Automatic band limited signal reconstruction

Marcus R. Wilson, Akshay Gulati and Robert J. Ferguson

ABSTRACT

We implement the popular Adaptive weights Conjugate gradient Toeplitz (ACT) algorithm for signal construction. This algorithm is fast and accurate, and we show its effectiveness in several typical trace regularization situations. This algorithm requires an estimate of the bandwidth as input, and overestimating the bandwidth can cause spurious high frequency noise in the reconstruction. As an improvement, we implement a modified version of ACT that is less well known Multi-level ACT performs automatic bandwidth detection on its input by performing ACT iteratively to estimate the optimum reconstruction bandwidth. We test this algorithm on a harmonic of unknown bandwidth, and results show that Multi-level ACT is effective when the signal bandwidth can not be accurately estimated. A toolbox has been assembled that can be requested from the authors.

INTRODUCTION

Trace interpolation is often employed in seismic data processing. The seismic wavefield is often sampled irregularly due to economic and physical constraints, as well as technical issues. In order to exploit many efficient numerical methods to process this data, it must be projected onto a regular grid. Countless techniques exist that attempt to reconstruct seismic signals from an uneven set of samples (see Gulati and Ferguson (2010) for several examples). However only a few reliable methods exist to reconstruct irregularly an irregularly sampled time series without any *a priori* information (Adorf, 1995).

In this paper we implement a "second generation" algorithm, due to Feichtinger et al. (1995) to estimate the Fourier components of an irregularly sampled band limited signal, or to resample a signal onto a regular grid, using conjugate gradients on a Toeplitz matrix derived from the Fourier transform. Our goal is to assemble a toolbox of numerical methods for use and study by staff, students and sponsors of CREWES.

THEORY

ACT Method

To derive the ACT algorithm we note that, given an irregularly sampled signal $s_j = s(t_j)$, for $j = 1, 2 \cdots N$, we can compute the simple DFT of the observed samples (Vio et al., 2000),

$$S_k = \sum_{j=1}^{N} s_j e^{-2\pi i t_j k/N}. \tag{1}$$

Now, the S_k are not the true Fourier coefficients of the underlying continuous signal $s(t)$, which can be related to the s_j by the inverse DFT equation,

$$\begin{aligned} s_j &= s(t_j) \\ &= \frac{1}{N}\sum_{m=-M}^{M}\hat{S}_m e^{2\pi i t_j m/N}, \end{aligned} \tag{2}$$

in the case that $s(t)$ is band limited by M. Substituting for s_j in Equation 1 gives us

$$\begin{aligned} S_k &= \frac{1}{N}\sum_{j=1}^{N}\sum_{m=-M}^{M}\hat{S}_m e^{2\pi i t_j m/N} e^{-2\pi i t_j k/N} \\ &= \frac{1}{N}\sum_{m=-M}^{M}\hat{S}_m \sum_{j=1}^{N} e^{2\pi i t_j (m-k)/N}. \end{aligned} \tag{3}$$

Writing Equation 3 with the true Fourier coefficient $\hat{S}_m$ as the unknowns input and the observed Fourier coefficients S_k as the known output, the problem becomes a matrix inversion on a Toeplitz matrix, with N rows and $2M+1$ columns. We can think of this as performing a discrete deconvolution on the observed Fourier coordinates, and the system can be solved for the true Fourier coefficients provided $2M+1 \leq N$. Furthermore, since the matrix is Toeplitz, it can be applied to a vector in $\mathcal{O}(n \log n)$ (Feichtinger et al., 1995), so we can expect a conjugate gradient inversion of this matrix to be fast.

This is just one of many ways to rewrite the DFT to derive a method for performing band limited signal reconstruction. Any other permutation would result in an algorithm that is technically equivalent, assuming perfect arithmetic. However in practice these methods will have different properties and one can be more effective than another in certain situations (Vio et al., 2000).

One drawback to this method occurs when the sampling pattern is very irregular, such as when the majority of samples are concentrated in one place, which causes the reconstruction to be biased towards this area. To combat this we add in a set of weights defined by the distance between a points two nearest neighbors. This will cause the densely sampled points to have lower weight in the inversion. The weights are given in Equation 4.

$$w_j(x) = \begin{cases} \frac{1+t_2}{2} & \text{if } j = 1 \\ \frac{N-t_{j-1}}{2} & \text{if } j = N \\ \frac{t_{j+1}-t_{j-1}}{2} & \text{otherwise} \end{cases} \tag{4}$$

Multiplying the s_j by these weights in Equation 2 gives us the weighted inversion function on which the ACT method is based, given by (Feichtinger et al., 1995),

$$S_k = \frac{1}{N}\sum_{m=-M}^{M}\hat{S}_m \sum_{j=1}^{N} w_j e^{2\pi i t_j (m-k)/N}. \tag{5}$$

This matrix equation is used to form the normal equations, which are solved by conjugate gradients (Shewchuk, 1994).

