Quasi-compressional ray propagation in a linearized general anisotropic medium

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ABSTRACT

Ray tracing for qP rays in a linearized general (21 parameter) anisotropic medium is investigated. A smoothly varying inhomogeneous medium is initially assumed and the axes of anisotropy within the layers are allowed to be oriented at arbitrary angles with respect to the model coordinates. As in almost all applications of ray tracing the two point problem is the most useful. That is given a source and receiver location within a anisotropic medium, determine the ray path(s) between the two points. This introduces two coupled nonlinear equations which must be solved. Initially, the medium will be chosen to be a smoothly varying inhomogeneous anisotropic medium with no rotations of the axes. These assumptions may appear to be quite restrictive; however, it would seem to be a reasonable foundation upon which to advance to more complicated media types within the context of qP ray propagation in an orthorhombic anisotropic structure. The density normalized anisotropic parameters used to describe an orthorhombic media, which have the dimensions of $(\text{velocity})^2$, will be in Voigt notation, A_{ij}, as this notation is familiar. One of the applications of this ray tracing method is for Born-Kirchoff migration, so that some attention will be afforded that topic.

INTRODUCTION

In the geophysical literature there have appeared a number of papers on linearized methods for obtaining phase velocities in a general anisotropic medium. An earlier derivation of the quasi-compressional (qP) linearized phase velocity for a media of this type may be found in Backus (1965). More recently, this problem has been the topic of a number of papers (see Pšenčík and Farra, 2005, and references contained within).

The exact derivations of ray tracing formulae for complex media types of higher complexity than orthorhombic is, as expected the mathematical treatment becomes overwhelming and not of particular use as the associating of empirical data observations with formulae does not follow. A less accurate, or rather possibly a less complicated manner of proceeding may be pursued if some assumptions of the medium are made that allow for use of approximations to the exact formulae such as linearization or the introduction of perturbation theory, a powerful procedural method where higher order approximations may be obtained from lower order terms. General perturbation theory may be found in the text by Nayfeh, (1973) while some applications specific to seismology appear in Jech and Pšenčík (1989).

The qP eikonal is a quasi-linear partial differential equations in the slowness vector components related to the phase function, $\tau(x_j)$, as $p_i = \partial\tau(x_j)/\partial x_i$ (Gassmann, 1964). The phase function describes the propagation of the wavefront through an elastic medium, so that $\mathbf{p} = \nabla \tau(x_j)$ indicates that slowness vector is normal to the surface of constant phase describing wavefront propagation in a medium. Employing standard partial differential equation solution methods, these eikonals may be used to obtain the components of the quasi-compressional (qP) group velocity (Courant and Hilbert, 1962, Červený, 2001, and Červený et al., 2007, as examples).

The slowness vector components are expressed in terms of phase velocity angles (θ, ϕ) and related velocities. For the qP type of wave propagation only 15 independent parameters of the 21 total anisotropic parameters are required to specify the medium (Pšenčík and Gajewski, 1998 and Pšenčík and Farra, 2005). In an orthorhombic anisotropic media type that is a reasonably complex subset of the general case for use in seismic hydrocarbon prospecting only 6 independent parameters are needed to specify a medium for qP wave propagation. This medium type is suitably complex to approximate most geological structures encountered in seismic exploration situations, especially if rotation of the anisotropic axes with respect to some local Cartesian coordinate system is introduced.

The algorithm presented in the (Pšenčík and Farra, 2005) was designated as FORT (First Order Ray Tracing) by those authors. Ray equations for the two coupled shear modes, qS_1 and qS_2, will not be dealt with in this work due to the added complexity of these coupled modes. This problem type was not treated in the 2005 work above but the theoretical development may be found in the following incomplete sequence of papers: Coates and Chapman, (1990), Bakker (2002), Farra (2005), Farra and Pšenčík (2008) and Farra and Pšenčík (2009). Other relevant citations may be found in these works.

