Ray-reflectivity method part 1: Theory for the P - S_V case

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ABSTRACT

Replacing the reflection and transmission coefficients at a solid-solid interface by thin layered transition zone analogues is considered. These thin layered zones are often associated with hydrocarbon deposits in sedimentary strata. This is done within the context of zero order asymptotic ray theory (ART) using the stationary phase method, which essentially produces the same results. The frequency dependent reflectivities and transmittivities that replace the reflection and transmission coefficients are derived using propagator matrix methods, the basis for the classical reflectivity method and incorporated into the zero order ART solution for a plane layered structure composed of thick layers interspersed with thin layered transition zones. ART is used in the thick layers to introduce geometrical spreading.

INTRODUCTION

In an earlier series of papers the hybrid ray-reflectivity method was presented for *SH* wave propagation in a halfspace composed of thick layers separated by thin layered or transition zones. (Daley and Hron, 1992, 1990 and 1982). A schematic of this for the P-S_V case is shown in Figure 1. The motivation for this manner of presentation was to communicate the theory without the added complexity introduced by considering the P-S_V case for a similar problem type. A relatively inaccessible booklet in Russian by Ratnikova (1976) deals with this problem but has several theoretical problems combined with an inordinate number of typographic errors. There are a significant number of references regarding the reflectivity method in Aki and Richards (1980) and Brekhovskikh (1980) apart from the tutorial on the subject by Muller (1986) and the comprehensive list of references contained there. Two of the more notable of these are Fertig and Muller (1978) and Fuchs and Muller (1971).

Combining asymptotic ray theory (ART) and aspects of the reflectivity method results in a hybrid method. This allows for the seismic modelling of many situations that are commonly found in hydrocarbon exploration situations. These include two layers whose thickness is large when compared to predominant wavelength, related to the predominant frequency of thee seismic wavelet, separated by one or more layers that can be classified as thin relative to the predominant wavelength. Layers on the order of half a wavelength or less will act as a loose definition of those layers classified as “thin”. This hybrid method uses standard ART when dealing with ray propagation in the thick layers and the thin layered or transition zones separating two thick layers will be assumed to be a pseudo-boundary at which reflection and transmission coefficient analogues are introduced. These analogues termed reflectivities and transmittivities are obtained using propagator matrix theory and are developed in a later section. The early work in this area was by Thomson (1950), Haskell (1953), Ewing et al. (1957) and Dunkin (1965) and especially the text by Kennett (1983).

The use of the stationary phase method rather than zero order asymptotic ray theory in the solution method is done to retain a consistent notation and methodology with that used in an earlier series of papers on the reflectivity method (Muller, 1986). The notation used in the theoretical development presented here for reflectivity theory related topics are similar to those used in paper by Fuchs (1968) (marginally inaccessible as it is in German) and Fuchs (1970) A similar manner of notation is employed for the reflectivity and transmittivity portion of the theory.

STATIONARY PHASE APPROXIMATION FOR REFLECTED AND TRANSMITTED WAVES IN A PLANE LAYERED MEDIUM

The coupled compressional – shear, $P-S_V$, wave propagation in an isotropic inhomogeneous half plane consisting of $n-1$ *thick* layers separated by *thin* layered or transition zones overlying a half space will be considered. The two halfspaces are denoted as *0* and *n* and the vertical *z* axis is chosen to be positive downwards with $z=0$ corresponding to the top of the thin layered zone. The intervening thin layers between the upper and lower half spaces are sequentially designated *1* to *n-1* (Figure 2).

A point source is assumed located at $z=-H$ and only waves polarized in the plane of incidence (*P* and S_V waves) will be considered. In cylindrical coordinates ($-\infty < z < \infty$, $0 \le r < \infty$, $0 \le \theta < 2\pi$) the radial, *r*, and horizontal, *z*, components of the displacement vector, u_m and w_m, in each layer or half space, can be expressed in terms of the two potentials, ϕ_m and ψ_m, as

$$u_m = \frac{\partial \phi_m}{\partial r} - \frac{\partial \psi_m}{\partial z}, \quad w_m = \frac{\partial \phi_m}{\partial z} + \frac{\partial \psi_m}{\partial r} + \frac{\psi_m}{r}, \quad \left(m = 0,1,\ldots,n\right). \tag{1}$$

Since the media are homogeneous, there are no lateral inhomogeneities and hence no θ dependence. The potentials in equation (1) must satisfy the conditions

$$\nabla^2 \phi_m - \alpha_m^{-2} \frac{\partial^2 \phi_m}{\partial t^2} = 0, \quad \nabla^2 \psi_m - \frac{\psi_m}{r^2} - \beta_m^{-2} \frac{\partial^2 \phi_m}{\partial t^2} = 0, \quad \left(m = 0,1,\ldots,n\right) \tag{2}$$

where α_m and β_m are the compressional (*P*) and shear $\left(S_V\right)$ wave velocities, respectively, in the m^{th} layer. After removing the time dependence of equations (2) by applying a Fourier time transform the following result

$$\nabla^2 \phi_m + \frac{\omega^2}{\alpha_m^2} \phi_m = 0, \quad \nabla^2 \psi_m - \frac{\psi_m}{r^2} + \frac{\omega^2}{\beta_m^2} \psi_m = 0, \quad \left(m = 0,1,\ldots,n\right). \tag{3}$$

If the point source at $z=-H$ is assumed to be a source of *P* waves only, the following time transformed quantities must be satisfied

$$\nabla^2\phi_0^0+\frac{\omega^2}{\alpha_0^2}\phi_0^0=F(\omega)\frac{\delta(r)}{2\pi r}\delta(z+H)\,. \tag{4}$$

The superscript on ϕ denotes the incident wave front and $F(\omega)$ is the Fourier time transform of the source wavelet $f(t)$.

The solution of (4) is given (Sommerfeld, 1949 and Aki and Richards, 1980) as

$$\phi_0^0(r,z,\omega)=\frac{F(\omega)\exp(-i\eta_0 R)}{R}=-iF(\omega)\int_0^\infty J_0(kr)\exp\left(-i\eta_0\left|z+H\right|\right)\frac{k\,dk}{\eta_0} \tag{5}$$

where

$R=\left[r^2+(z+H)^2\right]^{1/2}$,

$\omega -$ circular frequency,

$J_p(kr) -$ Bessel function of order p,

$$\eta_m - \begin{cases} \left(\dfrac{\omega^2}{\alpha_m^2}-\dfrac{\omega^2}{c^2}\right)^{1/2} = \left(k_{\alpha_m}^2-k^2\right)^{1/2}, & \alpha_m \le c \\ -i\left(k^2-k_{\alpha_m}^2\right)^{1/2}, & \alpha_m > c \end{cases},$$

$$\xi_m=\begin{cases} \left(k_{\beta_m}^2-k^2\right)^{1/2}, & \beta_m \le c \\ -i\left(k^2-k_{\beta_m}^2\right)^{1/2}, & \beta_m > c \end{cases}.$$

The quantity c is the horizontal velocity or equivalently, the inverse of the horizontal component of the slowness vector. The signs of η_m and ξ_m for α_m and β_m greater than c is chosen such that any exponential terms involving them vanish as $z\to\pm\infty$, satisfying physical radiation conditions.

