

# Radon transforms via truncated singular value decomposition

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## ABSTRACT

The Radon transform is a common tool used to attenuate multiple energy. The least-squares implementation commonly used introduces artifacts and alters the amplitudes of the primary events. A high-resolution methodology is developed based on the truncated singular value decomposition (TSVD). A synthetic example shows that the TSVD implementation is superior, leads to fewer artifacts, and removes multiples more effectively.

## INTRODUCTION

Most of the seismic migration methods used today in the industry are developed on the assumption that the seismic data contain only primary energy. However in reality the data are contaminated with different types of noise, coherent and incoherent. Multiples and ground roll are one example of coherent noise. The Radon transform (e.g. Yilmaz, 2001) is a standard tool used to remove coherent noise with linear, parabolic, and hyperbolic moveout. This process greatly improves the pre-stack migration velocity analysis. However it is well known that the current least-squares frequency domain implementation (Hampson, 1986) introduces artefacts in the data and alters the amplitudes of the primary events, which is an undesirable effect for AVO analysis (Kabir and Marfurt, 1999). In this paper, we review the theory of the least-squares Radon transform and we propose a new high-resolution implementation based on the truncated singular value decomposition (TSVD) (Aster et al., 2005). Through a synthetic example we show the TSVD implementation is superior to the least-squares one, leads to fewer artefacts and removes multiples more effectively.

## THEORY OF THE RADON TRANSFORM

The forward Radon transform  $u(\tau, q)$  of a 2-D function  $f(t, h)$  is defined by the integral expression (Beylkin, 1987; Yilmaz, 2001)

$$u(\tau, q) = \int_{-\infty}^{\infty} f(t = \tau + \varphi(q, h), h) dh \quad (1)$$

where  $t$  and  $h$  are input variables, and  $\tau$  and  $q$  are transform variables. The function  $\varphi(q, h)$  defines the integration path. In seismic applications,  $t$  is two-way travel time and  $h$  is offset. The inverse Radon transform is given by the integral expression

$$f(t, h) = \int_{-\infty}^{\infty} \rho(\tau) * u(\tau = t - \varphi(q, h), q) dq, \quad (2)$$

which is a convolution of  $u(\tau, q)$  with a rho filter. For 2-D data the rho filter is (Yilmaz, 2001)

$$\rho(\tau) = \sqrt{\omega} \exp(i\pi/4), \quad (3)$$

There are three forms of the Radon transform that are extensively used in seismic data analysis: linear, parabolic, and hyperbolic. The linear Radon transform is

$$u_{lin}(\tau, q) = \int_{-\infty}^{+\infty} d(t = \tau + qh, h) dh, \quad (4)$$

where  $q$  has the physical meaning of a ray parameter. It represents a summation along straight lines in the data domain. The parabolic Radon transform is

$$u_{par}(\tau, q) = \int_{-\infty}^{+\infty} d(t = \tau + qh^2, h) dh, \quad (5)$$

and represents a summation along parabolic curves. The hyperbolic Radon transform is defined by

$$u_{hyp}(\tau, q) = \int_{-\infty}^{+\infty} d(t = \sqrt{\tau^2 + h^2/v^2}, h) dh, \quad (6)$$

and represents a summation along hyperbolic curves dependant on the seismic velocity  $v$ .

It is important to stress that the Radon transform forward-inverse pair is a forward-adjoint pair. It means that by applying the forward transform and then inverting it does not recover the data exactly. The situation is worsened in practice because we must implement a discrete transform using a finite set of trajectories.

A practical solution to this problem was proposed by Thorson and Claerbout (1985) and Hampson (1986). In order to minimize the amplitude smearing on the conventional Radon space they used a least-squares formulation. For limited digital data the inverse Radon transform can be cast as a matrix operation

$$\mathbf{d}' = \mathbf{L}\mathbf{u}, \quad (7)$$

where  $\mathbf{d}'$  is the reconstructed data,  $\mathbf{u}$  is the Radon transform of the data  $\mathbf{d}$ , and  $\mathbf{L}$  is a linear operator. The forward Radon transform is then

$$\mathbf{u} = \mathbf{L}^T \mathbf{d}, \quad (8)$$

which is known as a low resolution solution. A least-squares solution is found by minimizing the error  $\mathbf{e}$  between the actual input data  $\mathbf{d}$  and the reconstructed data  $\mathbf{d}'$

$$\mathbf{e} = \mathbf{d} - \mathbf{d}' = \mathbf{d} - \mathbf{L}\mathbf{u}. \quad (9)$$