THEORY

Determining the linearized phase velocity in a general anisotropic medium for a quasi-compressional (qP) wave using the method, initially discussed by Backus (1965) for seismic related problems, yields, after some manipulation, the following first order linearized approximation in a weakly anisotropic medium, which is the standard that now appears in the literature for a range of disciplines of study.

$$\begin{aligned} v_{qP}^2(x_i, n_j) = A_{11}n_1^2 + A_{22}n_2^2 + A_{33}n_3^2 + E_{12}n_1^2n_2^2 + E_{13}n_1^2n_3^2 + E_{23}n_2^2n_3^2 + \\ 4\left(H_1n_1^2 + H_2n_2^2 + H_3n_3^2\right) \end{aligned} \tag{1}$$

Voigt notation obtained from the relationship involving the more general density normalized anisotropic parameters $a_{ijk\ell} = a_{ij}^{kl} \to A_{ij}$, where the 21 independent $A_{ij} = A_{ji}, (i, j = 1, 6)$ have the dimensions of velocity squared, is used here (see for example Gassmann, 1964, page 98). The symmetry relations $a_{ij\ell k} = a_{ji\ell k} = a_{ijk\ell} = a_{\ell kij}$

reduce the 81 anisotropic parameters $a_{ijk\ell}$ to 21. The direction of the normal to a wave front propagating in an anisotropic medium or equivalently the phase velocity propagation vector direction is defined as

$$\mathbf{n} = (n_1, n_2, n_3) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{2}$$

with θ being the polar angle measured from the positive x_3 (vertical) axis $(0 \le \theta \le \pi)$ and ϕ the azimuthal angle measured in a positive sense from the x_1 axis $(0 \le \phi < 2\pi)$, specifying the normal to the ray surface for some arbitrary ray.

Equation (2) is shown to be composed of three groups of terms: those specifying an ellipsoid and anellipsoidal deviation terms defining an orthorhombic anisotropic medium, plus an additional 9 parameters, the total of which define a general anisotropic medium. The ellipsoidal deviation terms, E_{ij}, are generally spatially dependent as the A_{ij} have so been assumed. They are defined through the relationships

$$E_{12} = 2(A_{12} + 2A_{66}) - (A_{11} + A_{22}) \tag{3}$$

$$E_{13} = 2(A_{13} + 2A_{55}) - (A_{11} + A_{33}) \tag{4}$$

$$E_{23} = 2(A_{23} + 2A_{44}) - (A_{22} + A_{33}) \tag{5}$$

and the additional terms that takes an orthorhombic to a general anisotropic medium are specified by

$$H_1(x_i, n_j) = (A_{14} + 2A_{56}) n_2 n_3 + A_{16} n_1 n_2 + A_{15} n_1 n_3 \tag{6}$$

$$H_2(x_i, n_j) = (A_{25} + 2A_{46}) n_1 n_3 + A_{26} n_1 n_2 + A_{24} n_2 n_3 \tag{7}$$

$$H_3(x_i, n_j) = (A_{36} + 2A_{45}) n_1 n_2 + A_{35} n_1 n_3 + A_{34} n_2 n_3 \tag{8}$$

Of the constituents of equations (6) – (7) there are only 9 parameters which may be obtained from the inversion qP travel times. These are A_{15}, A_{16}, A_{24}, A_{26}, A_{34} and A_{35}, together with $A_{14} + 2A_{56}$, $A_{25} + 2A_{46}$ and $A_{56} + 2A_{45}$. Thus with A_{11}, A_{22}, A_{33}, E_{12}, E_{13} and E_{23}, only 15 of the 21 total anisotropic parameters may be ascertained using only qP travel time data, the remainder requiring quasi – shear travel time data for a complete inversion. It is evident that these 15 parameters are not independent, 6 being expressed in terms of a combination of two or more of the A_{ij}. This should be compared with an orthorhombic medium where 9 anisotropic parameters are required to specify the medium. From qP travel time data only 6 of these may be recovered; A_{11}, A_{22}, A_{33}, E_{12}, E_{13} and E_{23}.

