$. Stability here means that after *m* repeated applications of $\tilde{W}$, the approximated operator, in a recursive scheme in a homogeneous medium, $\left|\hat{\tilde{W}}\right|^m \leq |1+\varepsilon|^m \sim 1+m\varepsilon$. When $\varepsilon = 0$ then $\tilde{W}$ is perfectly stable and technically non-stable otherwise. However, if $m\varepsilon \ll 1$, then $\tilde{W}$ is practically stable. Note that in the wavelike region, the Fourier transform of the desired extrapolator yields $\left|\hat{W}\right| = 1$, but gives $\left|\hat{W}\right| < 1$ in the evanescent region. The region $k(x) \geq k_x$ is called the wavelike region while the region $k(x) < k_x$ is called the evanescent region.

LEAST SQUARES METHOD FOR EXTRAPOLATOR DESIGN

Least squares can be used to design an accurate stable approximation to $\hat{W}(k_x, \omega)$ (Parks and Burrus, 1987). The least squares method minimizes the error, *E*, which is the sum of the squares of differences between the actual and desired Fourier transforms. This error can be defined as

$$E = \sum_{m=-(M-1)/2}^{(M-1)/2} \Upsilon(k_x) \left| \hat{W}(m\Delta k_x, \omega) - \hat{\tilde{W}}(m\Delta k_x, \omega) \right|^2, \tag{9}$$

where

$$\hat{\tilde{W}}(m\Delta k_x, \omega) = \Delta x \sum_{n=-(N-1)/2}^{(N-1)/2} \tilde{W}(n\Delta x, \omega) e^{im\Delta k_x n\Delta x}, \tag{10}$$

$$\Delta k_x = \frac{2\pi}{M\Delta x}, \tag{11}$$

$\Upsilon(k_x)$ is a non-negative error weighting function, *M* is the number of samples of the Fourier transform, and *N* is the number of filter coefficients (Parks and Burrus, 1987; Thorbecke et. al., 2004). The desired extrapolation operator $\hat{W}$ is symmetric with respect to k_x, which implies that the complex extrapolation filter coefficients $\tilde{W}$ should be even. This also means that

$$\tilde{W}(x_n) = \tilde{W}(x_{-n}). \tag{12}$$

This even-symmetry requirement suggests that *N* should be odd, with the coefficient index, *n*, bounded by (Hale, 1991)

$$-(N-1)/2 \leq n \leq (N-1)/2. \tag{13}$$

To obtain a least squares solution, $M > N$, so that there are more equations than unknowns. Equation (10) can be also expressed in matrix notation as in

$$\underline{\hat{\tilde{W}}} = \underline{F}\,\underline{\tilde{W}} \tag{14}$$

where F is the Fourier transformation matrix. A least squares solution that minimizes the error function can be given as

$$\underline{\tilde{W}} = \left[\underline{F}^H \underline{\Upsilon} \underline{F} \right]^{-1} \underline{F}^H \underline{\Upsilon} \underline{\hat{W}}, \tag{15}$$

where the superscript H denotes the complex-conjugate transpose. There are three methods for obtaining a least squares solution to the above approximation (Parks and Burrurs, 1987; Selesnick et. al., 1996)

i) Unweighted least squares.

ii) Weighted least squares with the use of a transition function to connect the wavelike and evanescent regions.

iii) Weighted least squares with the use of zero-weighted transition band placed between the wavelike and evanescent regions.

Unweighted least squares

Unweighted least squares can be obtained using

$$\underline{\tilde{W}} = \left[\underline{F}^H \underline{\Upsilon} \underline{F} \right]^{-1} \underline{F}^H \underline{\Upsilon} \underline{\hat{W}}, \tag{16}$$

with the diagonal entries of $\underline{\Upsilon}$ being ones. The equivalence (but not in a matrix notation) of the above equation is obtained by truncating the infinitely long operator with a boxcar as in

$$\tilde{W}\left(x - x^{'}, k\left(x\right), \Delta z\right) = \Omega\left(x - x^{'}\right) W\left(x - x^{'}, k\left(x\right), \Delta z\right), \tag{17}$$

where $\Omega\left(x - x^{'}\right)$ is a symmetric, compactly supported, boxcar window localized near $x = x^{'}$. This truncation leads to an oscillatory behavior that is more pronounced near the discontinuities. This behavior is known as *Gibbs phenomenon* and results from approximating a discontinuity in the desired transform and minimizing the squared error (Parks and Burrus, 1987).

Figure 1.a shows the amplitude and phase spectra of the desired transform. Figure 1.b shows the inverse Fourier transform of $\hat{W}$, which is very long. Figure 1.c shows the operators after truncation with two windows: boxcar and Hanning windows. The result of truncating $\hat{W}\left(k_x, k\left(x\right), \Delta z\right)$ with a boxcar window is that the amplitude spectrum of the Fourier transform of the truncated operator is unstable since the amplitude exceeds unity. On the other hand, when $\Omega\left(x - x^{'}\right)$ is a Hanning window, then the amplitude spectrum of the Fourier transform of $\tilde{W}\left(x - x^{'}, k\left(x\right), \Delta z\right)$ decays for some wavenumbers (Figure 1.e).

