Towards an analytic description of anelastic diffraction, reflection, and conversion phenomena

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ABSTRACT

In this note we lay some of the groundwork for a scattering theoretic description of anelastic wave propagation. The aim is to create a framework for (1) describing the diffraction and conversion of anelastic waves in heterogeneous media, and (2) directly inverting P, S, and converted wave data taken over dissipative media. Here we take the simple but important step of expressing reference and perturbed anelastic wave equations in diagonalized forms, which are then prepared for inclusion in an appropriate Scattering, or Lippmann-Schwinger equation. As a side note we also consider appropriate situations for the use of a popular relationship by which Q_P is related to Q_S.

INTRODUCTION

The purpose of this note is to begin constructing the mathematical ingredients required for a scattering description of anelastic waves. Our interests are twofold. First, modeling of the interaction of anelastic waves with heterogeneous media. Anelastic media are known to support certain inhomogeneous S-waves that have no elastic analogue; relatively complete theoretical descriptions of these waves have been presented for layered and homogeneous media (Borcherdt, 2009), but far less is understood about their behaviour in an arbitrary 2D and 3D Earth. Second, forming inverse procedures for data reflecting from arbitrary structures that dissipate seismic wave energy. Linear and nonlinear inverse scattering theory and processing procedures have been developed for anacoustic media (Innanen and Weglein, 2007; Innanen and Lira, 2010); however, for direct inverse procedures, which rely on seismic amplitudes, these basic developments must be extended to the anelastic case prior to field data application. Here we put together the first steps in the development, namely, the expression of the reference and perturbed anelastic wave equations, and their transformation to diagonalized form.

MODIFYING ELASTIC MODULI TO INCLUDE Q_P AND Q_S

In an elastic medium we have the following relationship between the [real] P- and S-wave phase velocities α and β and the [real] Lamé parameters λ and μ, and the density ρ:

$$\begin{aligned}
&\rho\alpha^2 = \lambda + 2\mu \equiv \gamma \\
&\rho\beta^2 = \mu, \\
&\rho, \alpha, \beta, \lambda, \mu \in \mathcal{R}.
\end{aligned} \tag{1}$$

Dissipation of the elastic potential energy held by a medium when it is under a shear and/or a volumetric strain may be incorporated with little apparent alteration to these relationships. To be given the opportunity to describe such processes, stress and strain, which are related via the quantities in equation (1), must become functions not only of each other's instantaneous values, but also of their past values. This in turn requires that the time domain influence of λ and μ must be convolutional rather than multiplicative, but if we focus on frequency domain wave descriptions, the multiplicative relationship, and the forms in equation (1), are recovered, albeit with complex, frequency dependent α, β, λ and μ (Borcherdt, 2009), signified with a $\hat{\cdot}$:

$$\begin{aligned}
&\rho\hat{\alpha}^2(\omega) = \hat{\lambda}(\omega) + 2\hat{\mu}(\omega) \equiv \hat{\gamma}(\omega) \\
&\rho\hat{\beta}^2(\omega) = \hat{\mu}(\omega), \\
&\rho \in \mathcal{R}, \\
&\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\mu} \in \mathcal{C}.
\end{aligned} \tag{2}$$

Fundamentally the complex nature of equation (2) is due to our alteration of λ and μ; the complexity of the phase velocities $\hat{\alpha}$ and $\hat{\beta}$ is then simply a mathematical necessity. Inertial density ρ must always remain real, or mass would not be locally conserved.

That said, in practice, convenient representation of the untold individual mechanisms of anelasticity that we must assume occur within the Earth as a wave passes, often takes the form of a re-casting of the phase velocities subject to several macroscopic requirements, principally linearity, causality and constant or nearly constant amplitude losses per cycle. Hence, in developing our description of anelastic wave scattering, we will not alter the moduli, which physics would seem to demand, but rather, for the sake of convenience, we will take as our starting point an anelastic alteration of the phase velocities, within which we will incorporate two "nearly constant Q" relationships. This will imply then that we treat anelasticity as a modification of *linear combinations* of the Lamé parameters, rather than the parameters themselves.

