Solving physics pde's using Gabor multipliers

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ABSTRACT

We develop the mathematical properties of Gabor multipliers, which are a nonstationary version of Fourier multipliers. Some difficulties with current practice are identified, a functional calculus for combining the operators is given, and some indications for including corrections terms are noted. These techniques are motivated by the need for nonstationary data processing methods to model seismic wave propagation in nonhomogeneous media.

INTRODUCTION

The Gabor transform and Gabor multipliers have been developed as nonstationary filtering techniques useful in a variety of seismic data processing applications such as spectral deconvolution, depth migration, and reverse time migration. Some recent work on this include (Ismail, 2008), (Ma and Margrave, 2007a), (Ma and Margrave, 2007b), (Henley and Margrave, 2007), (Montana and Margrave, 2006), (Margrave and Lamoureux, 2006), and (Grossman, 2005). Some foundational references include (Margrave et al., 2003b), (Margrave et al., 2003a) and (Margrave and Lamoureux, 2002). Gabor techniques are an extension of the Fourier transform methods applied to localized signals, allowing mathematical models with inhomogeneities in the physical material being studied.

The essential idea in the Gabor method is to break up a signal into small, localized packets by multiplying the signal with a window function. Typically, the window is a smooth "bump" function, such as a Gaussian, localized at the point of interest in the signal. The localized packet can then be analyzed or modified using Fourier techniques. This is done for a collection of windows, covering the entire extent of the signal. Finally, all the processed packets are re-assembled into one full, processed signal which is the result of the nonstationary filtering.

The goal of this paper is to establish basic mathematical properties of the Gabor multipliers as non-stationary filters, with the aim of improving current practice in using these filters. The motivation is that we see in practice some unusual, and undesired, behaviour for these filters. For instance, in Gabor deconvolution, there sometimes appears to be an unexpected phase rotation in the processed signal which is not physically realistic. It seems to be an artifact of the numerical technique. Similarly, extrapolation operators can become numerically unbounded unless a careful choice of windows is made.

It is possible to see errors at even the most basic level of the Gabor multiplier, in simple examples where the multiplier is used to approximate a derivative. For instance, in Figure 1, we see the result of numerically computing the first derivative of a sinusoid using both Fourier and Gabor multipliers. The results are identical. However, in Figure 2, the similar result of computing the second derivative of a sinusoid shows some clear errors in the Gabor

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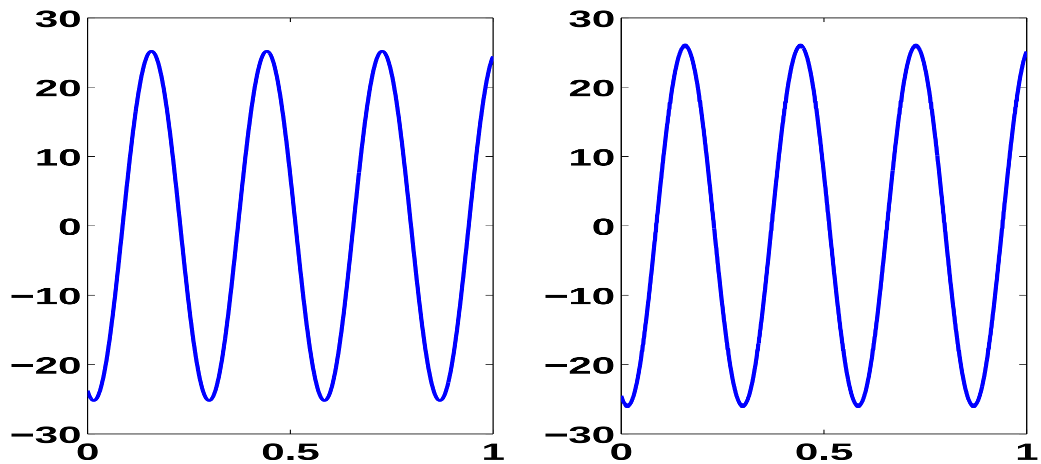