This example shows that simple truncation with window functions is suboptimal and most of the time leads to unstable operators.

Weighted least squares using a transition function

Thorbecke et al. (2004) use the weighted least squares with a transition function. Instead of using the desired transform, $\hat{W}$, which has discontinuities at the evanescent boundaries, they use a model-based function that approximates $\hat{W}$ only in the wavelike region. The amplitude and phase of the model-based function, $\hat{W}_s\left(k_x,\omega,\Delta z,k_\alpha\right)$, are defined as follows

$$\left\|\hat{W}_s\left(k_x,\omega,\Delta z,k_\alpha\right)\right\| = \begin{cases} 1.0 & \left|k_x\right| \le k_\alpha \\ spline & \left|k_x\right| > k_\alpha \\ 0.0 & \left|k_x\right| = \dfrac{\pi}{\Delta x} \end{cases} \tag{18}$$

and

$$\arg\left(\hat{W}_s\left(k_x,\omega,\Delta z,k_\alpha\right)\right) = \begin{cases} i\Delta k_z \Delta z & \left|k_x\right| \le k_\alpha \\ spline & \left|k_x\right| > k_\alpha \\ 0.0 & \left|k_x\right| = \dfrac{\pi}{\Delta \mathrm{x}} \end{cases}, \tag{19}$$

where $k_\alpha = \left(k\sin\alpha\right)$ and α is the maximum propagation angle. In the literature of filter design, a transition function is usually used only to connect the passband (wavelike) and stopband (evanescent) regions. However, in equations (18) and (19), the cubic *spline* functions go from k_α to the Nyquist wavenumber. Since the wavefield operator is applied recursively, the amplitudes in the evanescent region will decay after a number of applications as long as they are less than unity. This method, which we will call the weighted least squares using a transition function, (WLSTF), for designing a wavefield extrapolator can be obtained using

$$\underline{\tilde{W}} = \left[\underline{F}^H \Upsilon \underline{F}\right]^{-1} \underline{F}^H \Upsilon \underline{\hat{W}}_s, \tag{20}$$