The total compressional potential in the upper (0) half space has the form

$$\phi_0=\phi_0^0+\phi_0^+ \qquad \text{for } z<0 \tag{6}$$

where

$$\phi_0^+(r,z,\omega)=-iF(\omega)\int_0^\infty \tilde{R}_{PP}(\omega,k)\,J_0(kr)\exp\left[i\eta_0(z-H)\right]\frac{k\,dk}{\eta_0} \quad, z<0 \tag{7}$$

is the reflected potential and $\tilde{R}_{PP}(\omega,k)$ is the frequency dependent reflection coefficient or reflectivity at the quasi-interface. The tilde above the reflection coefficient and those which will follow in this report are to indicate that these are *potential* reflection and transmission coefficients as opposed to *particle displacement* coefficients. For the definition of the displacement in terms of potentials used here the difference is just the ratio of reflected/transmitted to incident velocities. Other characterizations of particle displacements in terms of potentials generally take on more complex forms.

The reflected shear potential resulting from a compressional source at $z=-H$ can analogously be written as

$$\psi_0^+(r,z,\omega)=F(\omega)\int_0^\infty \tilde{R}_{PS}(\omega,k)J_1(kr)\exp\left[i(\xi_0 z-\eta_0 H)\right]\frac{k\,dk}{\eta_0} \quad ,z<0 \qquad (8)$$

where again $\tilde{R}_{PS}(\omega,k)$ is the reflection coefficient.

Extending the method used above, it is possible to write expressions for the transmitted compressional and shear potentials in the lower (*n*) half space.

$$\phi_n^-(r,z,\omega)=-iF(\omega)\int_0^\infty T_{PP}(\omega,k)J_0(kr)\exp\left[-i\left(\nu_n(z-z_{n-1})+\nu_0 H\right)\right]\frac{k\,dk}{\nu_0} \quad ,z>z_{n-1} \quad (9)$$

$$\psi_n^-(r,z,\omega)=F(\omega)\int_0^\infty T_{PS}(\omega,k)J_1(kr)\exp\left[-i\left(\xi_n(z-z_{n-1})+\eta_0 H\right)\right]\frac{k\,dk}{\eta_0} \quad ,z>z_{n-1} \quad (10)$$

The expressions $\tilde{T}_{PP}(\omega,k)$ and $\tilde{T}_{PS}(\omega,k)$ are the quasi-transmission coefficients due to a plane *P* wave incident on the 0^{th} interface. These coefficients, together with $\tilde{R}_{PP}(\omega,k)$ and $\tilde{R}_{PS}(\omega,k)$ will be derived in a later section.

There are a number of difficulties associated with the numerical evaluation of the integrals involved in the equations for the reflected and transmitted potentials. As an example, for the reflected *PP* potential, ϕ_0^+, in equation (7), poles are encountered in the complex reflection coefficient $\tilde{R}_{PP}(\omega,k)$ for the wave numbers in the range $k>\omega/\alpha_0=k_{\alpha_0}$. These poles correspond to waves guided within the thin layered zone and propagating with a horizontal phase velocity $c<\alpha_0$. They correspond to Stonely waves guided at a first-order discontinuity between two elastic media. Their amplitudes decay with distance from the thin layered zone.

This difficulty may be overcome, assuming that the distance *H* from the source to the top of the thin layered zone is large when compared to the wavelength associated with the predominant frequency of the source wavelet. For $k>k_{\alpha_0}$, ν_0 in equation (7) will become negative and purely imaginary so that the exponential term (in part) is real and negative. Thus the integrand can be kept arbitrarily small for $k>k_{\alpha_0}$, if *H* (ω large – high frequency limit) is chosen sufficiently large. As a consequence, the contributions to the integral may be neglected for wave numbers $k > k_{\alpha_0}$ and the upper limit of integration replaced by $k_{\alpha_0} = \omega/\alpha_0$. With this assumption in place, equation (7) becomes[1]

$$\phi_0^+(r,z,\omega) = -iF(\omega)\int_0^{k_{\alpha_0}} \tilde{R}_{PP}(\omega,k)\, J_0(kr)\exp\left[i\eta_0(z-H)\right]\frac{k\,dk}{\eta_0} \quad , z<0 . \quad (11)$$

The singularity at $k = k_{\alpha_0}$ may be avoided through the change of variable

$$k = k_{\alpha_0}\sin\gamma \quad (12)$$

which is the relation between the wave number k and the angle of P wave incidence at the top of the thin layered zone. With this substitution, equation becomes

$$\begin{aligned}\phi_0^+(r,z,\omega) = -k_{\alpha_0}F(\omega)\int_0^{\pi/2} \tilde{R}_{PP}(\omega,k)\, J_0\left(\left(k_{\alpha_0}\sin\gamma\right)r\right)\times \\ \exp\left[ik_{\alpha_0}\cos\gamma(z-H)\right]\sin\gamma\, d\gamma,\ z<0\end{aligned} . \quad (13)$$

If the argument $kr\left(rk_{\alpha_0}\sin\gamma\right)$ is assumed large, the Bessel function $J_0(kr)$ can be approximated by its asymptotic formula for large argument (high frequency limit) given by

$$J_0(kr) = \frac{1}{\sqrt{2\pi kr}}\left\{\exp\left[i(kr-\pi/4)\right] + \exp\left[-i(kr-\pi/4)\right]\right\}. \quad (14)$$

As a result, equation (13) may be written as

$$\begin{aligned}\phi_0^+(r,z,\omega) = -iF(\omega)\left(\frac{k_{\alpha_0}}{2\pi r}\right)^{1/2}\times \\ \left\{\int_0^{\pi/2}\tilde{R}_{PP}(\omega,k)\ \exp\left[-ik_{\alpha_0}\left(\cos\gamma(H-z)+r\sin\gamma\right)\right]\sqrt{\sin\gamma}\,d\gamma + \right. \\ \left.\int_0^{\pi/2}\tilde{R}_{PP}(\omega,k)\ \exp\left[-ik_{\alpha_0}\left(\cos\gamma(H-z)-r\sin\gamma\right)\right]\sqrt{\sin\gamma}\,d\gamma\right\}\end{aligned} \quad (15)$$

One of the most common techniques used in the evaluation of integrals of the type in equation (15) is by using the method of stationary phase. To use this method to obtain an approximation to the integrals in (15), it will be assumed that the reflection coefficient $\tilde{R}_{P0P0}(\omega,k)$ varies slowly in amplitude but more importantly, in phase. Neglecting the

[1] At this point it would be convenient to transform from an integral in $J_0(\kappa)$ to one in $H_0^{(2)}(\kappa)$. However to maintain a consistent notation with the early papers related to the reflectivity method, the $J_0(\kappa)$ will be retained.

phase of $\tilde{R}_{P0P0}(\omega,k)$ in the stationary phase approximation has the effect of neglecting other possible arrivals associated with points of critical refraction (head waves), which may lie within the modified limits of integration.