The standard least-squares solution to this problem is:

$$\mathbf{u} = (\mathbf{L}^T \mathbf{L})^{-1} \mathbf{L}^T \mathbf{d}. \quad (10)$$

Generally the process of finding the inverse of  $(\mathbf{L}^T \mathbf{L})$  is unstable. To deal with the instability usually the damped least-squares (DLS) solution is used (Yilmaz, 2001)

$$\mathbf{u} = (\mathbf{L}^T \mathbf{L} + \beta \mathbf{I})^{-1} \mathbf{L}^T \mathbf{d}, \quad (11)$$

where  $\beta$  is a constant, called the damping factor, and  $\mathbf{I}$  represents the identity matrix

There is a problem however with the practical implementation of this solution. The dimensions of the matrix  $\mathbf{L}$  is  $N_T \times N_H$  by  $N_T \times N_Q$ , where  $N_T$  is the number of time samples,  $N_H$  is the number of the offsets in a CMP or shot gather (fold),  $N_Q$  number of  $q$  values in Radon space. For example the dimensions of  $\mathbf{L}$  could be 150 000 by 150 000. To overcome this problem Hampson (1986) suggests a solution in frequency domain. The linear, discrete Radon transform is:

$$d'(t, h) = \sum_p u(\tau = t - ph, h). \quad (12)$$

By transforming the time domain variable  $t$  into frequency domain (Fourier transform) we get

$$d'(f, h) = \sum_p u(f, p) \exp(-i2\pi f p h). \quad (13)$$

For each component of the frequency  $f$  we can write the equation

$$\mathbf{d}' = \mathbf{L} \mathbf{u}, \quad (14)$$

where  $\mathbf{L}$  is a complex matrix with dimensions  $N_H$  by  $N_Q$

$$\begin{pmatrix} \exp(-i2\pi f q_1 h_1) & \exp(-i2\pi f q_2 h_1) & \cdots & \exp(-i2\pi f q_{N_Q} h_1) \\ \exp(-i2\pi f q_1 h_2) & \exp(-i2\pi f q_2 h_2) & \cdots & \exp(-i2\pi f q_{N_Q} h_2) \\ \vdots & \vdots & \ddots & \vdots \\ \exp(-i2\pi f q_1 h_{N_H}) & \exp(-i2\pi f q_2 h_{N_H}) & \cdots & \exp(-i2\pi f q_{N_Q} h_{N_H}) \end{pmatrix}.$$

In a similar way we can define  $\mathbf{L}$  for the parabolic Radon transform

$$\begin{pmatrix} \exp(-i2\pi f q_1 h_1^2) & \exp(-i2\pi f q_2 h_1^2) & \cdots & \exp(-i2\pi f q_{N_Q} h_1^2) \\ \exp(-i2\pi f q_1 h_2^2) & \exp(-i2\pi f q_2 h_2^2) & \cdots & \exp(-i2\pi f q_{N_Q} h_2^2) \\ \vdots & \vdots & \ddots & \vdots \\ \exp(-i2\pi f q_1 h_{N_H}^2) & \exp(-i2\pi f q_2 h_{N_H}^2) & \cdots & \exp(-i2\pi f q_{N_Q} h_{N_H}^2) \end{pmatrix},$$

and the hyperbolic Radon transform

$$\begin{pmatrix} \exp(-i2\pi f h_1^2 / v_1^2) & \exp(-i2\pi f h_1^2 / v_2^2) & \cdots & \exp(-i2\pi f h_1^2 / v_{N_Q}^2) \\ \exp(-i2\pi f h_2^2 / v_1^2) & \exp(-i2\pi f h_2^2 / v_2^2) & \cdots & \exp(-i2\pi f h_2^2 / v_{N_Q}^2) \\ \vdots & \vdots & \ddots & \vdots \\ \exp(-i2\pi f h_{N_H}^2 / v_1^2) & \exp(-i2\pi f h_{N_H}^2 / v_2^2) & \cdots & \exp(-i2\pi f h_{N_H}^2 / v_{N_Q}^2) \end{pmatrix}.$$

The low resolution Radon space is

$$\mathbf{u} = \mathbf{L}^H \mathbf{d}, \quad (15)$$

where H denotes hermitian of a complex matrix  $\mathbf{L}$ .