where

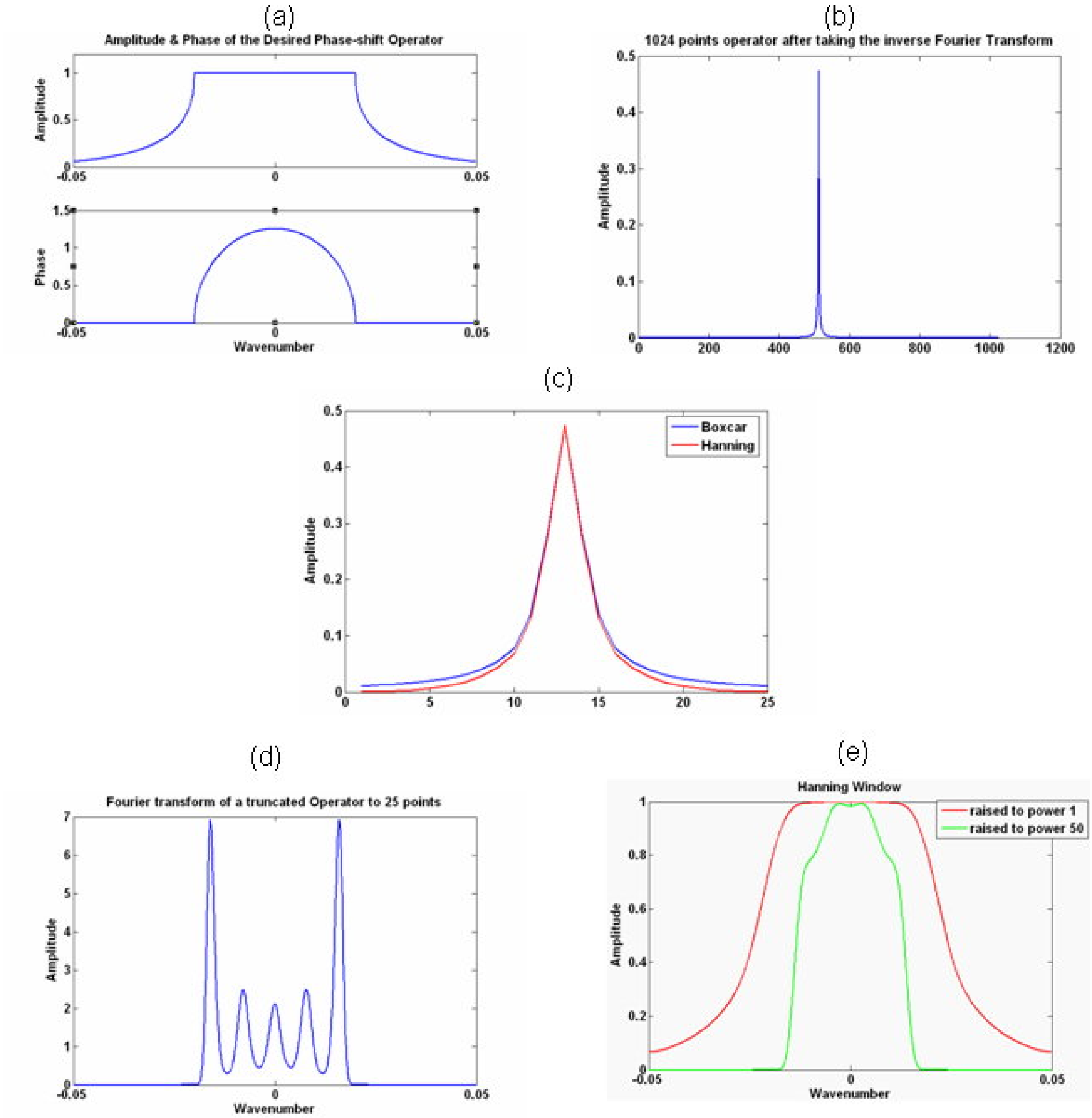


FIG. 1. Truncation effects where (a) shows the desired amplitude and phase spectra, (b) shows the inverse Fourier transform of the desired transform, which is very long, (c) shows two extrapolators that are truncated with boxcar and Hanning windows, (d) shows the amplitude spectrum of the Fourier transform of the truncated extrapolator with a boxcar window raised to a power of 50, and (e) shows the amplitude spectrum of the Fourier transform of the truncated extrapolator with a Hanning window. The parameters for these plots are vertical and spatial samplings of 10 m, *v*=2000 m/s, and frequency of 40 Hz.

$$\Upsilon_{mn} = wm\Delta k_x \delta_{mn}, \tag{21}$$

and w is a box-shaped weighting function with a weight of one in the wavelike region and a small dimensionless number (say 10^{-5}) in the evanescent region.

Weighted least squares using a transition band

One of most effective modifications of the least squares (LS) error design methods is to change the band of wavenumbers over which the minimization is carried out (Parks and Burrus, 1987). The band of wavenumbers for the transition region is simply removed from the error definition, and the region is called the transition band or "don't care" band. The error becomes

$$E = E_S + E_P, \tag{22}$$

where

$$E_P = \sum_{m=-p}^{p} \Upsilon(k_x) \left| \hat{W}(m\Delta k_x, \omega) - \hat{\tilde{W}}(m\Delta k_x, \omega) \right|^2, \tag{23}$$

$$E_S = \sum_{m=-(M-1)/2}^{-s} \Upsilon(k_x) \left| \hat{W}(m\Delta k_x, \omega) - \hat{\tilde{W}}(m\Delta k_x, \omega) \right|^2 + \sum_{m=s}^{(M-1)/2} \Upsilon(k_x) \left| \hat{W}(m\Delta k_x, \omega) - \hat{\tilde{W}}(m\Delta k_x, \omega) \right|^2, \tag{24}$$

$p\Delta k_x = k_\alpha$, $s\Delta k_x = |2k - k_\alpha|$, $k_\alpha = k\sin(\alpha)$, $k = \omega / v$, and the weight function, $\Upsilon(k_x)$ is defined as

$$\Upsilon(k_x) = \begin{cases} 1 & |k_x| \le k_\alpha \\ 0 & k_\alpha < |k_x| < |2k - k_\alpha| \\ \varepsilon & |2k - k_\alpha| < |k_x| < \dfrac{\pi}{\Delta x} \end{cases}, \tag{25}$$

such that

$$\varepsilon \ll 1. \tag{26}$$

In this method of weighted least squares using a transition band, WLSTB, there is no constraint placed on $\hat{W}(k_x, \omega)$ in the transition region. Further, there will be less squared error because the error of the transition region is not included (equation (21)). The final extrapolator can be obtained using

$$\underline{\tilde{W}} = \left[\underline{F}^H \underline{\Upsilon} \underline{F} \right]^{-1} \underline{F}^H \underline{\Upsilon} \underline{\hat{W}}. \tag{27}$$

DISCUSSION

Using a transition band in the weighted least squares filter design can lead to a more stable design than using a transition function for the transition region. To illustrate the difference between them, a comparison of the amplitude spectra of the weighed least squares with a transition function (WLSTF) and weighted least squares with a transition band (WLSTB) is shown in Figure 2. The same parameters that were used by Thorbecke et al. (2004) are used here to reproduce the same figure (Figure 2.d in Thorbecke et al., 2004) for comparison. The WLSTB shows a better approximation to the desired transform than the WLSTF extrapolator. Further, the oscillations of the WLSTB extrapolator, which is a potential source for instability, are less than the WLSTF extrapolator. On the other hand, the phases of WLSTF and WLSTB extrapolators show that both have a good phase control and they are similar to the phase of the desired transform (Figure 3).