Adopting the nearly constant Q model reviewed by Aki and Richards (2002) we consider then a version of equation (2) that reads

$$\begin{aligned}
&\rho\alpha^2 \left[1 + \frac{i}{2Q_P} - \frac{1}{\pi Q_P}\log\left(\frac{\omega}{\omega_P}\right)\right]^{-2} = \hat{\lambda}(\omega) + 2\hat{\mu}(\omega), \\
&\rho\beta^2 \left[1 + \frac{i}{2Q_S} - \frac{1}{\pi Q_S}\log\left(\frac{\omega}{\omega_S}\right)\right]^{-2} = \hat{\mu}(\omega), \\
&\rho, \alpha, \beta \in \mathcal{R}, \\
&\hat{\lambda}, \hat{\mu} \in \mathcal{C},
\end{aligned} \tag{3}$$

which, for weak attenuation, may alternatively be written

$$\begin{aligned}
&\rho\alpha^2 - \rho\alpha^2 \left[\frac{i}{Q_P} - \frac{2}{\pi Q_P}\log\left(\frac{\omega}{\omega_P}\right)\right] = \hat{\lambda}(\omega) + 2\hat{\mu}(\omega), \\
&\rho\beta^2 - \rho\beta^2 \left[\frac{i}{2Q_S} - \frac{2}{\pi Q_S}\log\left(\frac{\omega}{\omega_S}\right)\right] = \hat{\mu}(\omega).
\end{aligned} \tag{4}$$

Equation (4) is strongly suggestive that, in deciding how quality factors thus defined can be seen as alterations of the nominal elastic properties, we view Q_P and Q_S as parameters that generate deviations, in $\hat{\lambda}(\omega) + 2\hat{\mu}(\omega)$ and $\hat{\mu}(\omega)$ respectively, away from real reference values $\gamma = \lambda + 2\mu$ and μ:

$$\begin{aligned}\hat{\gamma}(\omega) &= \gamma \left[1 + \frac{F_P(\omega)}{Q_P}\right], \\ \hat{\mu}(\omega) &= \mu \left[1 + \frac{F_S(\omega)}{Q_S}\right],\end{aligned} \tag{5}$$

where

$$\begin{aligned}F_P(\omega) &= -i + \frac{2}{\pi}\log\left(\frac{\omega}{\omega_P}\right) \\ F_S(\omega) &= -i + \frac{2}{\pi}\log\left(\frac{\omega}{\omega_S}\right),\end{aligned} \tag{6}$$

and ω_P and ω_S are the reference frequencies at which the P-wave and the S-waves propagate at α and β respectively.

ANELASTIC WAVE EQUATIONS

Displacement formulation

Anelastic displacements in a volume of space absent sources, can be expressed in the space-frequency domain as an elastic wave operator $\mathcal{L}_A$ acting on the displacement vector $\mathbf{u} = \mathbf{u}(\mathbf{x}, \omega)$:

$$\mathcal{L}_{\text{AE}}\mathbf{u} = 0, \tag{7}$$

where $\mathbf{u} = (u_x, u_y, u_z)^T$, and $\mathbf{x} = (x, y, z)^T$, and

$$\mathcal{L}_{\text{AE}} = \begin{bmatrix} L_{xx} & L_{xy} & L_{xz} \\ L_{yx} & L_{yy} & L_{yz} \\ L_{zx} & L_{zy} & L_{zz} \end{bmatrix}. \tag{8}$$