FIG. 1. The first derivative of a sinusoid, computed using Fourier and Gabor multipliers. There is excellent agreement between the two results.

calculation. Little spikes appear in the smooth derivative, an artifact of the windowing process. A hint to where those artifacts come from is shown in Figure 3, which plots the graphs of a correction term for a second order differential operator. These errors are not the result of numerical roundoff, but the consequence of properties of the Gabor multiplier that appear with higher order derivatives. The spikes come from the window edges, and identify the errors that appear in Figure 2.

With this motivations in mind, this paper shows how we can use Gabor multipliers to accurately approximate more general partial differential equations, which are used to model a physical system. We specify a functional calculus for Gabor multipliers, including how they combine as sums, products, exponentials – and how well we can approximate nonconstant coefficient PDEs using these multipliers. The motivating idea is to make rigourous the use of Gabor multipliers to model seismic waves, creating both one-way wave operators and wavefield extrapolators for numerical experiments.

The model for this general behaviour of Gabor multipliers is the functional calculus for Fourier multipliers, which are used extensively for representing and solving constant coefficient PDEs.

The structure of the paper is to cover some background mathematics, including Fourier multipliers and their properties. We then describe the results for Gabor multipliers, giving precise error terms for the approximations that arise in combining multipliers, and in estimating non-constant coefficient PDEs.

BACKGROUND MATHEMATICS

Fourier multipliers

The technique of Gabor multipliers depends heavily on the well-known properties of Fourier multipliers, which we review here.

A Fourier multiplier is an operator that modifies a signal $f(x)$ by multiplying by some

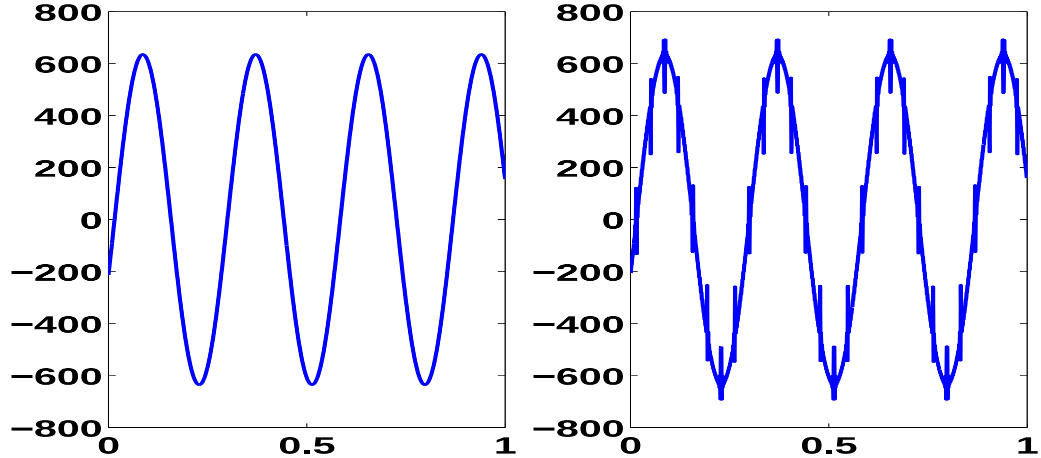

FIG. 2. The second derivative of a sinusoid, computed using Fourier and Gabor multipliers. The Gabor result on the right shows some obvious errors, due to the windowing.

