An acoustic description of nonlinearity in seismic exploration

Kris Innanen

ABSTRACT

In 2008, the Priddis pump-probe experiment, in which the Earth was simultaneously subjected to a vibrating source and a transient source, was carried out by CREWES with the hope of detecting nonlinear behaviour in an exploration seismology setting. In this paper we present a theoretical description of such behaviour, assuming an acoustic medium. Through a simple extension of the order arguments by which fluid equations are typically manipulated to form a linear wave equation on the pressure p_1, we find a nonlinear equation for a corrective term, p_2, such that in total the field, $p = p_1 + p_2$, is aware of the changes it itself makes in the medium through which it propagates.

INTRODUCTION

In 2008, CREWES researchers carried out and presented a discussion of the "Priddis pump-probe experiment" (Margrave et al., 2008), in which an Earth volume was simultaneously subjected to a strong vibrating source and a transient source. The point was to determine whether the Earth property variations induced by the vibrating background field would be of sufficient amplitude to alter the transient experiment, creating a different response than would have been generated had the transient experiment occurred in a quiet background.

At heart, this is a question of the presence or absence of nonlinear phenomena in exploration seismology as we carry it out today. Whether or not, that is, seismic waves influence themselves, scattering from each other and altering their own amplitudes (Figure 1), and whether or not such phenomena could conceivably rise above the noise level in our records. From a theoretical point of view, the idea is far from outlandish. Since "normal" seismic waves—which pass each other like ships in the night—are predicted by equations of motion that have been explicitly linearized, it is, in principle, only a matter of the low amplitude of non-linear seismic phenomena, rather than some fundamental issue, that best explains their current absence. This invisibility may or may not last. Ongoing advances in instrument sensitivity, survey design, and processing methods, in particular coupled with a newfound interest in recording seismic data from shots set off simultaneously (Beasely, 2010), suggest that a theoretical description of the non-linear interaction of seismic wave fields would be a timely contribution.

In this note we derive a simple acoustic description of interacting seismic waves—we make the ships, in other words, if not collide then at least pass each other during daylight hours. This is a starting point only. To validate it as a theory, we must next analyze the resulting equations and predict data variations which match with the results of experiment like that of the Priddis pump-probe. Since the results of that experiment were reported as ambiguous, we all appear to have some work left to do.

The only other theoretical work on nonlinear seismology recently reported in our com-

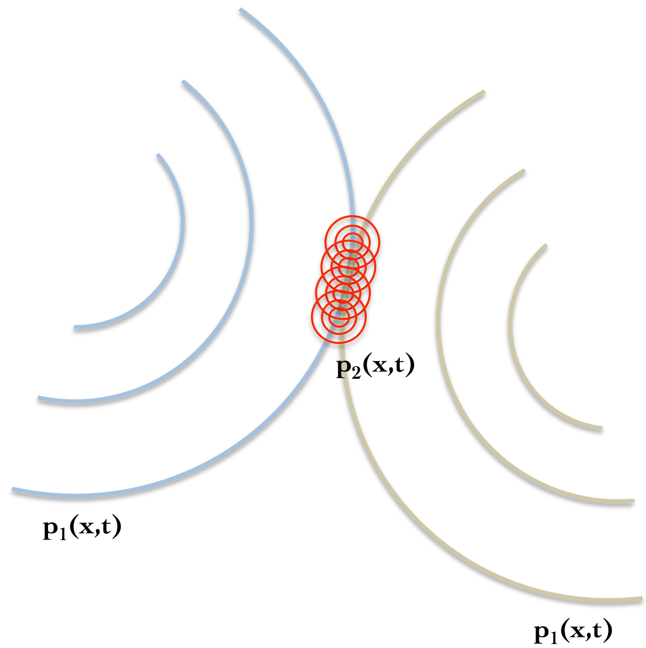

FIG. 1. Seismic waves alter the density and moduli of the Earth as they pass. In principle this means that two incident seismic fields might be seen to scatter from each other. Indeed the amplitude of a local portion of the seismic wave should be expected to influence itself, if its amplitudes are large enough. But if a wave, which depends on the properties of the medium it is in, changes those properties wherever it actually exists, how do you start the problem?