Points of stationary phase occur at those locations on the real axis where the exponential terms

$$L(\gamma)_{\mp} = -(H-z)\cos\gamma \mp r\sin\gamma \quad , z<0 \tag{16}$$

vary most slowly, i.e., where the first derivative with respect to γ is equal to zero

$$L'(\gamma)_{\mp} = (H-z)\sin\gamma \mp r\cos\gamma = 0 \quad , z<0\,. \tag{17}$$

It can be readily seen that the first integral in equation (15) has a stationary point at $\gamma = \tan^{-1}\left[r/(H-z)\right]$ and that the second integral does not have a stationary point within the limits of integration; assuming $H>z$. Therefore, the contribution of the second integral, for large $k_{\alpha_0} r$, may be neglected.

The general formula for the approximation of integrals using the stationary phase method is

$$\lim_{\ell\to\infty} \int_{\gamma_0-\varepsilon}^{\gamma_0+\varepsilon} P(\gamma)e^{i\ell L(\gamma)}d\gamma \approx \left(\frac{\pi}{\ell\left|L''(\gamma_0)\right|}\right)^{1/2} e^{i\ell L(\gamma_0)+\frac{i\pi}{4}\operatorname{sgn}\left(L''(\gamma_0)\right)} + O\left(\frac{1}{\ell}\right). \tag{18}$$

In the case being considered being considered here,

$$\begin{aligned} &L(\gamma) = L_{-}(\gamma) \\ &P(\gamma) = \tilde{R}_{P0P0}(\gamma)\sqrt{\sin\gamma}\,, \qquad \ell = k_{\alpha_0} = \omega/\alpha_0 \\ &L''(\gamma) = L_{-}(\gamma) = \left(H+|z|\right)\cos\gamma + r\sin\gamma > 0 \end{aligned} \tag{19}$$

The final form of the stationary phase approximation to equation (15) is given by

$$\phi_0^{+}(r,z,\omega) \approx \frac{F(\omega)}{R_1}\tilde{R}_{P0P0}(\omega,\gamma_0)e^{ik_{\alpha_0}R_1}, \qquad z<0 \tag{20}$$

with $R_1 = \left[\left(H+|z|\right)^2 + r^2\right]^{1/2}$.

It is possible, based on the derivation presented above, to evaluate the remaining integral expressions for the potentials of the reflected S_V wave and the transmitted P and S_V waves in an analogous manner, the results of which are as follows

$$\psi_0^+(r,z,\omega) \approx \frac{F(\omega)}{R_2}\tilde{R}_{P0S0}(\omega,\gamma_0)\exp\left[-i\left(\frac{k_{\alpha_0}H}{\cos\gamma_0}+\frac{k_{\beta_0}|z|}{\cos\zeta_0}\right)\right] \qquad (21)$$

where

$$\sin\zeta_0 = \frac{\beta_0\sin\gamma_0}{\alpha_0} \;,\; \gamma_0 \text{ being defined by}$$
$$H\tan\gamma_0 + |z|\tan\zeta_0 = r \;,\; \text{with}$$
$$R_2 = \frac{H}{\cos\gamma_0}+\frac{|z|}{\cos\zeta_0} \;.$$

$$\phi_n^-(r,z,\omega) \approx \frac{F(\omega)}{R_3}\tilde{T}_{PnPn}(\omega,\gamma_0)\exp\left[-i\left(\frac{k_{\alpha_0}H}{\cos\gamma_0}+\frac{k_{\alpha_n}(z-z_{n-1})}{\cos\gamma_n}\right)\right] \quad ,z>z_{n-1} \quad (22)$$

where

$$\sin\gamma_n = \frac{\alpha_n\sin\gamma_0}{\alpha_0} \;,\; \gamma_0 \text{ being determined from}$$
$$H\tan\gamma_0 + |z-z_{n-1}|\tan\gamma_n = r \;,\; \text{and}$$
$$R_3 = \frac{H}{\cos\gamma_0}+\frac{|z-z_{n-1}|}{\cos\gamma_n} \;.$$

$$\psi_n^-(r,z,\omega) \approx \frac{-F(\omega)}{R_4}\tilde{T}_{PnSn}(\omega,\gamma_0)\exp\left[-i\left(\frac{k_{\alpha_0}H}{\cos\gamma_0}+\frac{k_{\beta_n}(z-z_{n-1})}{\cos\zeta_n}\right)\right] \quad ,z>z_{n-1} \quad (23)$$

with

$$\sin\zeta_n = \frac{\beta_n\sin\gamma_0}{\alpha_0} \;,\; \gamma_0 \text{ obtained from}$$
$$H\tan\gamma_0 + |z-z_{n-1}|\tan\zeta_n = r \;,\; \text{and}$$
$$R_4 = \frac{H}{\cos\gamma_0}+\frac{|z-z_{n-1}|}{\cos\zeta_n} \;.$$

The vertical and horizontal components of displacement, $w_{qw}^m(r,z,\omega)$ and $u_{qw}^m(r,z,\omega)$ of the reflected and transmitted waves can be calculated using the definitions of the potentials in equation (1) in equations (20)-(23). Thus, in the upper half space the reflected particle displacements may be written as

$$u_{P0P0}^{0} = -ik_{\alpha_0} \sin\gamma_0\, \phi_0^{+} \tag{24}$$

$$w_{P0P0}^{0} = ik_{\alpha_0} \cos\gamma_0\, \phi_0^{+} \tag{25}$$

$$u_{P0S0}^{0} = -ik_{\beta_0} \cos\zeta_0\, \psi_0^{+} \tag{26}$$

$$w_{P0S0}^{0} = -ik_{\beta_0} \cos\zeta_0\, \psi_0^{+} \tag{27}$$

and in the lower half space the transmitted particle displacements have the form

$$u_{P0Pn}^{n} = -ik_{\alpha_n} \sin\gamma_n\, \phi_n^{-} \tag{28}$$

$$w_{P0Pn}^{n} = -ik_{\alpha_n} \cos\gamma_n\, \phi_n^{-} \tag{29}$$

$$u_{P0Sn}^{n} = ik_{\beta_n} \sin\zeta_n\, \psi_n^{-} \tag{30}$$

$$w_{P0Sn}^{n} = -ik_{\beta_n} \sin\zeta_n\, \psi_n^{-}\ . \tag{31}$$

The expressions obtained above discussion are very similar in form to those obtained using asymptotic ray theory for two half spaces in welded contact. The factor $1/R_i$ characterizes the geometrical spreading of the wave front and the exponential term defines the phase shift of the wave. The reflection and transmission coefficients, however, of a plane wave impinging on a first order interface are not frequency dependent while those in formulae (20)-(23) are dependent on frequency.

As the phase of the reflection and transmission coefficients was not taken into account in the stationary phase approximations, the integrals (20)-(23) are inexact everywhere the arguments of the coefficients vary rapidly with angle. The stationary phase approximation can then be assumed to be valid for values of an angle, γ, where the following conditions hold

$$\left| \frac{d\left(\arg \tilde{R}_{qw}\right)}{d\gamma} \right| << r\,\frac{dL}{d\gamma} \tag{32}$$

and

$$\left| \frac{d\left(\arg \tilde{T}_{qw}\right)}{d\gamma} \right| << r\,\frac{dL}{d\gamma}\ . \tag{33}$$

The inclusion of the phase of the reflection and transmission coefficients in the exponent used to calculate the points of stationary phase would be much more accurate. However, this further refinement would require the numerical calculation of the second derivative of the reflection and transmission coefficients which presents a computational problem whose solution, in computer time, approaches that of direct numerical differentiation.