The elements of $\mathcal{L}_e$ contain instances of the density, and spatial derivatives acting on spatially-variant Lamé parameters and quality factors, in addition to the elements of $\mathbf{u}$:

$$\begin{aligned}L_{xx} &= \partial_x\gamma\left[1 + \frac{F_P}{Q_P}\right]\partial_x + \partial_y\mu\left[1 + \frac{F_S}{Q_S}\right]\partial_y + \partial_z\mu\left[1 + \frac{F_S}{Q_S}\right]\partial_z + \rho\omega^2 \\ L_{yy} &= \partial_y\gamma\left[1 + \frac{F_P}{Q_P}\right]\partial_y + \partial_x\mu\left[1 + \frac{F_S}{Q_S}\right]\partial_x + \partial_z\mu\left[1 + \frac{F_S}{Q_S}\right]\partial_z + \rho\omega^2 \\ L_{zz} &= \partial_z\gamma\left[1 + \frac{F_P}{Q_P}\right]\partial_z + \partial_x\mu\left[1 + \frac{F_S}{Q_S}\right]\partial_x + \partial_y\mu\left[1 + \frac{F_S}{Q_S}\right]\partial_y + \rho\omega^2,\end{aligned} \tag{9}$$

and

$$
\begin{aligned}
L_{xy} &= \partial_x \left\{ \gamma \left[1 + \frac{F_P}{Q_P} \right] - 2\mu \left[1 + \frac{F_S}{Q_S} \right] \right\} \partial_y + \partial_y \mu \left[1 + \frac{F_S}{Q_S} \right] \partial_x \\
L_{xz} &= \partial_x \left\{ \gamma \left[1 + \frac{F_P}{Q_P} \right] - 2\mu \left[1 + \frac{F_S}{Q_S} \right] \right\} \partial_z + \partial_z \mu \left[1 + \frac{F_S}{Q_S} \right] \partial_x \\
L_{yx} &= \partial_y \left\{ \gamma \left[1 + \frac{F_P}{Q_P} \right] - 2\mu \left[1 + \frac{F_S}{Q_S} \right] \right\} \partial_x + \partial_x \mu \left[1 + \frac{F_S}{Q_S} \right] \partial_y \\
L_{yz} &= \partial_y \left\{ \gamma \left[1 + \frac{F_P}{Q_P} \right] - 2\mu \left[1 + \frac{F_S}{Q_S} \right] \right\} \partial_z + \partial_z \mu \left[1 + \frac{F_S}{Q_S} \right] \partial_y \\
L_{zx} &= \partial_z \left\{ \gamma \left[1 + \frac{F_P}{Q_P} \right] - 2\mu \left[1 + \frac{F_S}{Q_S} \right] \right\} \partial_x + \partial_x \mu \left[1 + \frac{F_S}{Q_S} \right] \partial_z \\
L_{zy} &= \partial_z \left\{ \gamma \left[1 + \frac{F_P}{Q_P} \right] - 2\mu \left[1 + \frac{F_S}{Q_S} \right] \right\} \partial_y + \partial_y \mu \left[1 + \frac{F_S}{Q_S} \right] \partial_z .
\end{aligned}
\tag{10}
$$

where F_P and F_S are the complex, frequency dependent functions defined in equations (6). If the medium is homogeneous, we have instead the simpler form

$$
\mathcal{L}^0_{\text{AE}} = \begin{bmatrix} L^0_{xx} & L^0_{xy} & L^0_{xz} \\ L^0_{yx} & L^0_{yy} & L^0_{yz} \\ L^0_{zx} & L^0_{zy} & L^0_{zz} \end{bmatrix},
\tag{11}
$$

where

$$
\begin{aligned}
L^0_{xx} &= \gamma \left[1 + \frac{F_P}{Q_P} \right] \partial_x^2 + \mu \left[1 + \frac{F_S}{Q_S} \right] (\partial_y^2 + \partial_z^2) + \rho\omega^2 \\
L^0_{yy} &= \gamma \left[1 + \frac{F_P}{Q_P} \right] \partial_y^2 + \mu \left[1 + \frac{F_S}{Q_S} \right] (\partial_x^2 + \partial_z^2) + \rho\omega^2 \\
L^0_{zz} &= \gamma \left[1 + \frac{F_P}{Q_P} \right] \partial_z^2 + \mu \left[1 + \frac{F_S}{Q_S} \right] (\partial_x^2 + \partial_y^2) + \rho\omega^2,
\end{aligned}
\tag{12}
$$