A linearized eikonal equation for this anisotropic medium type may be obtained by introducing the slowness vector components, which are given in terms of the phase velocity vector and the scalar phase velocity as

$$p_k = \frac{n_k}{v(x_i, n_k)} \tag{9}$$

may be written as

$$\begin{aligned} G_{qP}\left(x_i, p_j\right) &= A_{11}p_1^2 + A_{22}p_2^2 + A_{33}p_3^2 + \\ &\left[E_{12}p_1^2p_2^2 + E_{13}p_1^2p_3^2 + E_{23}p_2^2p_3^2 + 4\left(H_1p_1^2 + H_2p_2^2 + H_3p_3^2\right)\right] / \left(p_kp_k\right) = 1 \end{aligned} \tag{10}$$

where the E_{ij} are as given in equations (3) – (5), $\left(p_kp_k\right) = p_1^2 + p_2^2 + p_3^2$ and the H_j now have the form

$$H_1\left(x_i, p_j\right) = \left(A_{14} + 2A_{56}\right)p_2p_3 + A_{16}p_1p_2 + A_{15}p_1p_3 \tag{11}$$

$$H_2\left(x_i, p_j\right) = \left(A_{25} + 2A_{46}\right)p_1p_3 + A_{26}p_1p_2 + A_{24}p_2p_3 \tag{12}$$

$$H_3\left(x_i, p_j\right) = \left(A_{36} + 2A_{45}\right)p_1p_2 + A_{35}p_1p_3 + A_{34}p_2p_3 \tag{13}$$

Thus equation (10) is in general a function of only the slowness vector components and the spatial dependence of the A_{ij}.

The method of characteristics (Courant and Hilbert, 1962 and Červený, 2001) is used to determine the rays, along which the energy traverses between one point in the medium and another. The ray (group) velocity vector and corresponding slowness vector components are given generally in terms of the eikonal equation, $G\left(x_i, p_j\right)$, defined in (10) as

$$\frac{dx_i}{d\tau} = \frac{1}{2}\frac{\partial G(x_k, p_k)}{\partial p_i} \tag{14}$$

$$\frac{dp_i}{d\tau} = -\frac{1}{2}\frac{\partial G(x_k, p_k)}{\partial x_i}. \tag{15}$$

The independent parameter along the ray is chosen to be the propagation time specified by the variable τ in equations (14) and (15).

The initial value problem above is fully specified, given some initial conditions

$$\mathbf{x_0} = \mathbf{x}(\tau_o) \text{ and } \mathbf{p_0} = \mathbf{p}(\tau_o) \tag{16}$$

at a reference time τ_0. The progression of the ray in 3D Cartesian space as well as the magnitude and direction of the slowness vector at these points may be determined. The group velocity in terms of its components is given by

$$\frac{d\mathbf{x}}{d\tau} = \left(\frac{dx_1}{d\tau}, \frac{dx_2}{d\tau}, \frac{dx_3}{d\tau} \right) \tag{17}$$

with the magnitude defined as

$$\left| \frac{d\mathbf{x}}{d\tau} \right| = \left[\left(\frac{dx_1}{d\tau} \right)^2 + \left(\frac{dx_2}{d\tau} \right)^2 + \left(\frac{dx_2}{d\tau} \right)^2 \right]^{1/2} . \tag{18}$$

The ray velocity vector components are $dx_i/d\tau$ equivalent to those used in FORT (Pšenčík and Farra, 2005) may be given explicitly in Voigt notation as