As a final topic in this section the relationship between potential and particle displacement reflection and transmission coefficients will briefly be considered. Consider the PS displacement vector which may be written as

$$\mathbf{u}_{P_0S_0} = \left(u_{P_0S_0}, w_{P_0S_0}\right) \tag{34}$$

so that the argument of this complex vector has the form

$$\left|\mathbf{u}_{P_0S_0}\right| = \sqrt{\mathbf{u}_{P_0S_0} \cdot \mathbf{u}^*_{P_0S_0}} = k_{\beta_0}\psi_0^+ \tag{35}$$

where the superscript "*" indicates complex conjugate. The far field approximation to the incident *P* wave has the form

$$\mathbf{u}_0 = \left(u_0, w_0\right) \approx -ik_{\alpha_0}\left(\sin\gamma_0\,\phi_0^+, \cos\gamma_0\,\phi_0^+\right) \tag{36}$$

that has the argument

$$\left|\mathbf{u}_0\right| = \sqrt{\mathbf{u}_0 \cdot \mathbf{u}_0^*} = k_{\alpha_0}\phi_0^+ . \tag{37}$$

At any point on the interface at $z = H = 0$ the ratio of the reflected PS wave to the incident is just the particle displacement reflectivity, $R_{P0S0}\left(\omega,k\right)$ yielding

$$R_{P0S0}\left(\omega,k\right) = \frac{\beta_0}{\alpha_0}\tilde{R}_{P0S0}\left(\omega,k\right) . \tag{38}$$

Thus for the definition of the displacement in term of the potentials used here the particle displacement reflectivities and transmittivities are those derived using potentials scaled by a factor of the ratio of the velocity associated with the reflected or transmitted wave type to the incident wave type.

ANALOGUES OF REFLECTION AND TRANSMISSION COEFFICIENTS

A thin layered or transition zone composed of $n-1$ homogeneous isotropic layers will be assumed to lie between two homogeneous isotropic halfspaces which will be designated as $0\left(upper\right)$ and $n\left(lower\right)$. The vertical axis will be chosen positive downwards with $z=0$ coinciding with the top of the thin layered zone with this interface denoted as 0. The interfaces at layer boundaries between the two half spaces are numbered sequentially from 1 to $n-1$ with the interface n defining the interface with the bottommost thin layer and the underlying halfspace (Figure 3).

The $n+1$ thin layers and half spaces will be characterized by their P and S_V velocities, α_m and β_m as well as their densities, $\rho_m\left(0 \le m \le n\right)$. The $n-1$ thin layers have thicknesses of d_m; z_m and z_{m-1} are the depths of the lower and upper interfaces of the m^{th} layer. This has

$$d_m = z_m - z_{m-1} > 0 \quad (1 \le m \le n-1) \tag{39}$$

The compressional and shear wave velocities, α_m and β_m, may be expressed in terms of Lame's coefficients, λ_m and μ_m, and the densities, ρ_m in the standard manner

$$\alpha_m^2 = \frac{\lambda_m + 2\mu_m}{\rho_m} \tag{40}$$

$$\beta_m^2 = \frac{\mu_m}{\rho_m} \tag{41}$$

With the particle displacement vector in the m^{th} layer defined as $\mathbf{u}_m = (u_m, w_m)$

$$u_m = \frac{\partial \phi_m}{\partial x} - \frac{\partial \psi_m}{\partial z} \tag{42}$$

$$w_m = \frac{\partial \phi_m}{\partial z} + \frac{\partial \psi_m}{\partial x} \tag{43}$$

Utilizing these potentials, the equations of motion in each layer can be decomposed into the two following wave equations

$$\nabla^2 \phi_m - \frac{1}{\alpha_m^2}\frac{\partial^2 \phi_m}{\partial t^2} = 0 \quad (P \text{ wave propagation}) \tag{44}$$

$$\nabla^2 \psi_m - \frac{1}{\beta_m^2}\frac{\partial^2 \psi_m}{\partial t^2} = 0 \quad (S_V \text{ wave propagation}) \tag{45}$$

Defining the double Fourier transform and its inverse as

$$\overline{f}(k,z,\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,z,t)\exp\left[-i(\omega t + kx)\right] dx\, dt \tag{46}$$

$$f(x,z,t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \overline{f}(k,z,\omega)\exp\left[-i(\omega t + kx)\right] dk\, d\omega \tag{47}$$

the t and x dependence is removed from equations (44) and (45) so that the transformed solutions in the m^{th} $(0 \le m \le n)$ layer or half space may be written as

$$\bar{\phi}_m = A_m \exp\left[i\eta_m\left(z_m - z_{m-1}\right)\right] + B_m \exp\left[-i\eta_m\left(z_m - z_{m-1}\right)\right] = \phi_m^- + \phi_m^+ \tag{48}$$

$$\bar{\psi}_m = C_m \exp\left[i\xi_m\left(z_m - z_{m-1}\right)\right] + D_m \exp\left[-i\xi_m\left(z_m - z_{m-1}\right)\right] = \psi_m^- + \psi_m^+ \tag{49}$$

$$\eta_m = \begin{cases} \left(\dfrac{\omega^2}{\alpha_m^2} - k^2\right)^{1/2} & \dfrac{\omega}{\alpha_m} \geq k \\ -i\left(\left|\dfrac{\omega^2}{\alpha_m^2} - k^2\right|\right)^{1/2} & \dfrac{\omega}{\alpha_m} < k \end{cases} \tag{50}$$

$$\xi_m = \begin{cases} \left(\dfrac{\omega^2}{\beta_m^2} - k^2\right)^{1/2} & \dfrac{\omega}{\beta_m} \geq k \\ -i\left(\left|\dfrac{\omega^2}{\beta_m^2} - k^2\right|\right)^{1/2} & \dfrac{\omega}{\phi_m} < k \end{cases} \tag{51}$$

The choice of signs for ω/α_m and ω/β_m less than k was such that the physical condition is that the potentials vanish as $z \to \pm\infty$. The potentials ϕ_m^- and ψ_m^- refer to the upward propagating waves while ϕ_m^+ and ψ_m^+ represent downward propagating waves. The quantities A_m, B_m, C_m and D_m are complex valued functions of ω and k, and are also dependent on the layer's parameters α_m, β_m, ρ_m and d_m. These coefficients may be computed from the boundary conditions of continuity of shear and normal stress and the vertical and horizontal components of particle displacement.

Following the standard propagator matrix method described in Aki and Richards (1980) or Brekhovskikh (1980) and the numerous references contained in that work, two vectors can be defined, a potential vector

$$\Phi(z) = \left[\phi_m^-(z), \psi_m^-(z), \phi_m^+(z), \psi_m^+(z)\right]^T \tag{52}$$

and a stress-strain vector

$$S_m(z) = \left[\bar{u}_m(z), \bar{w}_m(z), \bar{\sigma}_{zz_m}(z), \bar{\sigma}_{xz_m}(z)\right]^T \tag{53}$$

for $z_{m-1} \leq z < z_m$.