munity (that we are aware of) is that of Chesnokov et al. (2009); it is not fully clear to us how the work of those authors relates to what is presented here, beyond that they have a very different superficial appearance. Beyond this, nonlinear models have been used to describe certain narrow seismic phenomena. For instance in the 1960s there was some consensus that anelastic losses were likely due to nonlinear constitutive relations (Kolsky, 1953; White, 2000); this consensus is nowhere to be seen now, perhaps because of the success of linear models, or perhaps instead because nonlinear models are more difficult to analyze. The approach we have taken is specifically designed to allow the nonlinearity to be expressed in terms of linear waves that are modified by a secondary field which overlays it. Beyond that, the theory here will have to distinguish itself from any others in the clarity with which it makes interpretable and measurable predictions.

As a starting point we adopt the basic framework of Landau (see, e.g., pp 1-6 & 245-246 of Landau and Lifshitz, 1959), which appears to have been used to good effect in deriving linear wave equations and solutions by De Santo (1992). If, instead of doing what we are about to do, one were to linearize the constitutive relations to follow, and treat only first-order variations in the field variables, the linear acoustic wave equations described by these authors would be recovered.

FLUID EQUATIONS

We assume, but for purposes of space do not include, space and time dependence on all fluid variables, i.e., $p(\mathbf{x}, t) = p$, etc. All of the linear and non-linear results we will provide in this note derive from the same precursors that give rise to the acoustic equations of motion, namely the fluid equations, first, Euler's equation, which is a modified statement of momentum conservation:

$$\frac{\partial p}{\partial x_j} = -\rho\left(\frac{\partial v_j}{\partial t} + v_i \frac{\partial v_j}{\partial x_i}\right), \tag{1}$$

and the equation of continuity which is a statement of mass conservation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_j)}{\partial x_j} = 0, \tag{2}$$

where p is the pressure, ρ is the mass density, and v_j is the vector fluid velocity.

ORDER

We distinguish the behaviour of the fluid medium over a likely wide range of scales and magnitudes by expressing the variables p, ρ, and v_i as expansions in orders of the parameter ϵ:

$$\begin{aligned} p &= p_0 + \epsilon p_1 + \epsilon^2 p_2 + ..., \\ v_i &= v_{i_0} + \epsilon v_{i_1} + \epsilon^2 v_{i_2} + ..., \\ \rho &= \rho_0 + \epsilon \rho_1 + \epsilon^2 \rho_2 + ... \quad . \end{aligned} \tag{3}$$

For convenience in the development to follow, we will generate these expansions through a replacement rule. For instance, the pressure will be initially expressed as

$$p = p_0 + \epsilon p_1, \tag{4}$$

which reproduces equation (3) to first order in ϵ, and is appropriate in distinguishing large scale (i.e., hydrostatic and hydrodynamic) aspects of the variables (p_0) from acoustic aspects (p_1). Then, when we "look closer" at the fields, to discuss their non-linear activity, we will replace p_1 in equation (4) with

$$p_1 \rightarrow p_1 + \epsilon p_2, \tag{5}$$

which reproduces equation (3) to second order in ϵ. This is as far as we will need to go in our analysis, but evidently the replacement rule

$$p_n \rightarrow p_n + \epsilon p_{n+1} \tag{6}$$

applied sequentially will correctly reproduce the $n + 1$'th order term in equation (3). The expansion in equations (4)–(6) is then repeated on ρ and v_i. As in standard acoustics, the key is that the same order parameter is used in the expansions of all three variables, hence the product $p_1\rho_1$, for example, is considered to have an amplitude of the same order as p_2.