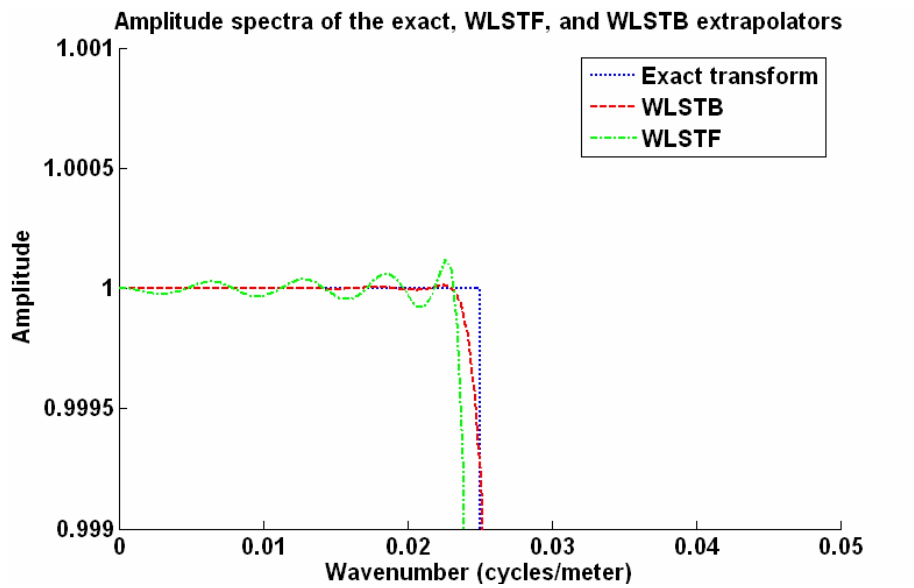


FIG. 2. Amplitude spectra of the exact, WLSTF, and WLSTB extrapolators where $\Delta x = 10$ m, $\Delta z = 2$ m, $f = 50$ Hz, $v = 2000$ m/s, and $\alpha = 70^{o}$.

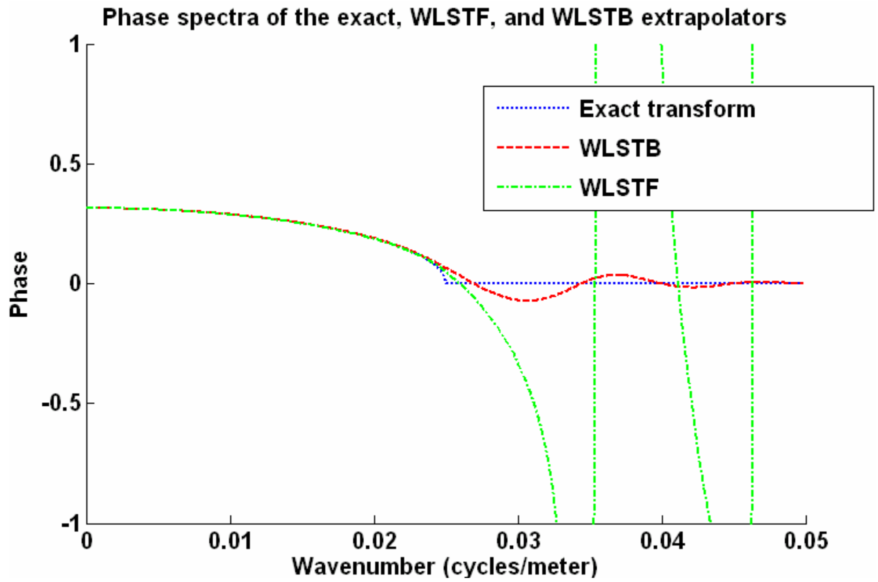


FIG. 3. Phase spectra of the exact, WLSTF, and WLSTB extrapolators where $\Delta x = 10$ m, $\Delta z = 2$ m, $f = 50$ Hz, $v = 2000$ m/s, and $\alpha = 70^{\circ}$.

Impulse responses of Phase shift, WLSTF, and WLSTB operators

The impulse responses of the phase shift operator, and the WLSTF and WLSTB extrapolators are shown in Figure 4. The parameters are $\Delta x = \Delta z = 10$ m, $\Delta t = 0.004$ ms, v=2000 m/s, $\alpha = 70^{\circ}$, and N=25. The impulse responses of the WLSTF and WLSTB are very similar and there is no noticeable difference between them. This shows that the WLSTB results are comparable with the WLSTF.