Here $\bar{\sigma}_{zz_m}(z)$ and $\bar{\sigma}_{xz_m}(z)$ are the transformed components of normal and shear stress, obtained by applying the transform defined in equation (46), which will be used in determining the boundary conditions. At every point in the m^{th} layer the two vectors defined above are related as

$$S_m(z) = \mathbf{T}_m \, \phi_m(z) \tag{54}$$

or solving for $\phi_i(z)$

$$\phi_m(z) = \mathbf{T}_m^{-1} \, S_m(z) \, . \tag{55}$$

The matrix $\mathbf{T}_m$ and its inverse $\mathbf{T}_m^{-1}$ have the following definition

$$\mathbf{T}_m = \begin{bmatrix} ik & -i\xi_m & ik & -i\xi_m \\ i\eta_m & ik & -i\eta_m & ik \\ \mu_m \ell_m & -2\mu_m k\xi_m & \mu_m \ell_m & 2\mu_m k\xi_m \\ -2\mu_m k\eta_m & -\mu_m \ell_m & 2\mu_m k\eta_m & -\mu_m \ell_m \end{bmatrix} \tag{56}$$

$$\mathbf{T}_m^{-1} = \frac{\beta_m^2}{2\mu_m \xi_m \eta_m \omega^2} \begin{bmatrix} -i2\mu_m k\xi_m\eta_m & i\mu_m \ell_m \xi_m & -\xi_m\eta_m & -k\xi_m \\ i\mu_m \ell_m \eta_m & -i2\mu_m k\xi_m\eta_m & -k\eta_m & \xi_i\eta_m \\ -i2\mu_m k\xi_m\eta_m & i\mu_m \ell_m \xi_m & -\xi_m\eta_m & k\xi_m \\ i\mu_m \ell_m \eta_m & -i2\mu_m k\xi_m\eta_m & k\eta_m & \xi_m\eta_m \end{bmatrix} \tag{57}$$

where ℓ_m is given by

$$\ell_m = 2k^2 - \frac{\omega^2}{\beta_m^2} \tag{58}$$

The vectors $\mathbf{\Phi}_m(z_m)$ and $\mathbf{\Phi}_m(z_{m-1})$ at the lower and upper boundaries of the m^{th} layer are related as

$$\mathbf{\Phi}_m(z_m) = \mathbf{E}_m \, \mathbf{\Phi}_m(z_{m-1}) \, . \tag{59}$$

The diagonal matrix $\mathbf{E}_m$ has the form

$$\mathbf{E}_i = \begin{bmatrix} e^{i\eta_m d_m} & 0 & 0 & 0 \\ 0 & e^{i\xi_m d_m} & 0 & 0 \\ 0 & 0 & e^{-i\eta_m d_m} & 0 \\ 0 & 0 & 0 & e^{-i\xi_m d_m} \end{bmatrix} \tag{60}$$

The boundary conditions for the continuity of displacement and stress at the n interfaces implies continuity of the vectors S_m at these interfaces such that

$$S_m(z_i) = S_{m+1}(z_m) \qquad (m = 0,1, \ldots n-1). \tag{61}$$

It shall be assumed that the only energy incident on the thin layered zone is due to a P wave incident in the upper half space $(\phi_0^+ \neq 0)$. Consequently,

$$\phi_n^- = \psi_n^- = 0 \tag{62}$$

and

$$\psi_0^+ = 0. \tag{63}$$

The treatments of the three other possible type of incidence at the quasi-interface closely resemble that of P wave incidence from the upper half space and will not be considered here.

If the amplitude of the incident P wave front is set equal to unity, the amplitudes of the reflected P and S_V waves in the upper half space at $z = 0$ are, from equations (48) and (49)

$$A_0 = \tilde{R}_{P0P0}(k,\omega) \tag{64}$$

$$C_0 = \tilde{R}_{P0S0}(k,\omega) \tag{65}$$

and the corresponding transmitted P and S_V waves in the lower halfspace at $z = z_n$ are

$$B_n = \tilde{T}_{P0Pn}(k,\omega) \tag{66}$$

$$D_n = \tilde{T}_{P0Sn}(k,\omega). \tag{67}$$

The coefficients $\tilde{R}_{P0P0}$ and $\tilde{R}_{P0S0}$ are the reflectivities of the thin layer zone due to an incident P wave in the upper half space while $\tilde{T}_{P0Pn}$ and $\tilde{T}_{P0Sn}$ are the corresponding transmittivities due the same incident wave type. These reflectivities and transmittivities are analogous to reflection and transmission coefficients at a sharp boundary where the elastic parameters change discontinuously. It is to be remembered that the assumption in all of this is that the individual layers within these zones are small when compared to the predominant wavelength associated with the source pulse. A less rigid assumption is that the total thickness of the thin layered zone should be less than, or of the order of, the predominant wavelength.

Using these coefficients the potential vectors $\Phi_0(0)$ and $\Phi_n(z_n)$ at the upper and lower edges of the transition zone may be written as

$$\Phi_0(0) = \left[\tilde{R}_{P0P0}, \tilde{R}_{P0S0}, 1, 0\right]^T \tag{68}$$

$$\Phi_n(z_n) = \left[0, 0, \tilde{T}_{P0Pn}, \tilde{T}_{P0Sn}\right]^T. \tag{69}$$

It will next be shown that these two vectors may be related by elementary matrix operations.

From equation (16) the potential vector $\Phi_0(0)$ at the upper boundary of the transition zone is related to the stress-strain vector $S_0(0)$ by

$$S_0(0) = \mathbf{T}_0 \, \Phi_0(0) \tag{70}$$

As $S_0(0)$ is continuous across the interface at $z = 0$, $S_1(0) = S_0(0)$, so that

$$S_1(0) = \mathbf{T}_0 \, \Phi_0(0) \tag{71}$$

In layer 1, S_1 is related to Φ_1 yielding

$$\Phi_1(0) = \mathbf{T}_1^{-1} S_1(0) = \mathbf{T}_1^{-1} \mathbf{T}_0 \, \Phi_0(0) \tag{72}$$

with the superscript "-1" indicating the inverse. Taking Φ_1 to the lower boundary of the first layer has

$$\Phi_1(z_1) = \mathbf{E}_1 \Phi_1(0) = \mathbf{E}_1 \mathbf{T}_1^{-1} \mathbf{T}_0 \, \Phi_0(0). \tag{73}$$

From this the stress-strain vector at $z = z_1$ is

$$S_1(z_1) = \left[\mathbf{T}_1 \mathbf{E}_1 \mathbf{T}_1^{-1}\right] \mathbf{T}_0 \, \Phi_0(0). \tag{74}$$