The least-squares solution in frequency domain is

$$\mathbf{u} = (\mathbf{L}^H \mathbf{L})^{-1} \mathbf{L}^H \mathbf{d}, \quad (16)$$

however in practice due to singularities in the matrix  $\mathbf{L}^H \mathbf{L}$  a constrained damped least-squares (DLS) is used

$$\mathbf{u} = (\mathbf{L}^H \mathbf{L} + \beta \mathbf{I})^{-1} \mathbf{L}^H \mathbf{d}. \quad (17)$$

This is a standard equation to compute the Radon transform.

### RADON TRANSFORM VIA TSVD

The stability problem of the matrix  $\mathbf{L}$  may be analysed by singular value decomposition (SVD) (Aster et al., 2005). Any matrix  $\mathbf{A}$  can be decomposed as

$$\mathbf{A} = \mathbf{U} \mathbf{S} \mathbf{V}^T, \quad (18)$$

where

$\mathbf{U}$  is an orthogonal matrix with column that are unit basis vectors spanning the data space

$\mathbf{V}$  is an orthogonal matrix with columns that are unit basis vectors spanning the model space

$\mathbf{S}$  is a diagonal matrix with nonnegative elements called singular values

Some of the singular values can be zeros or very close to zero. The existence of such values creates a stability problem for inverting  $\mathbf{L}^H \mathbf{L}$  and usually requires some kind of regularization. One way is the damped least-squares discussed previously. Another way to stabilize the solution is by truncated singular value decomposition (TSVD) (Aster et al., 2005). TSVD can recover a useful solution by truncating the singular values at some point when they approach zero. The lowest singular value can be picked by visually inspecting the singular values or by applying the discrepancy principal (Aster et al., 2005). Using the TSVD the Radon space can be found by the equation

$$\mathbf{u} = \mathbf{V}_k \mathbf{S}_k^{-1} \mathbf{U}_k^T \mathbf{d}, \quad (19)$$

where k is the index of the lowest singular value used to solve for  $\mathbf{u}$ .

### SYNTHETIC EXAMPLE

Hampson (1986) has shown that the parabolic radon transform can be used to attenuate multiples on moveout corrected CMP gathers. After moveout correction, the primary energy is flat, while the multiples have parabolic moveout. So by transforming the data into a parabolic radon space we can separate the primaries and the multiples,

mute the multiples and inverse back to time. This algorithm has been used in industry, however it introduces artefacts and fails to remove some of the multiple energy.

We developed a MATLAB code to implement, apply, and display three radon applications for multiple removal: low-resolution, damped least-squares, and TSVD. Figure 1 shows a model containing two primary events (Figure 1, a), one with constant amplitude and one with type II AVO anomaly. Three multiples with a parabolic moveout (Figure 1, b) were added to the primaries model (Figure 1, c). In practice we do not have uniformly sampled data in offset domain, so we have tested the algorithms with an incomplete model as well (Figure 1, d). Figure 2 shows the radon space of the complete model using low-resolution (a), DLS with regularization factor 0.05 (b), and TSVD using cutoff 0.05 (c). In theory a parabolic event should collapse to a point in radon space, however the low-resolution radon panel shows a large amount of smearing, while the DLS and TSVD solutions show much better focusing.

The energy above 150 ms was muted in radon space and the inverse radon transform was used to reconstruct the data (multiple attenuation). Figure 3 shows the reconstructed data with low-resolution solution (a), DLS (b), and TSVD solution (c). We can observe that most of the near offset multiple energy is still present in the near offsets in a), while both, the DLS and the TSVD solutions have attenuated the multiples successfully.

Figure 4, a) and b), shows the radon space of the incomplete model using the DLS and the TSVD solution. Although for the complete model the two methods showed similar results, the TSVD solution is superior to the least-squares on for the incomplete model. The energy is better focused and fewer artefacts are visible. Figure 4, c) and d), shows the reconstructed data after multiple attenuation. We can see that the TSVD solution has been much more effective in attenuating the multiples. Figure 5 is a display of the type II AVO anomaly before and after the multiple attenuation. The actual amplitudes are displayed in black, amplitudes with multiples in green, the DLS solution in blue, and the TSVD solution in red. We can notice the better fit of the TSVD result in the near offset range, i.e. the method introduces less artefacts on the primary amplitudes, which leads to better AVO analysis.

By altering the regularization parameter in the DLS method we manage to achieve a better multiple energy attenuation (Figure 6, regularization factor 40, TSVD cutoff 0.1). However the error between the actual type II AVO anomaly grew so large in near and far offsets for the DLS result, that any AVO analysis would be extremely inaccurate (Figure 7).