CONSTITUTIVE RELATIONS

To close the fluid equations requires a constitutive relation of the form

$$p = f(\rho). \tag{7}$$

To discuss non-linear influences we need to extend beyond the linearizations typical of acoustic theory. Using the order relations of the previous section, we have, initially,

$$p_0 + \epsilon p_1 = f\left(\rho_0 + \epsilon\rho_1\right). \tag{8}$$

Expanding f about ρ_0,

$$f(\rho_0 + \epsilon\rho_1) = f(\rho_0) + \epsilon f'(\rho_0)\rho_1 + \epsilon^2 \frac{1}{2} f''(\rho_0)\rho_1^2 + ..., \tag{9}$$

where the prime denotes the derivative with respect to ρ_0, equation (8) becomes

$$p_0 + \epsilon p_1 = f(\rho_0) + \epsilon f'(\rho_0)\rho_1 + \epsilon^2 \frac{f''(\rho_0)}{2}\rho_1^2 + ... \;. \tag{10}$$

This has exposed a portion of the second order behaviour of $p = f(\rho)$, through the term in ρ_1^2. However, because of our use of equation (4) and the framework of the previous section, other second order and higher behaviour is embedded in p_1 and ρ_1. To account for this explicitly, we replace ρ_1 and p_1 using equation (5), obtaining

$$\begin{aligned} &p_0 + \epsilon p_1 + \epsilon^2 p_2 \\ &\quad = f(\rho_0) + \epsilon f'(\rho_0)\rho_1 + \epsilon^2 f'(\rho_0)\rho_2 + \epsilon^2 \frac{f''(\rho_0)}{2}\rho_1^2. \end{aligned} \tag{11}$$

Since we have used a replacement rule, the variables with subscript 2 in fact contain second order and all higher order behaviour. We will from now on assume that variables at third order and higher in ϵ are negligible. Equating like orders, we obtain the relations

$$p_0 = f(\rho_0), \tag{12}$$

$$p_1 = f'(\rho_0)\rho_1, \tag{13}$$

and

$$p_2 = f'(\rho_0)\rho_2 + \frac{1}{2} f''(\rho_0)\rho_1^2. \tag{14}$$

From equation (12) we have that

$$f'(\rho_0) = \frac{\partial p_0}{\partial \rho_0}, \tag{15}$$

and this, along with equation (13), provides the relation

$$\rho_1 = \left(\frac{\partial \rho_0}{\partial p_0}\right) p_1, \tag{16}$$

by which first order density variations are typically eliminated in deriving a linear wave equation on the pressure. We now add to this a formula, derived from equations (12)–(15), by which second order density variations may be likewise eliminated:

$$\rho_2 = \left(\frac{\partial \rho_0}{\partial p_0}\right) p_2 - \frac{1}{2}\left(\frac{\partial^2 p_0}{\partial \rho_0^2}\right)\left(\frac{\partial \rho_0}{\partial p_0}\right)^3 p_1^2. \tag{17}$$

Defining

$$\begin{aligned} \frac{1}{c_1^2} &\equiv \left(\frac{\partial \rho_0}{\partial p_0}\right) \\ \frac{1}{c_2^2} &\equiv -\frac{1}{2}\left(\frac{\partial^2 p_0}{\partial \rho_0^2}\right)\left(\frac{\partial \rho_0}{\partial p_0}\right)^3, \end{aligned} \tag{18}$$

we can instead write more simply:

$$\rho_1 = \frac{1}{c_1^2} p_1, \tag{19}$$

and

$$\rho_2 = \frac{1}{c_1^2} p_2 + \frac{1}{c_2^2} p_1^2. \tag{20}$$

NON-LINEAR ACOUSTIC WAVE EQUATIONS

We begin to separate out acoustic motions from other motions of the fluid by substituting the forms in equations (3)–(5) into equations (1)–(2). From the continuity equation we derive three relations, at zero'th, first, and second order:

$$\frac{\partial \rho_0}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho_0 v_{j_0}\right) = 0, \tag{21}$$