Prestack depth migration of the Marmousi dataset using the WLSTF and WLSTB extrapolators

The 2-D acoustic Marmousi dataset was created at the Instut Francais du Petrole (IFP) (Bourgeois et al., 1991). With the presence of complex reflectors, steep dips and strong velocity gradients, it is widely recognized as an ideal synthetic dataset for testing migration algorithms. The dataset consists of 240 individual shot records of 96 traces each in a marine towed streamer configuration. The source and receiver intervals are 25 m and the highest coherent frequencies in the data are about 50 Hz. Prior to migration, we applied a wavelet shaping filter designed to whiten the signal spectrum and to remove an approximately 60 ms delay due to ghosting and water-bottom multiples. We also interpolated each shot to a receiver spacing of 8.3333 m.

Figure 5.a shows an approximation to the reflectivity of Marmousi, Figure 5.b shows the prestack result using WLSTF, and Figure 5.c shows the result obtained using the WLSTB. The two results are comparable in terms of their abilities to handle such a complicated subsurface. The parameters are $\Delta x = \Delta z = 8.3333$ m, and N=31. Figure 6 shows a detailed comparison of the shallow central sections of Figures 5.b and 5.c. The

imaging of the dipping events in the WLSTB is superior to that of the WLSTF. However, at the target level, there is no noticeable difference between them (Figure 7).