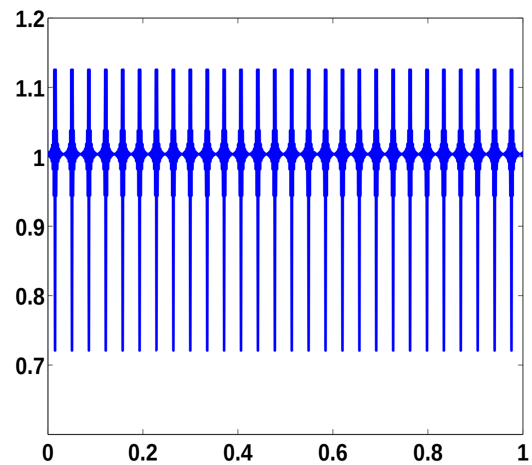

FIG. 3. A hint to what is causing the errors – the Gabor multiplier missing a correction term that identifies the edges of the window.

function $\alpha(\xi)$ in the Fourier transform domain. These operators are typically used as spatial or temporal filters and are familiar in seismic data processing.

We define the Fourier transform of a function $f(x)$ on $\mathbb{R}^n$ as the integral

$$\widehat{f}(\xi) = \int f(x) e^{-2\pi i x \cdot \xi} \, dx, \tag{1}$$

which has an inverse given by the integral

$$f(x) = \int \widehat{f}(\xi) e^{2\pi i x \cdot \xi} \, d\xi. \tag{2}$$

Given a function $\alpha(\xi)$ on the Fourier domain ξ, the Fourier multiplier F_α is the linear operator defined by first transforming f to the Fourier domain, $\widehat{f}$, multiplying by α, and then inverting back to the spatial domain, so

$$(F_\alpha f)(x) = \int \alpha(\xi) \widehat{f}(\xi) e^{2\pi i x \cdot \xi} \, d\xi. \tag{3}$$

In summary, the Fourier multiplier F_α is defined as the composition

$$F_\alpha = \mathcal{F}^{-1} M_\alpha \mathcal{F}, \tag{4}$$

where $\mathcal{F}$ is the Fourier transform operator, $\mathcal{F}^{-1}$ is its inverse, and M_α is the operation of multiplication by α. The function α is called the symbol of the multiplier F_α.

Operator norm

There is a close connection between the symbol α and the continuity properties of the operator F_α. The operator F_α is continuous if and only if the function α is bounded. The norm of the operator F_α is given as

$$||F_\alpha|| = \max_\xi |\alpha(\xi)|. \tag{5}$$

This bound is a useful measure of how the operator grows when repeated, such as in a wavefield extrapolation scheme.

Functional calculus

The multiplication M_α represents the Fourier multiplier F_α as an ordinary multiplication operator. As a result, we get a simple functional calculus for Fourier multipliers. Sums, differences, products, quotients, and even analytic extensions of Fourier multipliers are again Fourier multipliers, with the natural symbol. For instance, with symbols α, β, and real number t, it is easy to verify that the following combinations of operators hold:

$$\begin{aligned} tF_\alpha &= F_{t\cdot\alpha} && (6)\\ F_\alpha + F_\beta &= F_{\alpha+\beta} && (7)\\ F_\alpha - F_\beta &= F_{\alpha-\beta} && (8)\\ F_\alpha \cdot F_\beta &= F_{\alpha\cdot\beta} && (9)\\ F_\alpha (F_\beta)^{-1} &= F_{\alpha/\beta} && (10)\\ \exp(F_\alpha) &= F_{e^\alpha}, && (11) \end{aligned}$$

provided all the resulting combinations of operators make sense. (eg. no division by zero.)

This functional calculus is often used in seismic imaging. For instance, the wave equation can be represented using Fourier multipliers, provided the velocity field is constant. A one-wave wave operator is obtained by taking the square root of one of these operators, so the functional calculus gives us

$$(F_\alpha)^{1/2} = F_{\sqrt{\alpha}}. \tag{12}$$

A wavefield extrapolator is obtained by exponentiating the square root operator, so we obtain

$$\exp(tF_\alpha)^{1/2} = F_{e^{t\sqrt{\alpha}}}, \tag{13}$$

where t is the step size for the extrapolation.