$$\begin{aligned} \frac{dx_1}{d\tau} = p_1 A_{11} + \Big\{ p_1 \Big[\left(E_{12} + E_{13} - E_{23} \right) p_2^2 p_3^2 + \left(E_{12} p_2^4 + E_{13} p_3^4 \right) \Big] \Big\} \left(p_k p_k \right)^{-2} + \\ 2\Big\{ 2\left(A_{14} + 2A_{56} \right) p_1 p_2 p_3 \left(p_2^2 + p_3^2 \right) + \\ \left[3\left(p_2^2 + p_3^2 \right) + p_1^2 \right] p_1^2 \left[A_{16} p_2 + A_{15} p_3 \right] + \\ \left(p_2^2 + p_3^2 - p_1^2 \right) \Big[\left[\left(A_{25} + 2A_{46} \right) p_3 + A_{26} \right] p_2^3 + \\ \left[\left(A_{36} + 2A_{45} \right) p_2 + A_{35} \right] p_3^2 \Big] - \\ 2\left(A_{24} p_2^2 + A_{34} p_3^2 \right) p_1 p_2 p_3 \Big\} \left(p_k p_k \right)^{-2} \end{aligned} \tag{19}$$

$$\begin{aligned} \frac{dx_2}{d\tau} = p_2 A_{22} + \Big\{ p_2 \Big[\left(E_{12} + E_{23} - E_{13} \right) p_1^2 p_3^2 + \left(E_{12} p_1^4 + E_{23} p_3^4 \right) \Big] \Big\} \left(p_k p_k \right)^{-2} + \\ 2\Big\{ 2\left(A_{25} + 2A_{46} \right) p_1 p_2 p_3 \left(p_1^2 + p_3^2 \right) + \\ \left[3\left(p_1^2 + p_3^2 \right) + p_2^2 \right] p_2^2 \left[A_{26} p_1 + A_{24} p_3 \right] + \\ \left(p_1^2 + p_3^2 - p_2^2 \right) \Big[\left[\left(A_{14} + 2A_{56} \right) p_3 + A_{16} \right] p_1^2 + \\ \left[\left(A_{36} + 2A_{45} \right) p_1 + A_{34} \right] p_3^3 \Big] - \\ 2\left(A_{15} p_1^2 + A_{35} p_3^2 \right) p_1 p_2 p_3 \Big\} \left(p_k p_k \right)^{-2} \end{aligned} \tag{20}$$

$$\frac{dx_3}{d\tau} = p_3 A_{33} + \Big\{ p_3 \Big[\left(E_{13} + E_{23} - E_{12} \right) p_1^2 p_2^2 + \left(E_{13} p_1^4 + E_{23} p_2^4 \right) \Big] \left(p_k p_k \right)^{-2} \Big\} +$$
$$2\Big\{ 2\left(A_{36} + 2A_{45} \right) p_1 p_2 p_3 \left(p_1^2 + p_2^2 \right) +$$
$$\Big[3\left(p_1^2 + p_2^2 \right) + p_3^2 \Big] p_3^2 \left[A_{35} p_1 + A_{34} p_2 \right] + \tag{21}$$
$$\left(p_1^2 + p_2^2 - p_3^2 \right) \Big[\Big[\left(A_{14} + 2A_{56} \right) p_2 + A_{15} p_1 \Big] p_1^2 +$$
$$\Big[\left(A_{25} + 2A_{46} \right) p_1 + A_{24} p_2 \Big] p_2^2 \Big] -$$
$$2\left(A_{16} p_1^2 + A_{26} p_2^2 \right) p_1 p_2 p_3 \Big\} \left(p_k p_k \right)^{-2}$$

with the corresponding quantities $dp_i/d\tau$ having the form

$$\frac{dp_j}{d\tau} = \frac{1}{2}\Bigg\{ \frac{dA_{11}}{dx_j} p_1^2 + \frac{dA_{22}}{dx_j} p_2^2 + \frac{dA_{33}}{dx_j} p_3^2 +$$
$$\Bigg[\frac{dE_{12}}{dx_j} p_1^2 p_2^2 + \frac{dE_{13}}{dx_j} p_1^2 p_3^2 + \frac{dE_{23}}{dx_j} p_2^2 p_3^2 + \tag{22}$$
$$4\left(\frac{dH_1}{dx_j} p_1^2 + \frac{dH_2}{dx_j} p_2^2 + \frac{dH_3}{dx_j} p_3^2 \right) \Bigg] \Bigg/ \left(p_k p_k \right) \Bigg\}$$