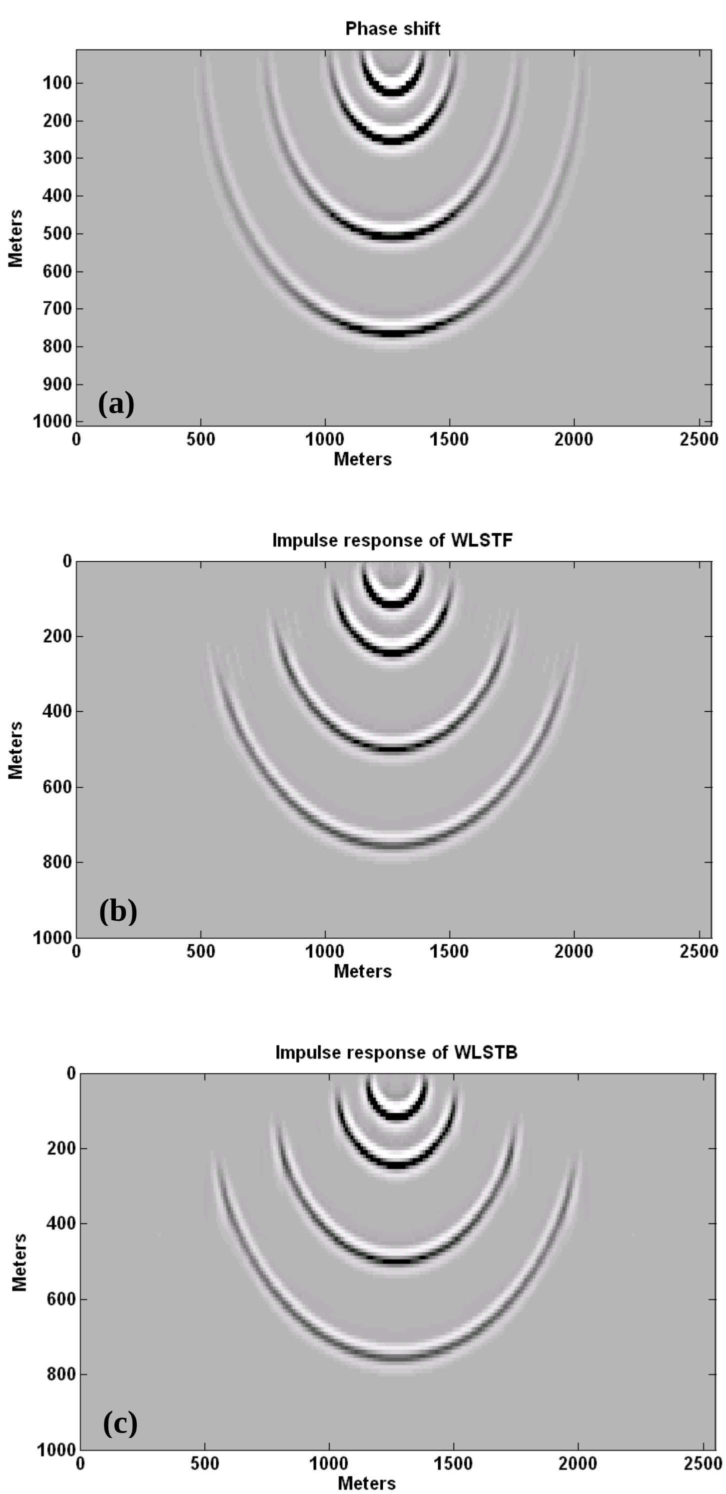
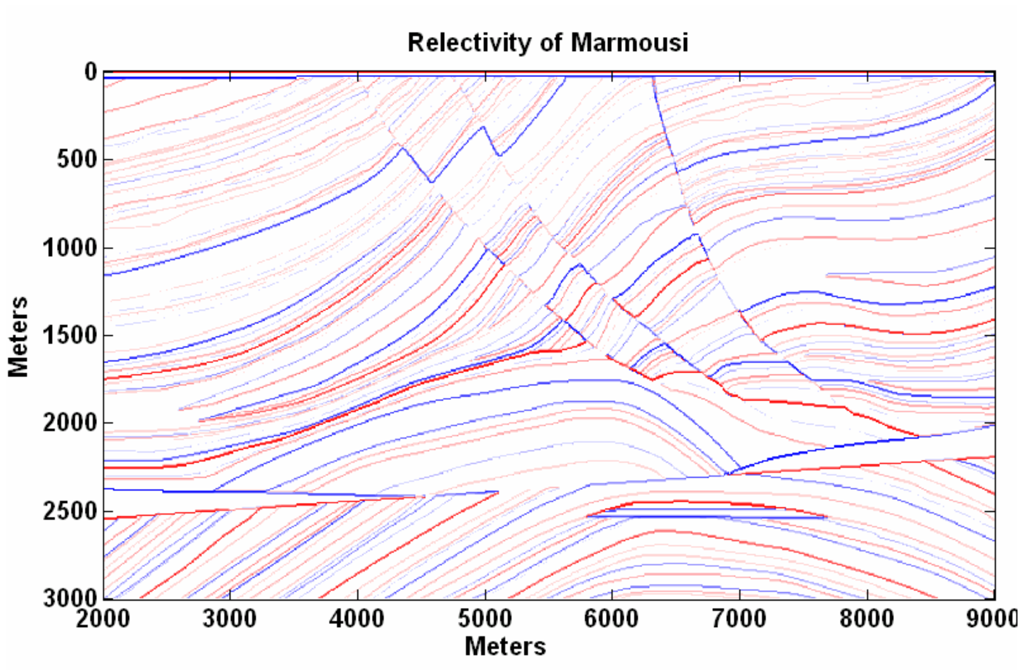
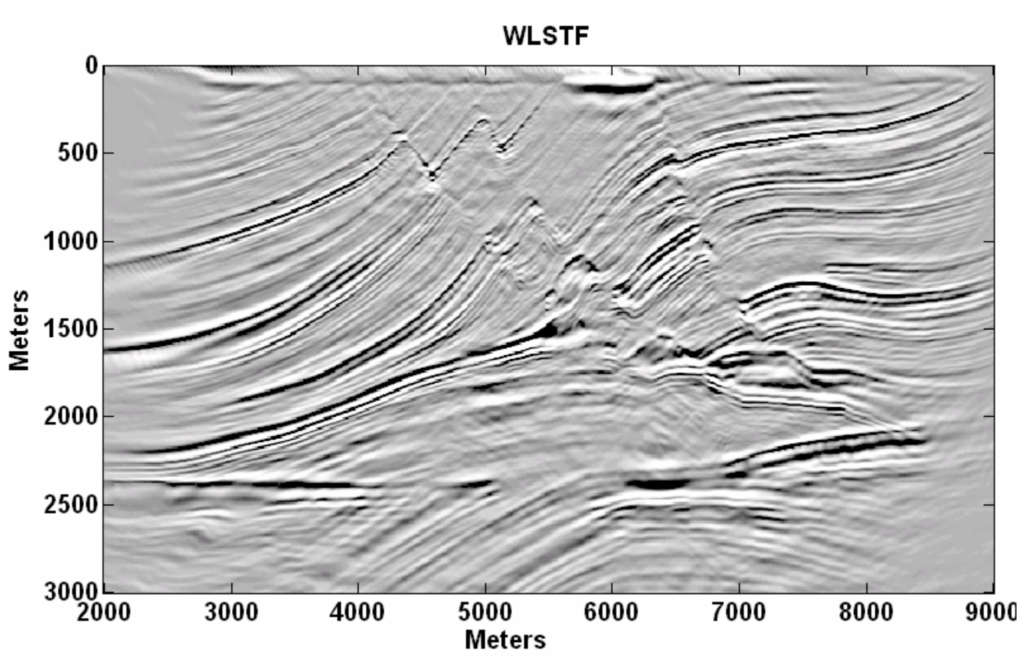


