A band-limited minimum phase calculation

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ABSTRACT

We look at the general example of computing a minimum phase signal with a band-limited spectrum, using an IIR filter. The key point is that no causal signal can be band-limited, so we find a stable alternative.

INTRODUCTION

This is a continuation of two previous papers in this CREWES report on minimum phase signals, wherein we have been discussing the mathematical structure of outer functions and their connection to minimum phase signals.

We address here the question of how to compute the minimum phase version of a band-limited signal. The basic difficulty is that somewhere in the typical Hilbert transform calculation, the logarithm of the amplitude spectrum must be calculated. With the zeros in the band, the logarithm is minus infinity, resulting in an integral that cannot be computed.

Standard practice is to add a stability factor to the amplitude spectrum, thus removing the zeros and allowing the calculation to proceed. However, it is worth noting that this stability factor is misnamed: while it allows the calculation to proceed, it does not produce a stable answer that is close to the desired minimum phase signal. In fact, no band-limited, minimum phase signal exists† so it is questionable as to what it means to calculate such a signal.

We take the following approach. Since the signal is band-limited, truncate its spectrum to a frequency interval $[-F_0, F_0]$ on which the amplitude spectrum is significantly non-zero. Use this amplitude data to create a sampled, minimum phase signal, with a Nyquist frequency of F_0. Then resample the data, passing it through a minimum phase lowpass filter to truncate any extraneous frequency components in the band above F_0.

This produces a minimum phase signal with spectrum equal to the desired valued in the frequency range $[-F_0, F_0]$, and with small amplitude spectra outside that range.

AN EXAMPLE

We demonstrate the procedure with a simple spectrum showing exponential decay, of the form

$$spec(\omega) = 100 * (\omega + .1)\exp(-20\omega).$$

The spectrum is shown in Figure 1, the top row indicating both spectrum (in normalized frequency), and the corresponding zero phase wavelet.

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†In fact, no band-limited, causal signals exist.

We truncate the spectrum at $\omega = 0.5$, having decided that the spectrum is close to zero outside that range. In the second row in Figure 1, we see the truncated and stretched out spectrum, and the corresponding minimum phase wavelet which was computed using the 'rceps' command in MATLAB. Note how this minimum phase wavelet is obviously too narrow a pulse, compared to the original zero phase wavelet. This is because it has been sampled at half the original sampling rate.

We now upsample this wavelet, by inserting a zero between every sample. The result is the widely oscillating waveform in row three, with a spectrum that shows mirror symmetry.

Finally, we lowpass filter the oscillating waveform, using in this case a 10th order Chebyshev II filter with 20dB decay in the stop band. The result is the waveform in row four, whose amplitude spectrum closely matches the spectrum of the zero phase wavelet.