It is convenient to introduce the group velocity angles, that is, the azimuthal and polar angles at which the ray propagates. The azimuthal angle, Φ, $\left(0 \le \Phi < 2\pi\right)$ may be determined from

$$\tan\Phi = \left[\frac{dx_2}{dx_1} \right] = \left[\frac{dx_2/d\tau}{dx_1/d\tau} \right]. \tag{23}$$

Defining the projection of the 3D group velocity vector onto the $\left(x_1, x_2 \right)$ plane as

$$\frac{dr}{d\tau} = \left[\left(\frac{dx_1}{d\tau} \right)^2 + \left(\frac{dx_2}{d\tau} \right)^2 \right]^{1/2}, \tag{24}$$

the group polar angle, Θ, $\left(0 \le \Theta \le \pi\right)$ is obtained from

$$\tan\Theta = \left[\frac{dr}{dx_3} \right] = \left[\frac{dr/d\tau}{dx_3/d\tau} \right]. \tag{25}$$

As the formulae presented here are linearized approximations to the exact ray propagation problem, it may be useful to obtain an appreciation of this problem, starting with the qP eikonal given in Schoenberg and Helbig (1996) and employ equations (14) and (15) above will produce the exact ray tracing equations after a moderate amount of basic mathematics. (See also, Gajewski and Pšenčík, 1987).

It should be clear that there is much more to the most general case of this problem than has been presented. However, the formulae derived are enough to undertake a preliminary investigation of tracing rays in a general anisotropic medium. A large section dealing with the 3D anisotropic analogue of Snell's Law has been deleted as the velocity model used in this report is parametric. Additionally, for both parametric and non-parametric interface specifications, the ray tracing program has been designed to rely heavily on the model building algorithm. A fairly general model building algorithm has not yet been implemented.

DISCUSSION OF METHOD

In a related report in this volume a similar problem is addressed using linearized qP group velocities so that there are only two levels of embedded equations that must be solved rather than the three for this problem specification. As mentioned in the other report, model building for some three dimensional structure can be as involved as the ray tracing procedure. For that reason, only an anisotropic halfspace will be considered in this section. The square roots of the anisotropic parameters (velocities) will be allowed linear gradients in all three Cartesian dimensions. The type of spatial variation just mentioned has the A_{ij} being specified as

$$A_{ij} = A_{ij}^{0}\left[1 + g_1\left(x_1 - x_1^{(0)}\right) + g_2\left(x_2 - x_2^{(0)}\right) + g_3\left(x_3 - x_3^{(0)}\right)\right]^2 \tag{26}$$

or more generally

$$A_{ij} = A_{ij}^{0}\left[1 + \mathbf{g}\cdot\left(\mathbf{x} - \mathbf{x}^{(0)}\right)\right]^2 \tag{27}$$

where $A_{ij}^{(0)}$ are the known values of the anisotropic parameters A_{ij} at some point $\mathbf{x}^{(0)}$ in a three dimensional $(3D)$ Cartesian space. The vector $\mathbf{g} = (g_1, g_2, g_3)$ defines the three dimensional velocity gradient common to all of the A_{ij} relative to the reference point $\mathbf{x}^{(0)}$. The point $\mathbf{x} = (x_1, x_2, x_3)$ is some point within some $3D$ volume at which the group velocity is required.

The model that will be considered is the weakly anellipsoidal orthorhombic material, whose anisotropic properties are similar in degree of anisotropy to transversely isotropic clay-shale associated with hydrocarbon deposits that could have been made orthorhombic (azimuthally anisotropic) through the introduction of vertical fracturing. The model is defined by the density normalized anisotropic parameters, A_{ij}, which have the dimensions of velocity squared $(km/s)^2$ and given in Table 1 and is similar to olivine.