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho_0 v_{j_1} + \rho_1 v_{j_0}\right) = 0, \tag{22}$$

$$\frac{\partial \rho_2}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho_0 v_{j_2} + \rho_1 v_{j_1} + \rho_2 v_{j_0}\right) = 0, \tag{23}$$

and likewise from Euler's equation:

$$\frac{\partial p_0}{\partial x_j} + \rho_0 \frac{\partial v_{j_0}}{\partial t} + \rho_0 v_{i_0} \frac{\partial v_{j_0}}{\partial x_i} = 0, \tag{24}$$

$$\frac{\partial p_1}{\partial x_j} + \rho_0 \frac{\partial v_{j_1}}{\partial t} + \rho_1 \frac{\partial v_{j_0}}{\partial t} + \rho_0 v_{i_0} \frac{\partial v_{j_1}}{\partial x_i} + \rho_0 v_{i_1} \frac{\partial v_{j_0}}{\partial x_i} + \rho_1 v_{i_0} \frac{\partial v_{j_0}}{\partial x_i} = 0, \tag{25}$$

and

$$\begin{aligned} &\frac{\partial p_2}{\partial x_j} + \rho_0 \frac{\partial v_{j_2}}{\partial t} + \rho_1 \frac{\partial v_{j_1}}{\partial t} + \rho_2 \frac{\partial v_{j_0}}{\partial t} + \rho_0 v_{i_0} \frac{\partial v_{j_2}}{\partial x_i} + \rho_0 v_{i_1} \frac{\partial v_{j_1}}{\partial x_i} \\ &\quad + \rho_0 v_{i_2} \frac{\partial v_{j_0}}{\partial x_i} + \rho_1 v_{i_0} \frac{\partial v_{j_1}}{\partial x_i} + \rho_1 v_{i_1} \frac{\partial v_{j_0}}{\partial x_i} + \rho_2 v_{i_0} \frac{\partial v_{j_0}}{\partial x_i} = 0. \end{aligned} \tag{26}$$

We next follow standard acoustic theory and set the hydrodynamic velocity $v_{i_0} = 0$, i.e., neglect large scale flow of the fluid medium. This simplifies equations (21)–(26) considerably, leaving

$$\frac{\partial \rho_0}{\partial t} = 0, \tag{27}$$

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial}{\partial x_j}\rho_0 v_{j_1} = 0, \tag{28}$$

$$\frac{\partial \rho_2}{\partial t} + \frac{\partial}{\partial x_j}\left(\rho_0 v_{j_2} + \rho_1 v_{j_1}\right) = 0, \tag{29}$$

$$\frac{\partial p_0}{\partial x_j} = 0, \tag{30}$$

$$\frac{\partial p_1}{\partial x_j} + \rho_0 \frac{\partial v_{j_1}}{\partial t} = 0, \tag{31}$$

$$\frac{\partial p_2}{\partial x_j} + \rho_0 \frac{\partial v_{j_2}}{\partial t} + \rho_1 \frac{\partial v_{j_1}}{\partial t} + \rho_0 v_{i_1} \frac{\partial v_{j_1}}{\partial x_i} = 0. \tag{32}$$

Again following standard derivation procedure, we note that since equation (30) disallows 0th order pressure gradients, and since equation (16) links ρ_0 and p_0 explicitly, it must be that $\partial \rho_0 / \partial x_j = 0$. This simplifies equation (28), just as it does in linear acoustics, and also, now, equation (29). Together with equations (31)–(32), we then have

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \frac{\partial v_{j_1}}{\partial x_j} = 0, \tag{33}$$

$$\frac{\partial \rho_2}{\partial t} + \rho_0 \frac{\partial v_{j_2}}{\partial x_j} + \frac{\partial}{\partial x_j}\rho_1 v_{j_1} = 0. \tag{34}$$

$$\frac{\partial p_1}{\partial x_j} + \rho_0 \frac{\partial v_{j_1}}{\partial t} = 0, \tag{35}$$

$$\frac{\partial p_2}{\partial x_j} + \rho_0 \frac{\partial v_{j_2}}{\partial t} + \rho_1 \frac{\partial v_{j_1}}{\partial t} + \rho_0 v_{i_1} \frac{\partial v_{j_1}}{\partial x_i} = 0. \tag{36}$$