## CONCLUSIONS

The low-resolution Radon transform is a very poor choice and fails even with complete data. The DLS and the TSVD solutions are very close for a complete data set (all offsets). However, for incomplete data (missing offsets) the TSVD shows superior properties for multiple attenuation. The TSVD approach may lead to better results in other areas of Radon transform usage, like data interpolation and velocity analysis.

## REFERENCES

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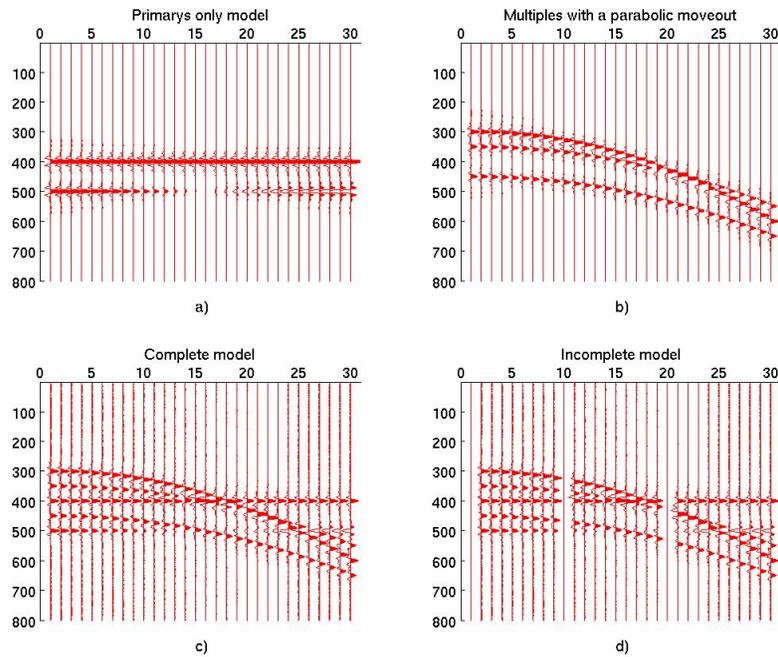


FIG. 1. Test model to evaluate parabolic radon transform. a) primaries b) multiples c) complete model primaries + multiples d) incomplete model with missing offsets.

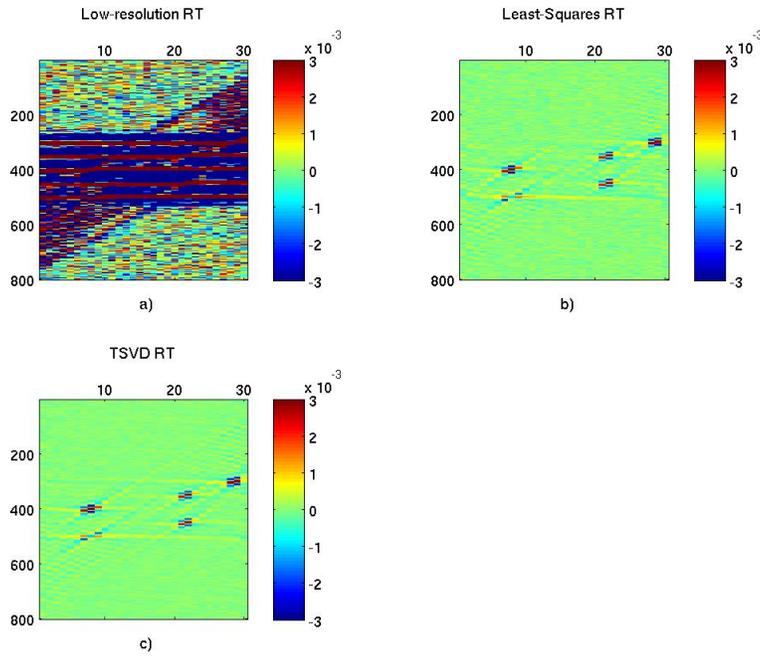


FIG. 2. Radon space using a) low-resolution, b) DLS solution, and c) TSVD solution of the complete data set.