A cone of rays from a point source locate at the surface are shot at azimuthal phase angles $\left(0 \le \phi < 360^{\circ}\right)$ at 5° increments. Polar angles $\left(-80^{\circ} \le \theta \le 80^{\circ}\right)$ at 5° increments are used. The rays are forced to lie within the spatial volume $(-10 \le x \le 10)$,

$(-10 \le y \le 10)$, $(0 \le z \le 20)$. The dimensions are in *km*. The velocity gradient used is $\mathbf{g} = (0.01 km/s, 0.01 km/s, 0.01 km/s)$. The rays generated are shown in Figure 1, with a scatter plot of those rays which arrive at the reference plane at a depth of $z = 20km$ given in Figure 2. The corresponding travel times for these arrivals are given in Figure 3.

CONCLUSIONS AND FUTURE WORK

A quasi-compressional (*qP*) linearized eikonal is presented for elastic wave propagation in a general 21 parameter parameter medium. Using Hamilton's method, six coupled nonlinear ordinary differential equations are derived for determining the 3 Cartesian components of the spatial locations of points along the ray as well as the Cartesian components of the corresponding slowness vector in slowness space. There exists standard (free) software for this solution method. However, a problem specific algorithm was written to provide better control of the solution, particularly the size of the time step, time being the dependent parameter. The above code is the nucleus of a two point ray tracing scheme. A significant amount of other code is required for a true two point ray tracing program, but the part described here is basic.

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$\mathbf{a}_{11}^{(0)}$	$\mathbf{a}_{22}^{(0)}$	$\mathbf{a}_{33}^{(0)}$	$\mathbf{a}_{44}^{(0)}$	$\mathbf{a}_{55}^{(0)}$	$\mathbf{a}_{66}^{(0)}$	$\mathbf{a}_{12}^{(0)}$	$\mathbf{a}_{13}^{(0)}$	$\mathbf{a}_{23}^{(0)}$
9.779	5.970	7.103	1.952	2.359	2.388	2.006	2.163	2.284

Table 1. Anisotropic parameters at the surface (*z=0km*). The dimensions are (*km/s*)^2.

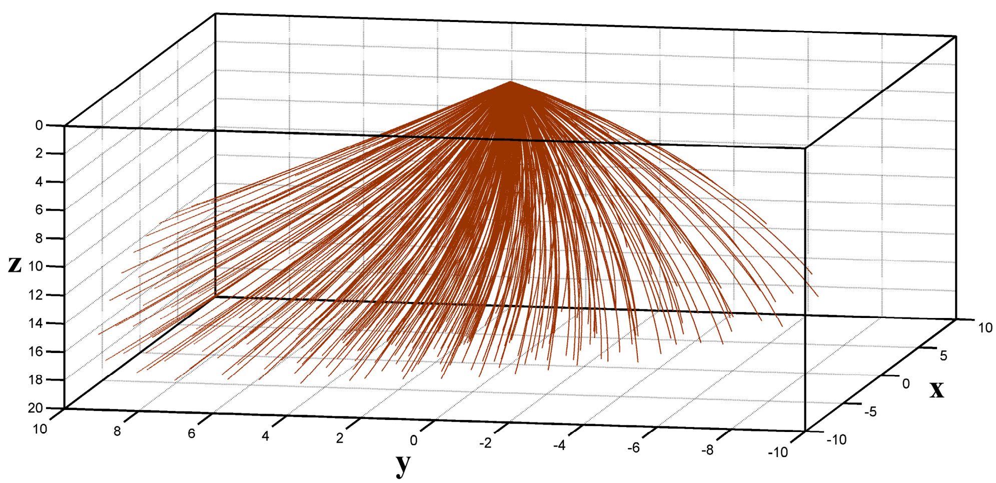

Fig. 1 Rays from a point source locate at the surface at azimuthal phase angles $\left(0 \le \phi < 360^{\circ}\right)$ at 5° increments. Polar angles $\left(-80^{\circ} \le \theta \le 80^{\circ}\right)$ at 5° increments are used. The rays are forced to lie within the spatial volume $(-10 \le x \le 10)$, $(-10 \le y \le 10)$, $(0 \le z \le 20)$. The dimensions are in *km*.