Representing a constant coefficient PDE

The Fourier multiplier operators can be used to represent any constant coefficient partial differential equation. An example will demonstrate the idea.

The acoustic wave equation, for a medium with constant velocity c, is given by

$$\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x_1^2} - \frac{\partial^2 f}{\partial x_2^2} - \frac{\partial^2 f}{\partial x_3^2} = g, \tag{14}$$

where the functions f, g depend on both time t and spacial variables x_1, x_2, x_3. Using the Fourier inversion formula, the derivatives can be taken under the integral sign, and they differentiate the exponential, giving factors $-4\pi^2\omega^2, -4\pi^2\xi_1^2, -4\pi^2\xi_2^2, -4\pi^2\xi_3^2$, where ω is temporal frequency and ξ_1, ξ_2, ξ_3 are spacial frequencies. Thus the differential operator is given by a single Fourier multiplier F_α with symbol

$$\alpha(\omega, \xi_1, \xi_2, \xi_3) = -\frac{4\pi^2}{c^2}\omega^2 + 4\pi^2\xi_1^2 + 4\pi^2\xi_2^2 + 4\pi^2\xi_3^2. \tag{15}$$

The differential equation is succinctly written in operator form as

$$F_\alpha f = g, \tag{16}$$

where F_α is the Fourier multiplier.

We are looking for similar results with Gabor multipliers which will allow us to work with non-constant velocity fields.

Gabor multipliers

A Gabor multiplier is a localized version of Fourier multipliers; it modifies the signal in the Gabor domain, by multiplying the transformed signal by a function (or symbol) of two variables, $\alpha(k, \xi)$, where k roughly indicates location in space, and ξ is spacial frequency.[†]

[†]This localization helps us deal with varying velocity fields in seismic, but also changes the elegant functional calculus of the Fourier multipliers.

The Gabor transform is defined by first selecting two families of window functions $\{v_k(x)\}_{k=1}^M, \{w_k(x)\}_{k=1}^M$, non-negative functions on $\mathbb{R}^n$, that satisfy the partition of unity condition,

$$\sum_k v_k(x)w_k(x) = 1, \qquad \text{for all } x. \tag{17}$$

In practice, the windows may be selected to be copies of a single bump function, translated around to cover the region of interest in space. Or it could be a collection of boxcar windows (or indicator functions), each one constant on some region when the physical parameters of what we are modeling are mainly constant. There is great freedom in the choice of windows, provided one respects the partition of unity condition.

A signal $f(x)$ is localized by multiplying with window $w_k(x)$, and the Gabor transform is defined as a series of Fourier transforms for these localized signals. The Gabor transform $\mathcal{G}f$ of function f is itself a function of two variables, given as

$$(\mathcal{G}f)(k,\xi) = \int f(x)w_k(x)e^{-2\pi i x\cdot\xi}\,dx. \tag{18}$$

Equivalently, in operator notation we have

$$(\mathcal{G}f)(k,\xi) = \mathcal{F}(w_k f)(\xi) = (\mathcal{F}M_{w_k}f)(\xi). \tag{19}$$

The function f can be recovered from its Gabor transform as

$$f(x) = \sum_k v_k(x)\mathcal{F}^{-1}(\mathcal{F}M_{w_k}f), \tag{20}$$

because of the partition of unity condition on the windows.

The Gabor multiplier G_α is obtained by inserting as multiplier the function $\alpha(k,\xi)$ into the above inversion formula, thus modifying the signal f in the Gabor domain. Notice that the Gabor symbol $\alpha(k,\xi)$ is a function of two variables, and when we insert it into the sum,we should use a function that depends only on the frequency variable ξ. We let α_k denote the function of one variable, with $\alpha_k(\xi) = \alpha(k,\xi)$. Thus we define the Gabor multiplier operator as

$$G_\alpha f = \sum_k M_{v_k}\mathcal{F}^{-1}M_{\alpha_k}\mathcal{F}M_{w_k}f. \tag{21}$$

In operator notation, we thus have

$$G_\alpha f = \sum_k M_{v_k}\mathcal{F}^{-1}M_{\alpha_k}\mathcal{F}M_{w_k}f = \sum_k M_{v_k}F_{\alpha_k}M_{w_k}f, \tag{22}$$

where we replaced the operator $\mathcal{F}^{-1}M_{\alpha_k}\mathcal{F}$ with its Fourier multiplier F_{α_k}.