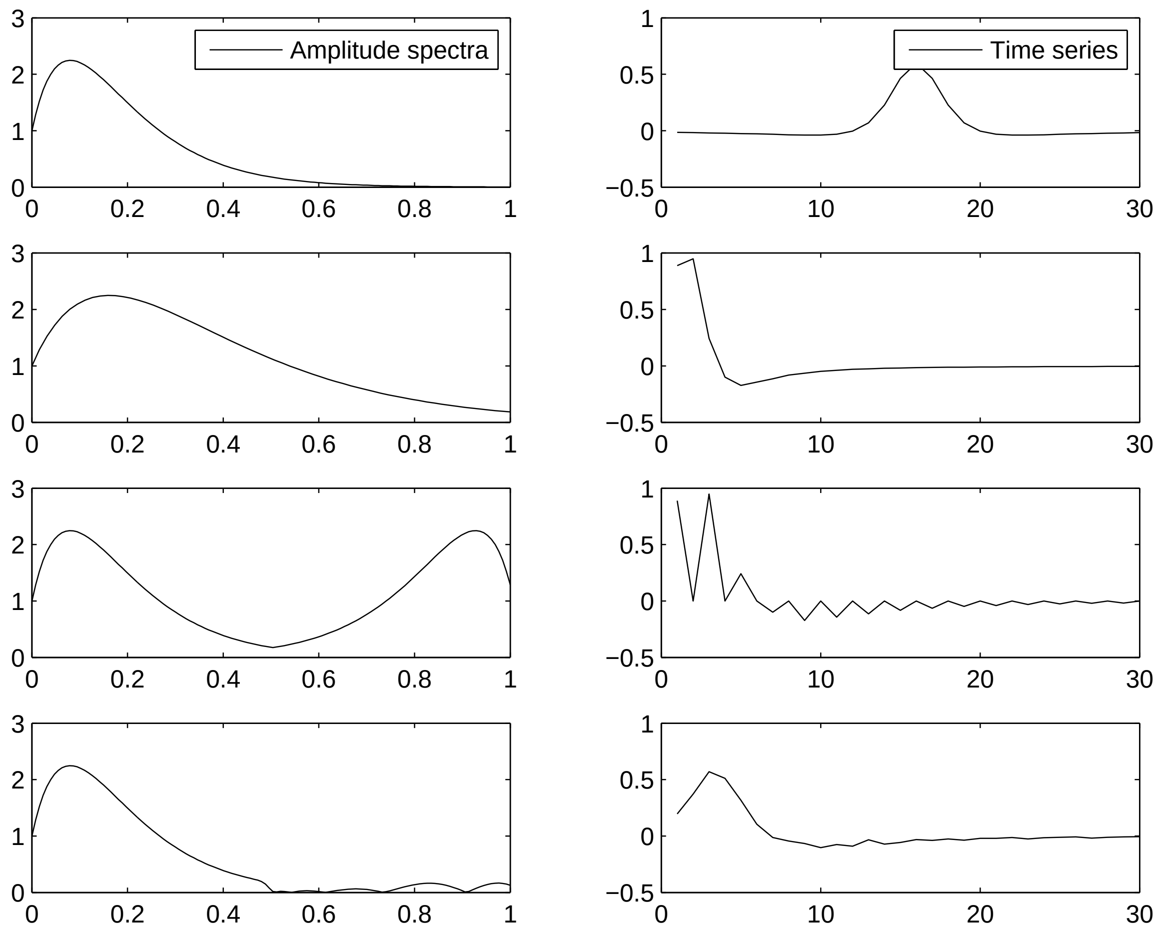

FIG. 1. Step-by-step creation of the band-limited minimum phase signal. Start with a linear phase pulse, whose spectrum shows exponential decay (first row). Truncate the spectrum to 0.5 Nyquist, where the decay is not too pronounced, and compute the minimum phase equivalent (second row). Upsample by zero sample insertion (third row) to produce an oscillatory waveform whose spectrum in range [0,.5] matches the spectrum of the original signal. Then lowpass filter through a minimum phase filter, to obtain the correct minimum phase signal (fourth row).

In Figure 2, we see a close up view of the bandlimited zero phase wavelet and its minimum phase counterpart.