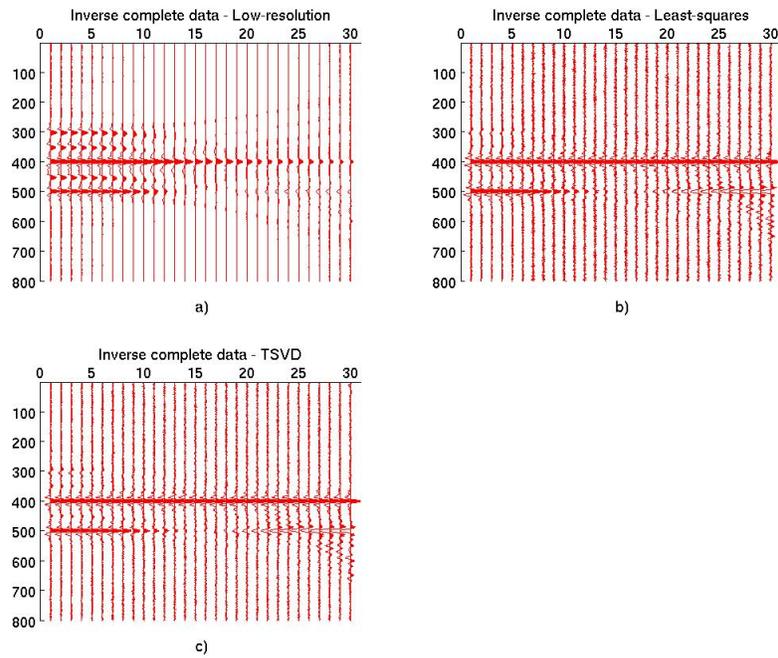


FIG. 3. Multiple attenuation using a) low-resolution, b) DLS solution, and c) TSVD solution of the complete data set.

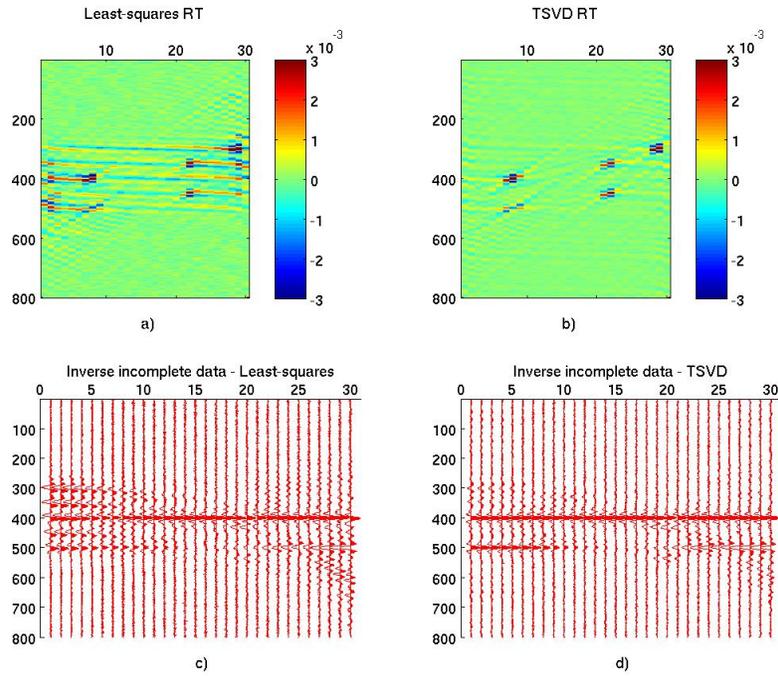


FIG. 4. Radon space of a) DLS solution, and b) TSVD solution. Multiple attenuation using c) DLS solution, and d) TSVD solution. DLS regularization factor 0.05, TSVD cutoff 0.05.

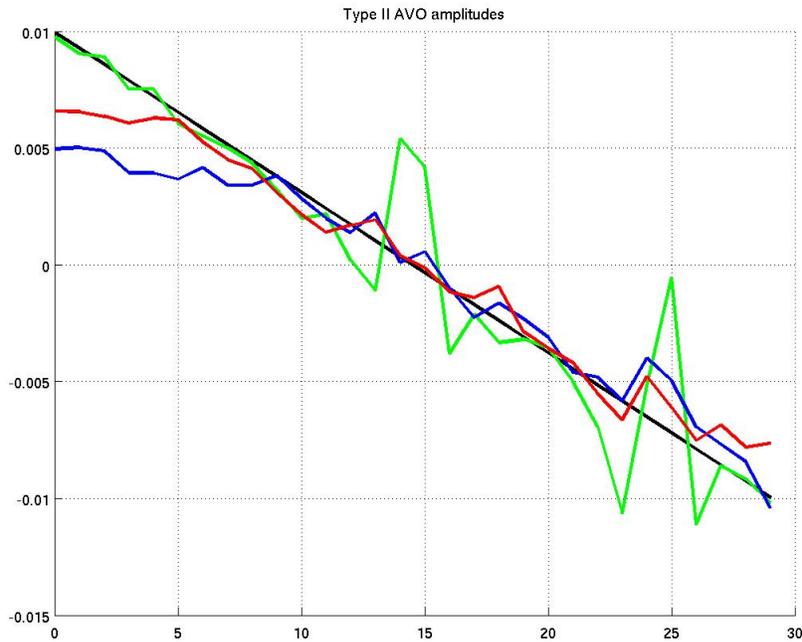


FIG. 5. Type II AVO anomaly amplitudes, actual in red, with multiples in green, DLS solution in blue, and TSVD solution in red. DLS regularization factor 0.05, TSVD cutoff 0.05.