We have arrived at a very compact form for the Gabor multiplier G_α as a sum of localized Fourier multipliers,

$$G_\alpha = \sum_k M_{v_k}F_{\alpha_k}M_{w_k}. \tag{23}$$

RESULTS

Operator norm - for Gabor

It is useful to know how large in norm these Gabor multipliers will be. When an operator is iterated, it is important to keep the norm below one, to prevent exponential growth in the result, and to minimize the accumulation of numerical errors.

Unfortunately, for the general Gabor multiplier, it is easy to cook up realistic examples where the norm of the operator grows with the number of windows. In fact, we can find growth on the order of $M^{1/2}$,

$$||G_\alpha|| \approx \sqrt{M} \max_{k,\xi} |\alpha(k,\xi)|, \tag{24}$$

where M is the number of windows. This is an unfortunate result. It shows the norm of the Gabor multiplier depends not only on the symbol α, but also on the particular choice of windows.

There is one case, though, that the operator norm is well behaved. We can state it as a theorem: If the windows are chosen symmetrically, so $v_k = w_k$ for each k, then we have that the Gabor multiplier is bounded above by the maximum of its symbol, so

$$||G_\alpha|| \leq \max_{k,\xi} |\alpha(k,\xi)|. \tag{25}$$

This is very much like the Fourier multiplier result, where the norm of the Fourier multiplier actually equals the maximum of α.

Functional calculus - for Gabor

What happens when you add or subtract Gabor multipliers? They behave as you expect: the result is a Gabor multiplier, whose symbol is the sum or difference of the first two symbols. That is

$$G_\alpha + G_\beta = G_{\alpha+\beta}, \tag{26}$$
$$G_\alpha - G_\beta = G_{\alpha-\beta}. \tag{27}$$

Similarly, if you scale a Gabor multiplier by a fixed number λ, the result is a new Gabor multiplier with the scaled symbol:

$$\lambda G_\alpha = G_{\lambda\alpha}. \tag{28}$$

These three results are summed up by saying the representation of symbols as Gabor multipliers is linear.

Other combinations of Gabor multipliers are not so well-behaved. We only get approximations to the expected result. So, for instance the product of two Gabor multipliers G_α, G_β is only approximately a Gabor multipler whose symbol is the product of the symbols α, β:

$$G_\alpha G_\beta \approx G_{\alpha\beta}. \tag{29}$$

Similarly, the square of a Gabor muliplier with symbol α is approximately a multiplier with symbol α^2:

$$(G_\alpha)^2 \approx G_{\alpha^2}; \tag{30}$$

the square root is given approximately as

$$(G_\alpha)^{1/2} \approx G_{\sqrt{\alpha}}; \tag{31}$$

and the multiplier with symbol α^{-1} acts as an approximate inverse, with

$$G_\alpha G_{\alpha^{-1}} \approx I. \tag{32}$$

We also might expect that the exponential of a Gabor multiplier is approximated as a multiplier with exponential symbol:

$$\exp(G_\alpha) \approx G_{e^\alpha}. \tag{33}$$

However, we are not yet able to show this rigorously.