The quantity

$$\mathbf{G}_1 = \mathbf{T}_1 \mathbf{E}_1 \mathbf{T}_1^{-1} \tag{75}$$

is propagator matrix of the first layer (Aki and Richards, 1980) so that (75) may be written as

$$S_1(z_1) = \mathbf{G}_1 \mathbf{T}_0 \, \Phi_0(0) \tag{76}$$

Extending this technique, the stress-strain vector $S_i(z_i)$ at the lower edge of the m^{th} layer is related to the potential $\Phi_0(0)$ through the relation

$$S_m(z_m) = [\mathbf{G}_m \mathbf{G}_{m-1} \mathbf{G}_{m-2} \cdots \mathbf{G}_3 \mathbf{G}_2 \mathbf{G}_1] \mathbf{T}_0 \, \Phi_0(0) \qquad (1 \le i \le n-1) \tag{77}$$

where the propagator matrix in some arbitrary j^{th} layer is defined as

$$\mathbf{G}_j = \mathbf{T}_j \mathbf{E}_j \mathbf{T}_j^{-1} \tag{78}$$

The elements of the 4×4 propagator matrix $\mathbf{G}_i$ are explicitly given as:

$$\begin{aligned}
g_m^{11} &= g_m^{44} = -\gamma_m \cos P_m + (\gamma_m + 1)\cos Q_m \\
g_m^{11} &= g_m^{44} = -\gamma_m \cos P_m + (\gamma_m + 1)\cos Q_m \\
g_m^{12} &= g_m^{34} = i\left[\left((\gamma_m + 1)\sin P_m \right) / r_{\alpha_m} + \gamma_m r_{\beta_m} \sin Q_m \right] \\
g_m^{13} &= g_m^{24} = i\left[(\cos Q_m - \cos P_m) / (\rho_m c\omega) \right] \\
g_m^{14} &= \left(\sin P_m / r_{\alpha_m} + r_{\beta_m} \sin Q_m \right) / (\rho_m c\omega) \\
g_m^{21} &= g_m^{43} = -i\left[\gamma_m r_{\alpha_m} \sin P_m + (\gamma_m + 1)\sin Q_m / r_{\beta_m} \right] \\
g_m^{22} &= g_m^{33} = (\gamma_m + 1)\cos P_m - \gamma_m \cos Q_m \\
g_m^{23} &= -\left[\left(r_{\alpha_m} \sin P_m - \sin Q_m / r_{\beta_m} \right) / (\rho_m c\omega) \right] \\
g_m^{31} &= g_m^{42} = i\rho_m c\omega\gamma_m (\gamma_m + 1)(\cos P_m - \cos Q_m) \\
g_m^{32} &= \rho_m c\omega\left[\left((\gamma_m + 1)^2 / r_{\alpha_m} \right)\sin P_m + \gamma_m^2 r_{\beta_m} \sin Q_m \right] \\
g_m^{41} &= -\rho_m c\omega\left[\gamma_m^2 r_{\alpha_m} \sin P_m + \left((\gamma_m + 1)^2 / r_{\beta_m} \right)\sin Q_m \right]
\end{aligned} \tag{79}$$

$$c = \frac{\omega}{k} = \frac{1}{p} \tag{80}$$

$$P_m = k r_{\alpha_m} d_m \tag{81}$$

$$Q_m = k r_{\beta_m} d_m \tag{82}$$

$$\gamma_m = \frac{-2\beta_m^2}{c^2} = -2\beta_m^2 p^2 \tag{83}$$

$$r_{\alpha_m} = \begin{cases} \left(\dfrac{c^2}{\alpha_m^2} - 1 \right)^{1/2} & c \ge \alpha_m \\ -i\left(1 - \dfrac{c^2}{\alpha_m^2} \right)^{1/2} & c < \alpha_m \end{cases} \tag{84}$$

$$r_{\beta_m} = \begin{cases} \left(\dfrac{c^2}{\beta_m^2} - 1\right)^{1/2} & c \geq \beta_m \\ -i\left(1 - \dfrac{c^2}{\beta_m^2}\right)^{1/2} & c < \beta_m \end{cases} \tag{85}$$

Thus at the interface defined by the lower boundary of the thin layered zone and the upper boundary of the lower half space the following holds

$$S_n\left(z_{n-1}\right) = S_{n-1}\left(z_{n-1}\right) = \mathbf{G}_{n-1}\mathbf{G}_{n-2}\cdots\mathbf{G}_2\mathbf{G}_1\mathbf{T}_0\,\Phi_0\left(0\right) \tag{86}$$

By transforming to the potential vector $\Phi_n\left(z_{n-1}\right)$ the following desired vector-matrix relationship with $\Phi_0\left(0\right)$ is obtained

$$\Phi_n\left(z_{n-1}\right) = \mathbf{M}\,\Phi_0\left(0\right) \tag{87}$$

with

$$\mathbf{M} = \mathbf{T}_n^{-1}\mathbf{G}_{n-1}\mathbf{G}_{n-2}\cdots\mathbf{G}_2\mathbf{G}_1\mathbf{T}_0 \,. \tag{88}$$

The elements of the 4×4 matrix $\mathbf{M}$ may be computed through a sequence of matrix multiplications. Substituting the expressions for $\Phi_0\left(0\right)$ and $\Phi_n\left(z_{n-1}\right)$ from equations (68) and (69) into (87) results in the following relationship between $\tilde{R}_{P0P0}$, $\tilde{R}_{P0S0}$, $\tilde{T}_{P0Pn}$ and $\tilde{T}_{P0Sn}$ is obtained which allows for these quantities to be computed as

$$\begin{bmatrix} 0 \\ 0 \\ \tilde{T}_{P0Pn} \\ \tilde{T}_{P0Sn} \end{bmatrix} = \mathbf{M} \begin{bmatrix} \tilde{R}_{P0P0} \\ \tilde{R}_{P0S0} \\ 1 \\ 0 \end{bmatrix} \tag{89}$$

Writing the 4×4 matrix in terms of the 2×2 sub-matrices $\mathbf{M}_{ij}$ as

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} \tag{90}$$

results in the following two equations for the unknown reflectivities and transmittivities

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{M}_{11} \begin{bmatrix} \tilde{R}_{P0P0} \\ \tilde{R}_{P0S0} \end{bmatrix} + \mathbf{M}_{12} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{91}$$

and

$$\begin{bmatrix} \tilde{T}_{P0Pn} \\ \tilde{T}_{P0Sn} \end{bmatrix} = \mathbf{M}_{21} \begin{bmatrix} \tilde{R}_{P0P0} \\ \tilde{R}_{P0S0} \end{bmatrix} + \mathbf{M}_{22} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{92}$$

The above two equations yield

$$\begin{bmatrix} \tilde{R}_{P0P0} \\ \tilde{R}_{P0S0} \end{bmatrix} = -\mathbf{M}_{11}^{-1}\,\mathbf{M}_{21} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{93}$$

and

$$\begin{bmatrix} \tilde{T}_{P0Pn} \\ \tilde{T}_{P0Sn} \end{bmatrix} = \left(-\mathbf{M}_{21}\,\mathbf{M}_{11}^{-1}\,\mathbf{M}_{12} + \mathbf{M}_{22} \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{94}$$

Defining the inverse of the sub-matrix $\mathbf{M}_{11}$ in terms of its determinant and transposed cofactor, indicated by the "^" circumflex, results in

$$\mathbf{M}_{11}^{-1} = \frac{\hat{\mathbf{M}}_{11}}{\det\left(\mathbf{M}_{11}\right)}. \tag{95}$$

The transposed cofactor is defined in terms of individual components of the matrix $\mathbf{M}_{11}$ is of the form

$$\hat{\mathbf{M}}_{11} = \begin{bmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{bmatrix} \tag{96}$$

with the determinant given in standard form as

$$\det\left(\mathbf{M}_{11}\right) = m\Big|_{12}^{12} = m_{11}m_{22} - m_{12}m_{21} \tag{97}$$