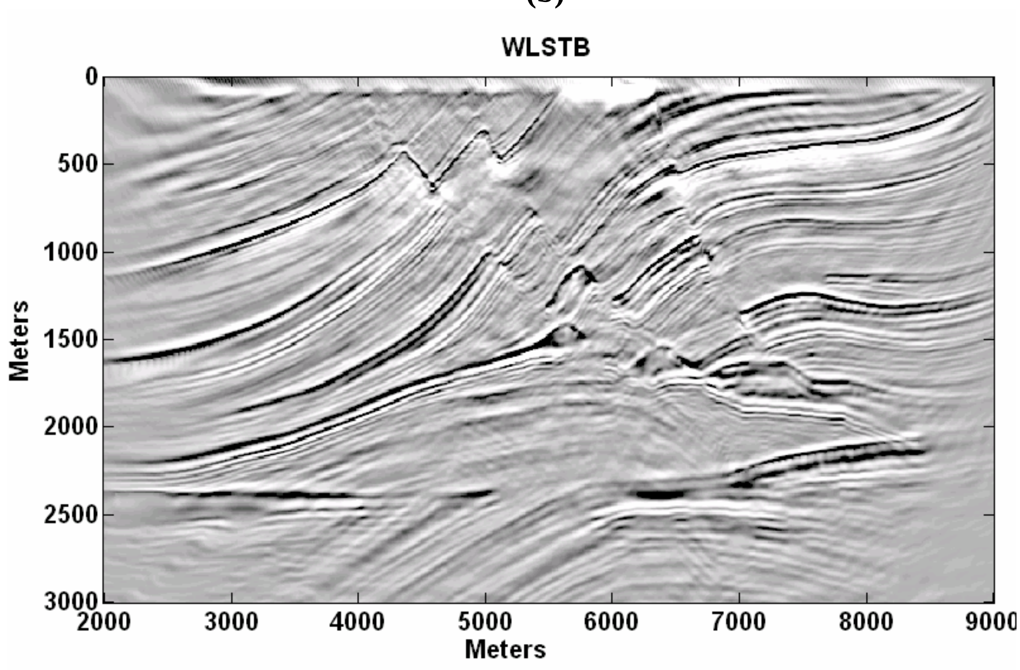
FIG. 4. Impulse responses of (a) the phase shift operator, (b) the WLSTF extrapolator, and (c) the WLSTB extrapolator using $\Delta x = \Delta z = 10$ m, $\Delta t = 0.004$ ms, *v=2000*m/s, $\alpha = 70^{\text{o}}$, and *N=25*.


Relectivity of Marmousi
Meters
0
500
1000
1500
2000
2500
3000
2000
3000
4000
5000
6000
7000
8000
9000
Meters

(a)

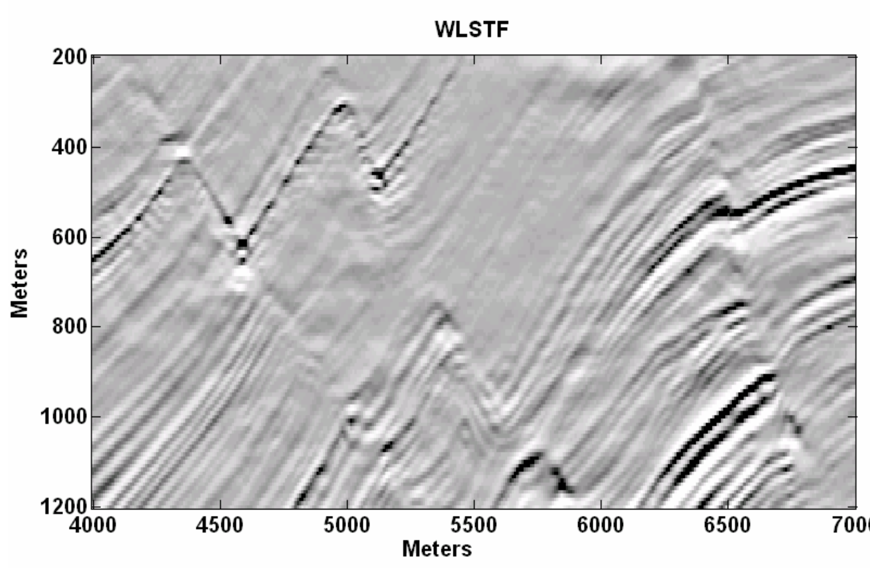


(b)

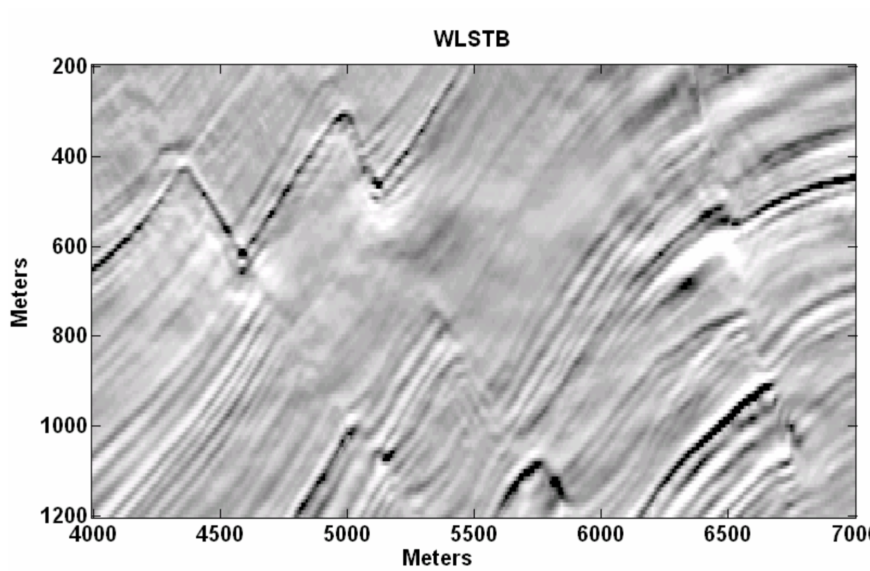


(c)

FIG. 5. Prestack depth migration results where (a) is the Marmousi velocity model, (b) is the WLSTF result, and (c) is the WLSTB result.