To verify these approximations, it is instructional to start with a simple case. Assume the windows w_k are indicator functions (i.e. boxcar functions, taking value 1 on set Ω_k, zero elsewhere), and let us use symmetric dual windows for the multipliers, so

$$G_\alpha = \sum_k M_{w_k} C_{\alpha_k} M_{w_k}. \tag{34}$$

The product of two such operators will give

$$\begin{aligned} G_\alpha G_\beta &= (\sum_k M_{w_k} C_{\alpha_k} M_{w_k})(\sum_j M_{w_j} C_{\beta_j} M_{w_j}) && (35)\\ &= \sum_{j,k} M_{w_k} C_{\alpha_k} M_{w_k} M_{w_j} C_{\beta_j} M_{w_j}. && (36) \end{aligned}$$

Since we have boxcar windows, the product in the middle, $M_{w_k} M_{w_j}$, is zero, except when $j = k$, at which point it is just M_{w_k}. The double sum collapses to

$$\begin{aligned} G_\alpha G_\beta &= \sum_k M_{w_k} C_{\alpha_k} M_{w_k} C_{\beta_k} M_{w_k} && (37)\\ &= \sum_k M_{w_k} (C_{\alpha_k} M_{w_k} - M_{w_k} C_{\alpha_k} + M_{w_k} C_{\alpha_k}) C_{\beta_k} M_{w_k} && (38)\\ &= \sum_k M_{w_k} [C_{\alpha_k}, M_{w_k}] C_{\beta_k} M_{w_k} + \sum_k M_{w_k} C_{\alpha_k}) C_{\beta_k} M_{w_k} && (39)\\ &= \sum_k M_{w_k} [C_{\alpha_k}, M_{w_k}] C_{\beta_k} M_{w_k} + \sum_k M_{w_k} C_{(\alpha\beta)_k}) M_{w_k}, && (40)\\ &= \Delta + G_{\alpha\beta}, && (41) \end{aligned}$$

where we recognize the second sum in the next-to-last line as the multiplier $G_{\alpha\beta}$, and the remaining term we call the error term Δ.

Thus, the error in the approximation $G_\alpha G_\beta \approx G_{\alpha\beta}$ is given as

$$\Delta = \sum_k M_{w_k}[C_{\alpha_k}, M_{w_k}]C_{\beta_k}M_{w_k}, \tag{42}$$

where the bracket $[\cdot,\cdot]$ in that sum is a shorthand notation for the commutator of two operators, written as

$$[C_{\alpha_k}, M_{w_k}] = C_{\alpha_k}M_{w_k} - M_{w_k}C_{\alpha_k}. \tag{43}$$

The error Δ is an amalgamation of operators $[C_{\alpha_k}, M_{w_k}]C_{\beta_k}$ and since we are using symmetric windows, we can bound the size of the error as

$$||\Delta|| \leq \max_k ||[C_{\alpha_k}, M_{w_k}]C_{\beta_k}||. \tag{44}$$

The key to controlling the size of the error Δ is in controlling the commutators $[C_{\alpha_k}, M_{w_k}]$.

On the other hand, from the form of the error Δ, we note that the errors in the operator approximation are typically concentrated near the edges of the support of the windows. That is, near the points where window functions jump between zero and one. The commutator is non-zero near the places where the window is non-constant. (And "nearness" is measured by the width of the convolution operators.)

The approximation $(G_\alpha)^2 \approx G_{\alpha^2}$ follows from the previous calculation, replacing symbol β in the product with α. In this case, the error term is

$$\Delta = \sum_k M_{w_k}[C_{\alpha_k}, M_{w_k}]C_{\alpha_k}M_{w_k}. \tag{45}$$

The approximation $(G_\alpha)^{1/2} \approx G_\alpha$ also follows from the product calculations, replacing symbols α, β in the product with $\sqrt{\alpha}$. In this case, the error term is

$$\Delta = \sum_k M_{w_k}[C_{\sqrt{\alpha}_k}, M_{w_k}]C_{\sqrt{\alpha}_k}M_{w_k}. \tag{46}$$

The approximate inverse given as $G_\alpha G_\alpha^{-1} \approx I$ also follows from the product calculations In this case, the error term is

$$\Delta = \sum_k M_{w_k}[C_{\alpha_k}, M_{w_k}]C_{\alpha_k^{-1}}M_{w_k}. \tag{47}$$

Finding the error for exponentiating a Gabor multiplier is left for future work.