In what follows the notation abbreviation

$$m\Big|_{k\ell}^{ij} = m_{ik}m_{j\ell} - m_{i\ell}m_{jk} \tag{98}$$

will be used for the sub-determinant of order 2 (second order minors). With this notation the reflection coefficients (reflectivities) may be written as

$$\tilde{R}_{P0P0} = \frac{m\Big|_{23}^{12}}{m\Big|_{12}^{12}} \tag{99}$$

$$\tilde{R}_{P0S0} = -\frac{m\Big|_{13}^{12}}{m\Big|_{12}^{12}} \tag{100}$$

and the related transmission coefficients (transmittivities) as

$$\tilde{T}_{P0Pn} = m_{31}\tilde{R}_{P0P0} + m_{32}\tilde{R}_{P0S0} + m_{33} \tag{101}$$

$$\tilde{T}_{P0Sn} = m_{41}\tilde{R}_{P0P0} + m_{42}\tilde{R}_{P0S0} + m_{43} \tag{102}$$

In a similar manner to the procedure used above it is possible to compute the reflectivities and transmittivities for the three other possible cases of incidence at a thin layered boundary. Without any derivation they may be written as

For S_V incidence from the upper $\left(0^{th}\right)$ layer:

$$\tilde{R}_{S0S0} = -\frac{m_{14}^{12}}{m_{12}^{12}} \tag{103}$$

$$\tilde{R}_{S0P0} = \frac{m_{24}^{12}}{m_{12}^{12}} \tag{104}$$

$$\tilde{T}_{S0Pn} = m_{31}S_0P_0 + m_{32}S_0S_0 + m_{34} \tag{105}$$

$$\tilde{T}_{S0Sn} = m_{41}S_0P_0 + m_{42}S_0S_0 + m_{44} \tag{106}$$

For P incidence from the lower $\left(n^{th}\right)$ layer:

$$R_{PnPn} = \frac{m_{21}^{23}}{m_{12}^{12}} \tag{107}$$

$$R_{PnSn} = -\frac{m_{21}^{24}}{m_{12}^{12}} \tag{108}$$

$$\tilde{T}_{PnP0} = m_{31}\tilde{R}_{PnPn} + m_{32}\tilde{R}_{PnSn} + m_{33} \tag{109}$$

$$\tilde{T}_{P0Sn} = m_{41}\tilde{R}_{PnPn} + m_{42}\tilde{R}_{PnSn} + m_{43} \tag{110}$$

For S_V incidence from the lower $\left(n^{th}\right)$ layer:

$$\tilde{R}_{SnSn} = \frac{m_{12}^{14}}{m_{12}^{12}} \tag{111}$$

$$\tilde{R}_{SnPn} = -\frac{m_{13}^{12}}{m_{12}^{12}} \tag{112}$$

$$\tilde{T}_{SnP0} = m_{31}\tilde{R}_{SnPn} + m_{32}\tilde{R}_{SnSn} + m_{34} \tag{113}$$

$$\tilde{T}_{SnP0} = m_{41}\tilde{R}_{SnSn} + m_{42}\tilde{R}_{SnPn} + m_{43} \tag{114}$$

As mentioned several times in the text of this report, the tildes above the reflectivities and transmittivities indicate that these coefficients are potential coefficients and not particle displacement coefficients. If v_I is the velocity associated with the incident wave front type and $v_{R/T}$ is the velocity of the reflected or transmitted wave front type, then the particle displacement reflectivities and transmittivities may be written in terms of their potential counterparts (for the potential definitions of particle displacement used here) as

$$R_{I,R/T} = \frac{v_{R/T}}{v_I}\tilde{R}_{I,R/T} \qquad (115)$$

Two simple examples of equation (115) are

$$R_{P0P0} = \frac{\alpha_0}{\alpha_0}\tilde{R}_{P0P0} = \tilde{R}_{P0P0} \qquad (116)$$

and

$$R_{SnSn} = \frac{\beta_n}{\beta_n}\tilde{R}_{SnSn} = \tilde{R}_{SnSn} \qquad (116)$$

Other examples, where the velocity ratio is not unity, are the displacement reflectivities and transmittivities, R_{S0P0} and T_{PnS0}, which are given in terms of their corresponding potential counterparts as

$$R_{S0P0} = \frac{\alpha_0}{\beta_0}\tilde{R}_{S0P0} \qquad (117)$$

and

$$T_{PnS0} = \frac{\beta_0}{\alpha_n}\tilde{T}_{PnS0} \qquad (118)$$

In a similar manner all of the 16 possible displacement reflectivities and transmittivities may be ex pressed in terms of the derived quantities, the displacement potential reflectivities and transmittivities.

CONCLUSIONS

Utilizing what have been presumed to be some of the best qualities of both asymptotic ray theory and the reflectivity method, a hybrid method has been developed. It is employed to consider the seismic response from a plane layered isotropic structure composed of thick layers, where asymptotic ray methods are used, separated by thin layered zones, where the validity of asymptotic ray theory is questionable and reflectivity (matrix) methods are used. The fact that the computation time is increased as the computation of synthetic sections must be done in the frequency domain is somewhat compensated for when models with large number of layers are considered as the number of rays that are required to be used can be significantly reduced.

This method incorporates the flexibility of asymptotic ray theory, including the ability to identify individual *thick* layered arrivals, with accuracy comparable to the reflectivity method without the need for numerical integration. As the reflectivity method produces the total wave field response arrival identification is difficult without resorting to producing travel time tables usually by a two point ray tracing algorithm which is a significant part of the cost of using the ray reflectivity method.

To adequately indicate the potential of this method separate reports are being prepared for presenting numerical results.

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REFERENCES

Aki, K. and Richards, P.G., 1980, Quantitative seismology: W.H. Freeman and Co.

Brekhovskikh, L.M., 1980, Waves in layered media, Academic Press.

Daley, P.F. and Hron, F., 1992, Application of the ray-reflectivity method to multilayered thin bed structures, Bull. Seism. Soc. Am., **82**, 914-933.

Daley, P.F. and Hron, F., 1990, The ray-reflectivity method with critically refracted waves, Bull. Seism. Soc. Am., **80**, 995-1013.

Daley, P.F. and Hron, F., 1990, Ray-reflectivity method for SH waves in stacks of thick and thin layers, Geophys. J. Roy. astr. Soc., **69**, 527-536.

Dunkin, J.W., 1965, Computation of modal solutions in layered elastic media at high frequencies, Bull. Seism. Soc. Am., **55**, 335-358.

Ewing, W.N., Jardetzky, W.S. and Press, F., 1957, Elastic waves in layered media, McGraw-Hill.

Fertig, J. and Muller, G., 1978, Computations of synthetic seismograms for coal seams with the reflectivity method, Geoph. Prosp., **26**, 868-878.

Fuchs, K., 1968, The reflectivity and transmittivity of a medium with arbitrary depth-dependent elastic moduli and density for plane waves at arbitrary incidence, Z, Geophys., **34**, 389-413. (In German.)

Fuchs, K., 1970, The method of stationary phase applied to the reflection of spherical waves from transition zones with arbitrary depth-dependent elastic moduli and density, Z, Geophys., **37**, 89-117.