Next, we eliminate all 1st and 2nd order density variations in favour of pressure, using the constitutive relations in equations (19) and (20), obtaining

$$\frac{1}{c_1^2}\frac{\partial p_1}{\partial t} + \rho_0 \frac{\partial v_{j_1}}{\partial x_j} = 0, \tag{37}$$

$$\frac{\partial p_1}{\partial x_j} + \rho_0 \frac{\partial v_{j_1}}{\partial t} = 0, \tag{38}$$

$$\frac{1}{c_1^2}\frac{\partial p_2}{\partial t} + \frac{1}{c_2^2}\frac{\partial p_1^2}{\partial t} + \rho_0 \frac{\partial v_{j_2}}{\partial x_j} + \frac{1}{c_1^2}\frac{\partial}{\partial x_j} p_1 v_{j_1} = 0, \tag{39}$$

$$\frac{\partial p_2}{\partial x_j} + \rho_0 \frac{\partial v_{j_2}}{\partial t} + \frac{1}{c_1^2} p_1 \frac{\partial v_{j_1}}{\partial t} + \rho_0 v_{i_1} \frac{\partial v_{j_1}}{\partial x_i} = 0. \tag{40}$$

Equations (37)–(38) are the well-known basic linear acoustic equations, and equations (39)–(40) are our non-linear additions. By forming

$$\frac{\partial}{\partial x_j}[\text{equation } 38] - \frac{\partial}{\partial t}[\text{equation } 37],$$

we eliminate velocity and produce an equation on first order variations in pressure p_1, which is, as expected, the wave equation:

$$\left(\frac{\partial^2}{\partial x_j^2} - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right) p_1 = 0. \tag{41}$$

We may also eliminate second order velocity variations from the non-linear equations (39)–(40) in favour of pressure. Forming

$$\frac{\partial}{\partial x_j}[\text{equation } 40] - \frac{\partial}{\partial t}[\text{equation } 39],$$

we obtain

$$\left(\frac{\partial^2}{\partial x_j^2} - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right) p_2 = \frac{1}{c_2^2}\frac{\partial^2}{\partial t^2}(p_1^2) + \frac{1}{c_1^2}\frac{\partial}{\partial x_j}\left(\frac{\partial p_1}{\partial t} v_{j_1}\right) - \rho_0 \frac{\partial}{\partial x_j}\left(v_{i_1}\frac{\partial v_{j_1}}{\partial x_i}\right). \tag{42}$$

This may be further simplified by isolating the time derivative of p_1 in equation (37) and substituting it into the right-hand side:

$$\left(\frac{\partial^2}{\partial x_j^2} - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right) p_2 = \frac{1}{c_2^2}\frac{\partial^2}{\partial t^2}(p_1^2) - \rho_0 \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\left(v_{i_1} v_{j_1}\right). \tag{43}$$

This makes a relatively clean elimination of velocity possible, which we do by solving for $\partial v_i/\partial t$ in equation (38) and integrating. This results in, finally,

$$\left(\frac{\partial^2}{\partial x_j^2} - \frac{1}{c_1^2}\frac{\partial^2}{\partial t^2}\right) p_2 = \mathcal{S}(p_1), \tag{44}$$

where

$$\mathcal{S}(p_1) \equiv \frac{1}{c_2^2}\frac{\partial^2}{\partial t^2}(p_1^2) + \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\left(\int_{-\infty}^{t} \frac{\partial p_1}{\partial x_i} dt \int_{-\infty}^{t} \frac{\partial p_1}{\partial x_j} dt\right). \tag{45}$$

DISCUSSION & CONCLUSIONS

Equations (41) & (44) are the coupled non-linear acoustic wave equations on the pressure, with the former being recognizable as the linear wave equation, and the latter being the non-linear addition. The second equation looks like a "normal," linear wave equation on p_2, but with a time and space varying *source function* $\mathcal{S}$ that depends on the linear result p_1. This may be an intuitively useful expression if we wish to predict a second order wave (p_2) arising from the interaction of two first order waves (p_1).

Clearly the next step is to begin to study the nature and detail of solutions arising from these equations. There are two qualitatively different types of result we will attempt to produce. First, an examination of the kinematic behaviour of p_2 may confirm or refute the idea that wave components we would identify in our data as *new events* might be created by nonlinear effects. Second, study of the amplitudes may provide order of magnitude predictions of the relative importance of nonlinear v. linear seismology, possibly leading to predictions regarding the likelihood of detecting these behaviours in a seismic record.

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