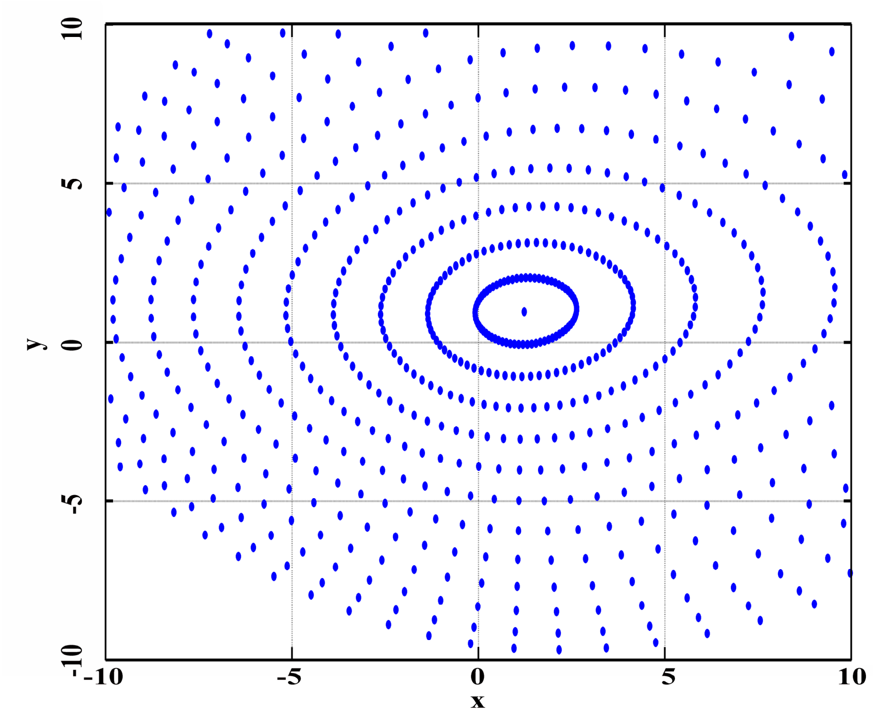

Fig. 2 Scatter plot of the *(x,y)* locations of those rays that arrive at the reference depth level of 20*km*.

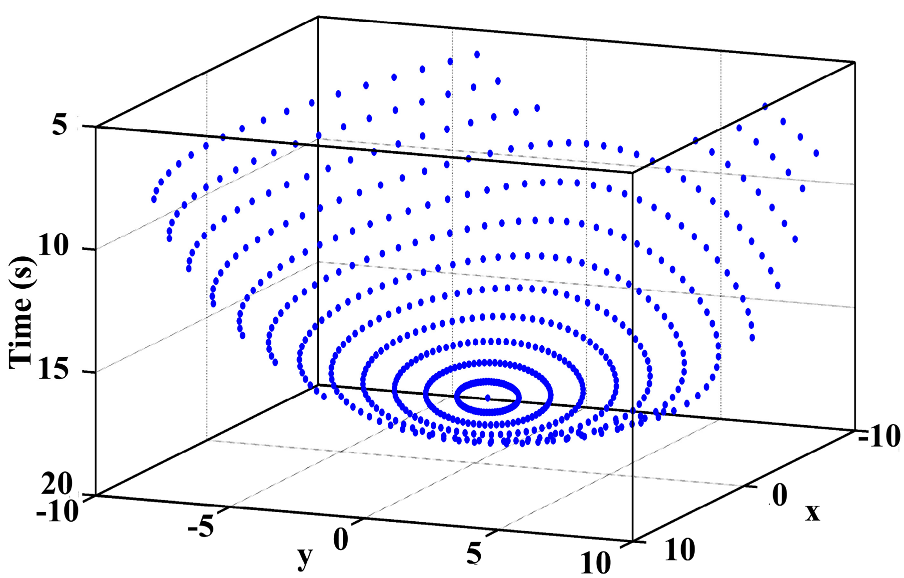

Fig. 3 Ray travel times at the reference depth of *z=20km*.