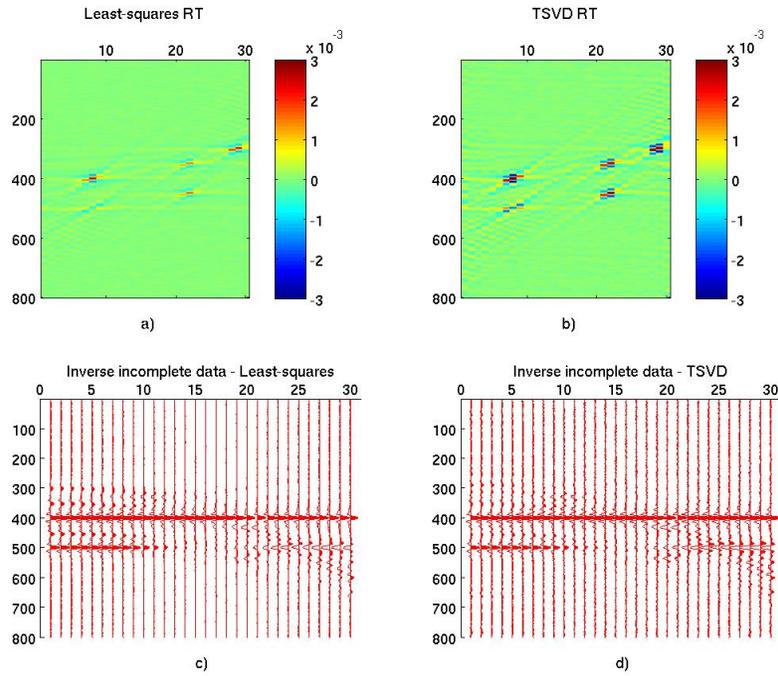


FIG. 6. Radon space of a) DLS solution, and b) TSVD solution. Multiple attenuation using c) DLS solution, and d) TSVD solution. DLS regularization factor 40, TSVD cutoff 0.1.

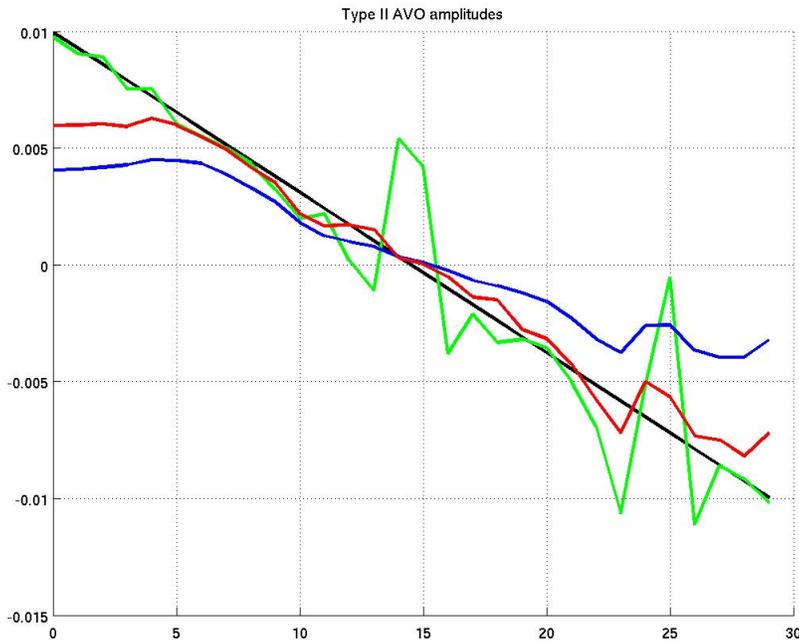


FIG. 7. Type II AVO anomaly amplitudes, actual in red, with multiples in green, DLS solution in blue, and TSVD solution in red. DLS regularization factor 40, TSVD cutoff 0.1