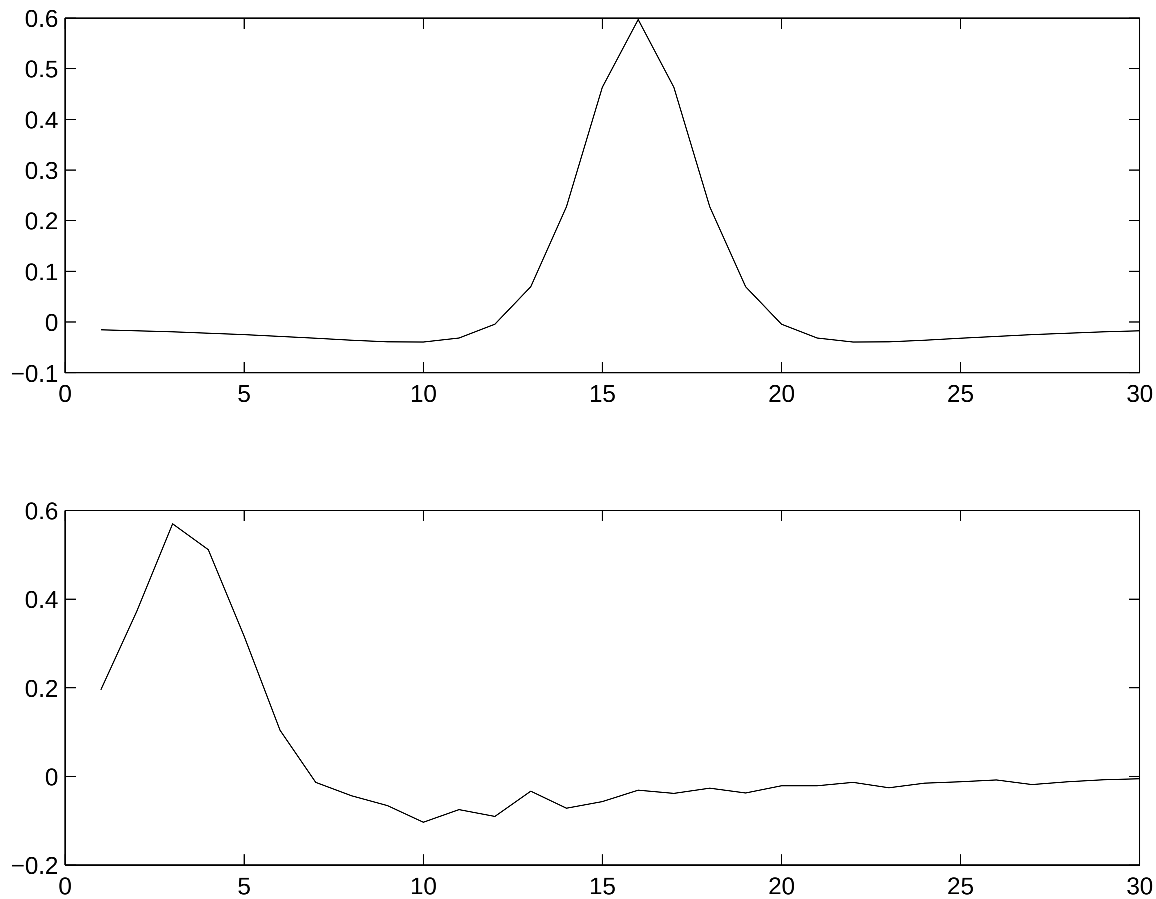

FIG. 2. A comparison of the zero phase band-limited wavelet and its roughly bandlimited minimum phase counterpart.

EFFECT OF THE STOP BAND

We recall that in the standard calculation of a bandlimited minimum phase signal, as the stability factor decreases to zero, the main pulse of the signal moves to the right. The same phenomena happens in our alternative formulation, except the rightward shift of the signal occurs due to the IIR lowpass filtering of the signal. Interpolation (rather than zero sample insertion and filtering) will also cause a shift; the higher the order, the more the shift. Essentially, it is the bandlimiting that causes the shift, not the details of the minimum phase calculation (rceps, or the Hilbert transform).

As an example, we see in Figure 3 the results of the same calculation in the last section, except we choose three different Chebyshev filters with a cutoff of 20dB, 60dB and 100dB in the stop band. (We also increased the order of the filter to 20 in all cases, so the constraints in the stop band could be met.) As we see in the figure, the main pulse moves to the right as the cutoff strength increases.