Functional calculus - special case

Sometimes it is necessary to combine a Fourier multiplier with a Gabor multiplier. This occurs, for instance, in Gabor deconvolution, where the source wavelet is represented by a single Fourier multiplier.

In the special case where the synthesis windows v_k are all equal to one (so the w_k form a partition of unity on their own), then we have

$$F_\alpha G_\beta = G_{\alpha\beta}. \tag{48}$$

Thus, the Fourier multiplier times the Gabor multiplier is another Gabor multiplier, whose symbol $\alpha\beta$ is simply the product of the two symbols α, β.

Notice in this formulation, α is a function of one variable, β is a function of two variables, and so

$$(\alpha\beta)(k, \xi) = \alpha(\xi)\beta(k, \xi). \tag{49}$$

Also notice that the Fourier multiplier F_α appears on the left in the product $F_\alpha G_\beta$; this has to do with our choice of the windows v_k being constant one.

Approximating a non-constant coefficient PDE - with Gabor

The typical PDE involves sums of differential operators of the form

$$a(x)\frac{\partial^N}{\partial x_1^{n1}\partial x_2^{n_2}\cdots\partial x_n^{n_n}}, \tag{50}$$

each of which can be understood as a multiplier M_a times a simple differential operator D.

By simple, we mean a differential operator of the form

$$D = \frac{\partial^N}{\partial x_1^{n1}\partial x_2^{n_2}\cdots\partial x_n^{n_n}}. \tag{51}$$

Such an operator is represented exactly by the Fourier multiplier F_α with symbol

$$\alpha = (-2\pi i)^N \xi_1^{n_1}\xi_2^{n_2}\cdots\xi_n^{n_n}. \tag{52}$$

Using this Fourier multiplier, we create a Gabor multiplier that represents D exactly.

There are three ways to do this. The first method chooses the synthesis windows to be constant one, $v_k \equiv 1$. In this case, the w_k form a partition of unity, so the identity operator is expressed as a sum, $I = \sum_k M_{w_k}$. For the differential operator D, we have $D = F_\alpha = F_\alpha \sum_k M_{w_k} = \sum_k F_\alpha M_{w_k}$. So we have

$$D = G_\alpha = \sum_k M_{v_k} F_\alpha M_{w_k}, \qquad \text{with } v_k \equiv 1. \tag{53}$$

The second method is to choose the analysis windows to be constant one, $w_k \equiv 1$. Now the v_k form a partition of unity, so $I = \sum_k M_{v_k}$. As in the first case, we get the result

$$D = G_\alpha = \sum_k M_{v_k} F_\alpha M_{w_k}, \qquad \text{with } w_k \equiv 1. \tag{54}$$

With smooth, symmetric windows ($v_k = w_k$), it is easy to verify that

$$F_\alpha = G_\alpha = \sum_k M_{w_k} F_\alpha M_{w_k} + \text{ lower order multipliers.} \tag{55}$$

As an example, we check a first order operator, $D = \partial/\partial x_1$. By the product rule,

$$M_{w_k} D M_{w_k} f = w_k({w_k}' f + w_k f') = w_k {w_k}' f + (w_k)^2 f'. \tag{56}$$

Summing over k gives

$$G_\alpha f = \sum_k M_{w_k} D M_{w_k} f = (\sum_k w_k {w_k}') f + (\sum_k (w_k)^2) f'. \tag{57}$$

By the partition of unity, the second sum on the right is one, and the first sum is zero, as the derivative of the first sum. Thus we have

$$G_\alpha f = Df, \tag{58}$$

so the Gabor multiplier $G_\alpha = D$ represents this first order differential operator exactly.