and

$$
\begin{aligned}
L^0_{xy} &= \left\{ \gamma \left[1 + \frac{F_P}{Q_P} \right] - \mu \left[1 + \frac{F_S}{Q_S} \right] \right\} \partial_x \partial_y \\
L^0_{xz} &= \left\{ \gamma \left[1 + \frac{F_P}{Q_P} \right] - \mu \left[1 + \frac{F_S}{Q_S} \right] \right\} \partial_x \partial_z \\
L^0_{yz} &= \left\{ \gamma \left[1 + \frac{F_P}{Q_P} \right] - \mu \left[1 + \frac{F_S}{Q_S} \right] \right\} \partial_y \partial_z \\
L^0_{yx} &= L^0_{xy} \\
L^0_{zx} &= L^0_{xz} \\
L^0_{zy} &= L^0_{yz} .
\end{aligned}
\tag{13}
$$

P- and S-wave formulation

A unified scheme developed by Stolt and Weglein, which diagonalizes all of the important mathematical objects involved in elastic scattering, leading to quantities directly expressed in terms of P-waves and S-waves, has been used to good effect in a number of applications (e.g., Matson, 1997; Zhang, 2006). Since the anelastic modifications we have so far introduced affect the time-frequency dependence of the equations of motion, and only involve new space dependences in the sense that Q_P and Q_S are heterogeneous properties of the medium, this same apparatus can be employed now with little alteration.

The key ingredient in the diagonalization is the operator Π, which performs a decomposition of any 3×1 vector into Helmholtz potentials:

$$\Pi = \begin{bmatrix} \partial_x & \partial_y & \partial_z \\ 0 & \partial_z & -\partial_y \\ -\partial_z & 0 & \partial_x \\ \partial_y & -\partial_x & 0 \end{bmatrix}, \tag{14}$$

and, for all cases that will affect us, possesses the inverse

$$\Pi^{-1} = (\Pi^T\Pi)^{-1}\Pi^T = (\partial_x^2 + \partial_y^2 + \partial_z^2)^{-1} \begin{bmatrix} \partial_x & 0 & -\partial_z & \partial_y \\ \partial_y & \partial_z & 0 & -\partial_x \\ \partial_z & -\partial_y & \partial_x & 0 \end{bmatrix}. \tag{15}$$

In Appendix A we demonstrate by recovering $(\mathcal{L}^0_{\text{AE}})_{11}$ that $\mathcal{L}^0_{\text{PS}}$, where

$$\mathcal{L}^0_{\text{PS}} = \begin{bmatrix} L_P & 0 & 0 & 0 \\ 0 & L_S & 0 & 0 \\ 0 & 0 & L_S & 0 \\ 0 & 0 & 0 & L_S \end{bmatrix}, \tag{16}$$

and

$$\begin{aligned} L_P &= \rho\alpha^2 \left[1 - \frac{F_P}{2Q_P}\right]^{-2} \left\{\nabla^2 + \frac{\omega^2}{\alpha^2}\left[1 - \frac{F_P}{2Q_P}\right]^2\right\} \\ L_S &= \rho\beta^2 \left[1 - \frac{F_S}{2Q_S}\right]^{-2} \left\{\nabla^2 + \frac{\omega^2}{\beta^2}\left[1 - \frac{F_S}{2Q_S}\right]^2\right\}, \end{aligned} \tag{17}$$

is the consequence of applying the Π operators to $\mathcal{L}^0_{\text{AE}}$:

$$\mathcal{L}^0_{\text{AE}} = \Pi \mathcal{L}^0_{\text{PS}} \Pi^{-1}. \tag{18}$$

Hence the P- and S-wave decomposition of equation (7) is of the form

$$\mathcal{L}^0_{\text{PS}} \begin{bmatrix} \nabla \cdot \mathbf{u} \\ (\nabla \times \mathbf{u})_x \\ (\nabla \times \mathbf{u})_y \\ (\nabla \times \mathbf{u})_z \end{bmatrix} = 0, \tag{19}$$

which, defining $\phi = \nabla \cdot \mathbf{u}$ and $\overline{\psi} = [(\nabla \times \mathbf{u})_x, (\nabla \times \mathbf{u})_y, (\nabla \times \mathbf{u})_z]^T$, represents the four equations

$$\begin{aligned} \rho\alpha^2 \left[1 - \frac{F_P}{2Q_P}\right]^{-2} \left\{\nabla^2 + \frac{\omega^2}{\alpha^2}\left[1 - \frac{F_P}{2Q_P}\right]^2\right\} \phi = 0, \\ \rho\beta^2 \left[1 - \frac{F_S}{2Q_S}\right]^{-2} \left\{\nabla^2 + \frac{\omega^2}{\beta^2}\left[1 - \frac{F_S}{2Q_S}\right]^2\right\} \overline{\psi} = 0, \end{aligned} \tag{20}$$

which is really only three since

$$\nabla \cdot \overline{\psi} = 0. \tag{21}$$

ON THE INTERDEPENDENCE OF Q_P AND Q_S

We end with a side note. A medium involving a completely elastic, i.e., non-attenuative, bulk modulus, can still support attenuating P-waves if the shear modulus behaves anelastically; in fact under those circumstances, a simple Q_P-Q_S relationship exists. Beginning again with

$$\alpha^2\rho = \lambda + 2\mu, \quad \kappa = \lambda + \frac{2}{3}\mu, \tag{22}$$

we may write down expressions relating the two wave velocities α and β with the two moduli κ and μ:

$$\begin{aligned} \alpha^2\rho &= \kappa + \frac{4}{3}\mu, \\ \beta^2\rho &= \mu. \end{aligned} \tag{23}$$

Again if we take Q_P and Q_S to parametrize causal attenuation and dispersion in the P-wave and the S-wave respectively, anelastic versions of equation (23) would read, for weak attenuation,

$$\begin{aligned} \rho\alpha^2 \left[1 - \frac{i}{Q_P} + \frac{2}{\pi Q_P}\log\left(\frac{\omega}{\omega_P}\right)\right] = \kappa + \frac{4}{3}\mu, \\ \rho\beta^2 \left[1 - \frac{i}{Q_S} + \frac{2}{\pi Q_S}\log\left(\frac{\omega}{\omega_S}\right)\right] = \mu, \end{aligned} \tag{24}$$

where ω_P and ω_S are the reference frequencies at which the P-wave and the S-wave propagate with real phase velocities α and β respectively. Eliminating complex μ from equation (24), *enforcing κ to be real*, and taking the imaginary part of the result, constrains Q_P to take on a finite value that is fixed by α, β and Q_S:

$$\frac{1}{Q_P} = \frac{4}{3}\frac{1}{Q_S}\left(\frac{\beta}{\alpha}\right)^2. \tag{25}$$

This is consistent with the relationship discussed by Waters (1978). Taken out of context, equation (25) might sensibly be construed as an equation that allows Q_S to be fixed given a Q_P value. But the derivation above cautions against such an interpretation – the Q_P in that relationship is incomplete, so to speak, representing only attenuation that has leaked into the P-wave from the anelasticity of the shear modulus, and saying nothing about actual compressive anelasticity. *This relationship must only be used if it is felt that all attenuation in the seismic disturbance arises from loss mechanisms associated with the shear modulus.* Waters himself says this, but in slightly roundabout language, and the danger of misuse seems real.

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