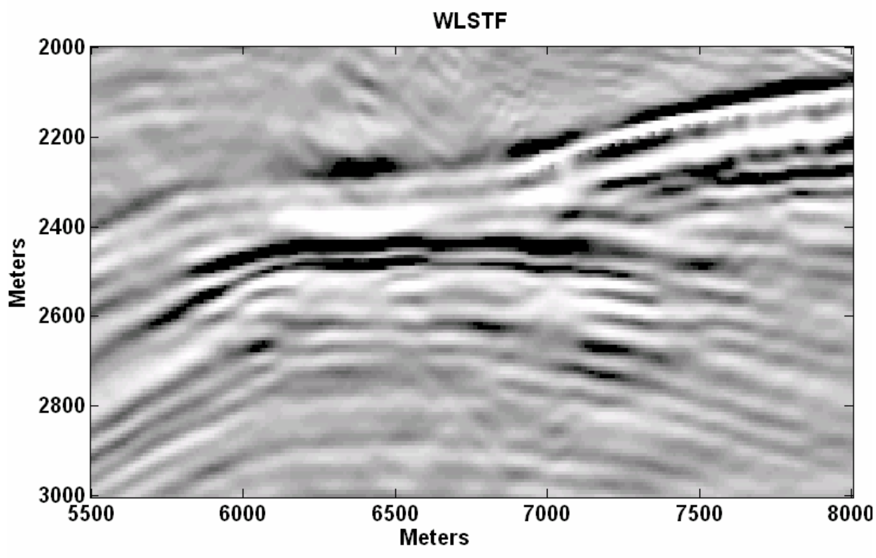


(a)

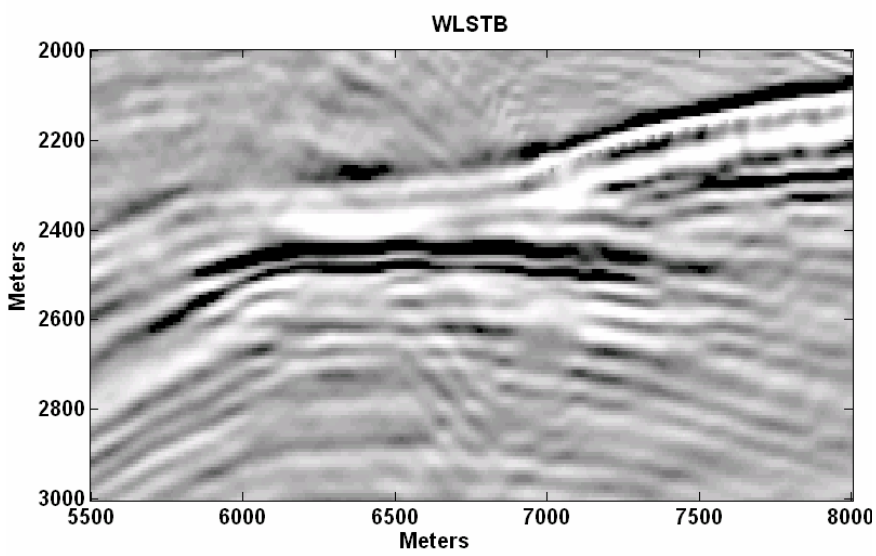


(b)

FIG. 6. Zoomed sections of the shallow central parts of Figures 5.b and 5.c where (a) is the WLSTF result and (b) is the WLSTB result.



(a)



(b)

FIG. 7. Zoomed sections of the deep parts of Figures 5.b and 5.c where (a) is the WLSTF result and (b) is the WLSTB result.

CONCLUSIONS

The desired transform of the extrapolator has discontinuities at the evanescent boundaries. Approximating discontinuities leads to *Gibbs phenomenon* or oscillatory behavior that is more pronounced near the discontinuities. Using weighted least squares along with changing the desired transform so that it has a smooth function can remove or reduce the overshoot. In the WLSTF design, the model based extrapolator matches the exact operator in the wavelike region and it is a spline function that goes smoothly to zero at the Nyquist wavenumber. Further, more weight is put on the wavefield region than the evanescent region to increase stability. This method can be used to design practical extrapolators.

In this report, we have introduced another method that is also based on weighted least squares but differs from WLSTF in that the band of wavenumbers belonging to the transition region is simply removed from the error definition. After removing these wavenumbers, this region is called the transition band so that this is referred to as the weighted least squares transition band (WLSTB) method. The preliminary results of WLSTB were comparable with those of WLSTF, which shows that the WLSTB approach can be used in extrapolator filter design.

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