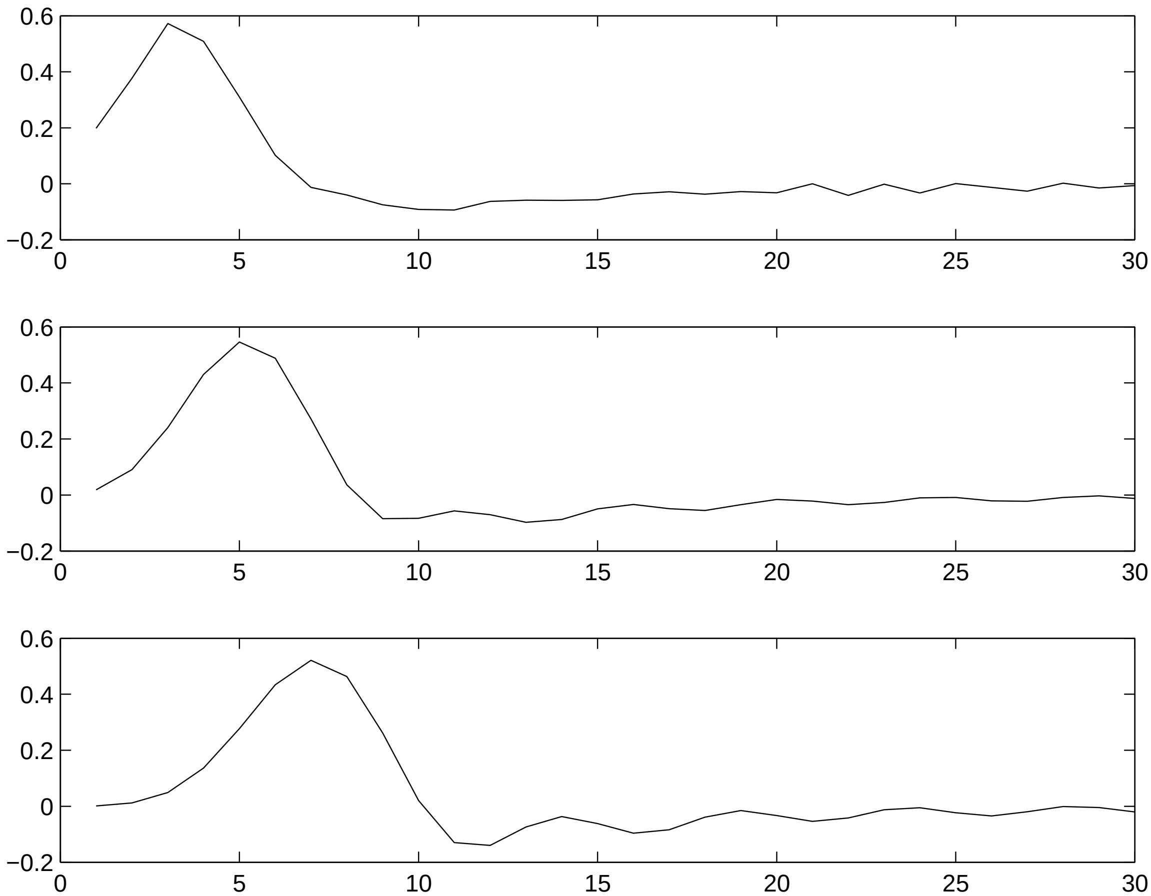

FIG. 3. Three computations of a bandlimited version of a minimum phase signal, using three different IIR filters. Top signal used a Chebyshev II with 20dB decay in the stop band; middle signal with 40dB decay; bottom signal with 100dB decay. Observe the main hump of the signal moves to the right as the cutoff strength increases. This is a property of the filter, not the minimum phase calculation.

SUMMARY

We have demonstrated how to use an IIR filter to build an approximate minimum phase version of a band-limited signal, avoiding the stabilization factors necessary in the Hilbert transform. Our numerical technique used the 'rceps' command in MATLAB (a Hilbert transform method) on a non-zero amplitude obtained by wrapping around the desired spectrum, and upsampling with minimum phase IIR filters generated. Delays in the resulting signal are shown to be a function of the IIR upsampling filter, not a property of the signal itself.

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