A second order operator has a correction term. With operator $D = \dfrac{\partial^2}{\partial x_i \partial x_j}$, and symbol $\alpha(\xi) = -4\pi^2 \xi_i \xi_j$, a calculation as above shows that

$$G_\alpha = D + M_d, \tag{59}$$

where $d(x) = \sum_k w_k(x) \frac{\partial^2}{\partial x_i \partial x_j} w_k(x)$ is the correction term coming from the product rule. The operator M_d is simply multiplication (in the spacial domain) by the function $d(x)$ and is considered a zero-th order operator, and thus of lower order than D.

Even in the constant velocity wave equation, there is a correction term. The (spacial) Laplacian, a second order operator, is given by a Fourier multiplier, and

$$\nabla^2 = F_\alpha = G_\alpha - M_d \tag{60}$$

where $\alpha(\xi) = -4\pi^2 |\xi|^2$ is the symbol for Fourier multiplier of the Laplacian, and $d(x) = \sum_k w_k \nabla^2 w_k$ is the symbol for the lower order correction term.

It is worth noting that the correction term M_d is a multiplication operator, with support on the transition areas of the windows: where the windows are not constant. This correction is easy to introduce in numerical computations.

Besides the simple differential operators, the PDEs involves multiplication operators M_a. Provided the function $a(x)$ is slowly varying, it can be approximated by a Gabor multiplier as follows. With well-chosen windows, we can assume that $a(x)$ is nearly constant on the support of window products $v_k(x)w_k(x)$, for each k. Say $a(x)$ is close to the value a_k on the support of $v_k(x)w_k(x)$. Then

$$a(x) v_k(x) w_k(x) \approx a_k v_k(x) w_k(x), \tag{61}$$

and as operators, we have

$$M_a M_{v_k} M_{w_k} \approx a_k M_{v_k} M_{w_k}. \tag{62}$$

Summing over k, and using the partition of unity condition, gives

$$M_a \approx \sum_k M_{v_k} a_k I M_{w_k}. \tag{63}$$

This last sum is a Gabor multiplier, with symbol $\alpha(k,\xi) = a_k$. Thus we have the approximation

$$M_a \approx G_\alpha. \tag{64}$$

More generally, the differential operator $M_a D$, where D is a simple differential operator as described above, can be approximated by a Gabor multiplier. We obtain the approximation

$$M_a D \approx G_\alpha + \text{ lower order terms}, \tag{65}$$

where the symbol is given as $\alpha(k,\xi) = a_k \alpha_0(\xi)$, using α_0 as the Fourier multiplier symbol corresponding to D.

For instance, in the non-constant velocity case, the wave equation has the Laplacian multiplied by coefficient $a(x) = \frac{1}{c(x)^2}$. We then can write

$$M_a \nabla^2 = M_a F_{\alpha_0} = M_a (G_{\alpha_0} - M_d) \approx G_\alpha - M_{ad}, \tag{66}$$

where $\alpha(k,\xi) = -4\pi^2 a_k |\xi|^2$ is the Gabor symbol and $d(x) = \sum_k w_k \nabla^2 w_k$ gives the correction term.

Again, it is worth pointing out that the wave equation, represented by the Gabor multiplier G_α, requires a correction term M_{ad}.

CONCLUSIONS

We have presented Gabor multipliers as a localized version of Fourier multipliers, which allows for nonstationary filtering of data signals. The Gabor multipliers are expressed as sums of composition of multiplications and convolutions (Fourier multipliers). From this representation, we obtain a functional calculus for the Gabor multipliers, showing how sums, products, quotients, and square roots are calculated, including correction terms. We also show how Gabor multipliers are used to represent partial differential operators, using the same symbol as the Fourier multiplier representations, plus lower order correction terms.

Future work will include applying these correction terms to specific seismic data processing algorithms.

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