Fuchs, K. and Muller, G., 1971, Computation of synthetic seismograms with the reflectivity method and comparison with observations, Geophys. J. Roy. astr. Soc., **23**, 417-438.

Haskell, N.A., 1953, The dispersion of surface waves in multilayered media, Bull. Seism. Soc. Am., **43**, 17-34.

Kennett, B.L.N., 1983, Seismic wave propagation, Cambridge University Press.

Muller, G. 1985, The reflectivity method: a tutorial, J. Geophys., **58**, 153-174.

Ratnikova, L.I., 1973, Methods of computation for seismic waves in thin layered media, NAUKA. (In Russian.)

Sommerfeld, A., 1949, Partial differential equations in physics, Academic Press.

Thomson, W.T., 1950, Transmission of elastic waves through a stratified solid medium, J. Appl. Phys., **21**, 89-93.

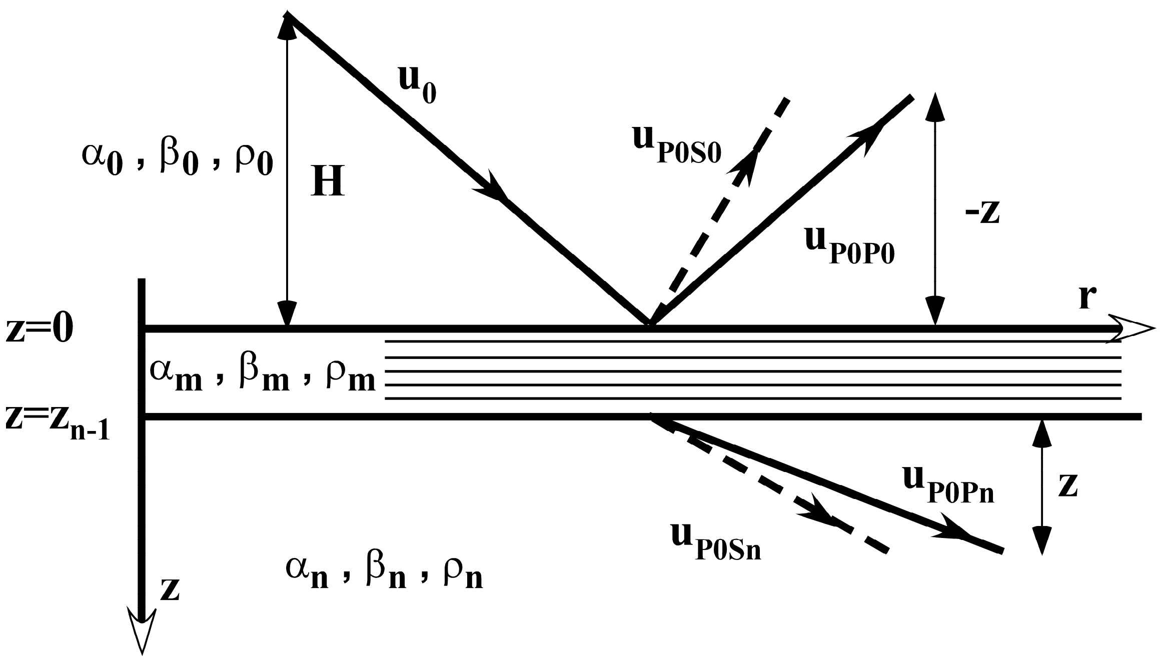

Figure 1. Geometry of problem with the reflected and transmitted particle displacements shown due to the incidence of a P wave from the upper halfspace on the thin layered zone.

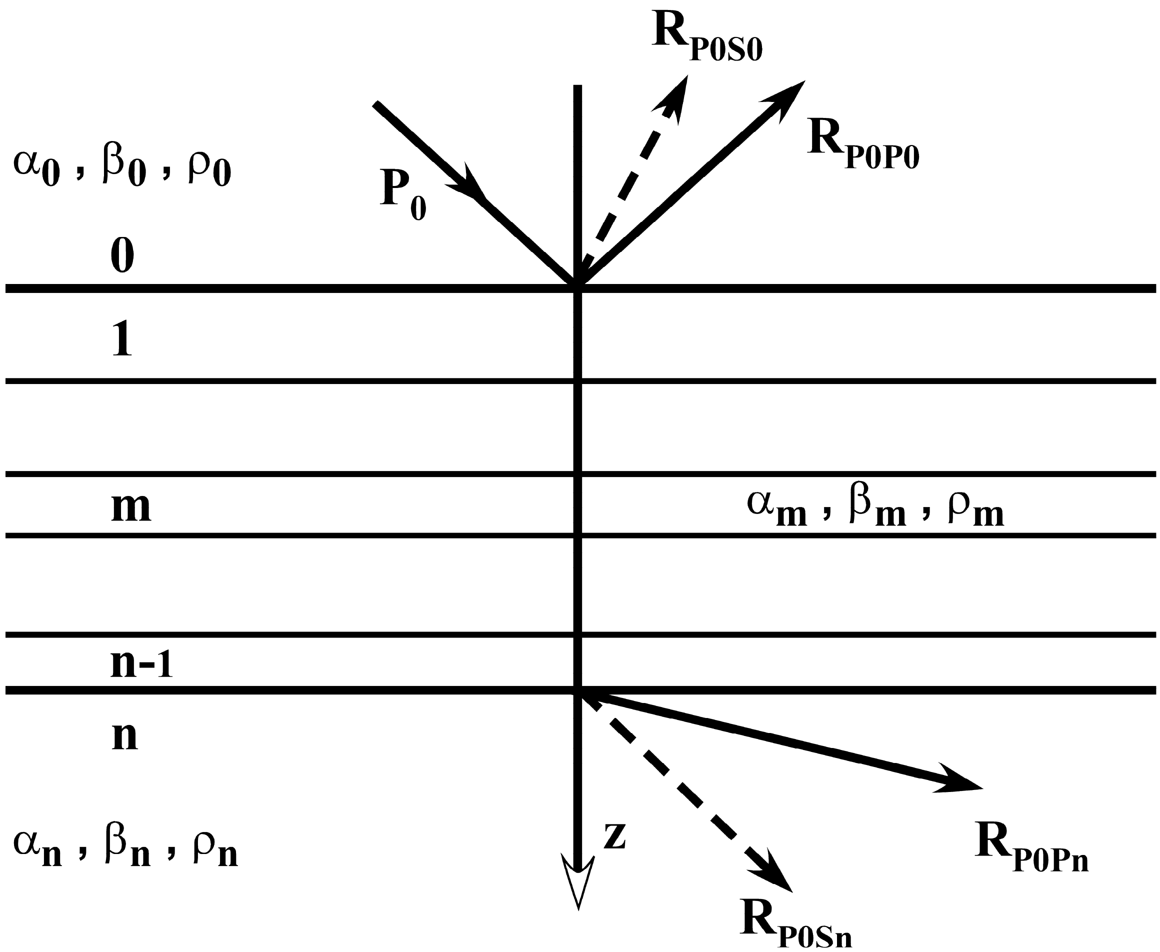

Figure 2. Notation used for the computation of the reflectivitities and transmittivities at a thin layered zone as a result of P wave incidence from the upper (0) halfspace.