MLACT Method

For signals for which the signal bandwidth is known, ACT is an effective regularization method. However, when the signal bandwidth is not known or cannot be measured, the temptation is to play it safe by setting M equal to the Nyquist frequency. This can cause any random noise to be modelled by the algorithm as high frequency signal, which can cause wild fluctuation in the resulting regularization that are not consistent with the signal. Likewise, choosing M too small will give a smoother result but the value of the output may not be accurate at known locations, as the inversion will fail to catch the higher frequency components of the system. To overcome this problem we can use Multi-level ACT, which will perform the reconstruction iteratively to estimate the optimum bandwidth. We run ACT starting with a bandwidth of $M = 0$, and increase M until the output agrees with the known samples to within a user-defined tolerance (Vio et al., 2000).

EXAMPLES

To showcase the properties of the ACT method, we will test its reconstruction performance for both uniform and random decimation. For regular decimation, we can expect to note the presence of coherent noise in our reconstruction. This noise, if present, will be highly structured with strong amplitudes. For random decimation, we expect that any error in our reconstruction will also be random, and the power will be concentrated at a few Fourier coefficients. We will restrict our analysis to one dimensional signals, where the problem can be thought of as the reconstruction of a time series.

Uniform Decimation

Figure 1(a) shows a simple signal composed of two superimposed harmonics. The top panel shows the true signal, and the lower panel shows a uniform sampling of 50% of the signal. Figure 1(b) shows the discrete Fourier transform of the decimated signal. Note that the distortion of the spectrum is highly structured with high amplitude aliases. Inserting zeros into the signal to denote the missing traces results in the Fourier spectrum in Figure 1(e). Note that the spectrum is the same but with more detail. Figure 1(b) shows the reconstructed harmonic after two iterations, and Figure 1(d) shows the Fourier spectrum. The ACT method converges linearly in relative error to the solution in the case of uniform decimation, and this simple signal is perfectly reconstructed.

The top of Figure 2(a) shows a seismic signal composed of 650 samples from a 25Hz Ricker wavelet convolved with a random reflectivity series, and the bottom panel shows the same signal with 50% of the samples set to zero. Figure 2(b) shows the reconstruction of the signal in the time domain. The reconstructed signal agrees quite well with the original signal. As with the last example, the ACT method linearly converges to the solution (Figure 2(c), so it is very effective for uniform decimation.

Random Decimation

Figure 3(a) shows the same harmonic, but with a random selection of 40% set to zero. Figure 3(b) shows the reconstructed harmonic, which agrees with the original almost everywhere. Figure 3(c) shows that the relative error of the solution decays exponentially with the number of iterations. This is less desirable than the linear convergence we noted in the uniform decimation examples. Figures 4 5 and 6 show three different random decimations of 70% of the traces. The ACT method performs well on the first trial, but breaks down on the second and third trials, resulting in significant spurious events. If we observe the sampling density in Figures 5(a) and 6(a), and the corresponding reconstructions in Figures 5(b) and 6(b), the large anomalous peaks in the output correspond to large gaps in signal coverage. Note also that the residual error in Figure 5(c) and Figure 6(c) decays exponentially at first, but then increases in peaks in the later iterations. Figure 7 shows a good reconstruction for the seismic trace, randomly decimated by 30%, although the reconstruction departs from the original signal in some places. At 50% decimation this method begins to fail on the seismic trace, because the algorithm starts to map the noise to the higher frequencies (Figure 8).

MLACT

To showcase the benefit of using MLACT when the bandwidth is unknown, Figure 9(a) shows a harmonic with 50% decimation. We perform ACT with an input bandwidth that is below the bandwidth of the signal. The result is the blue curve in Figure 9(b). The Multi-level ACT with automatic bandwidth detection catches the second harmonic and returns a better result, as represented by the black curve in Figure 9(b).

CONCLUSION

We find the ACT method to be a fast and accurate signal reconstruction method that is effective at interpolating stationary signals with up to 50% of the samples missing. The method begins to fail even on simple signals when decimation is increased to 70%, although the reconstruction can be successful if the gaps in signal coverage are not too extreme. The MLACT algorithm is a welcome improvement that can be used to estimate the bandwidth of an irregularly sampled signal. A toolbox has been assembled that can be requested from the authors.

ACKNOWLEDGEMENTS

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REFERENCES

Adorf, H. M., 1995, Interpolation of irregularly sampled data series – a survey: Astronomical Data Analysis Software and Systems IV, **77**, 460.

Feichtinger, H., Gröchenig, K., and Strohmer, T., 1995, Efficient numerical methods in non-uniform sampling theory: Numerische Mathematik, **69**, No. 4, 423–440.

Gulati, A., and Ferguson, R. J., 2010, Kaiser bessel gridding kernel for seismic data regularization: CREWES

Research Report, **22**.

Shewchuk, J. R., 1994, An introduction to the conjugate gradient method without the agonizing pain, Tech. rep., Carnegie Mellon University, Pittsburgh, PA, USA.

Vio, R., Strohmer, T., and Wamsteker, W., 2000, On the reconstruction of irregularly sampled time series: Publications of the Astronomical Society of the Pacific, **112**, 74 – 90.

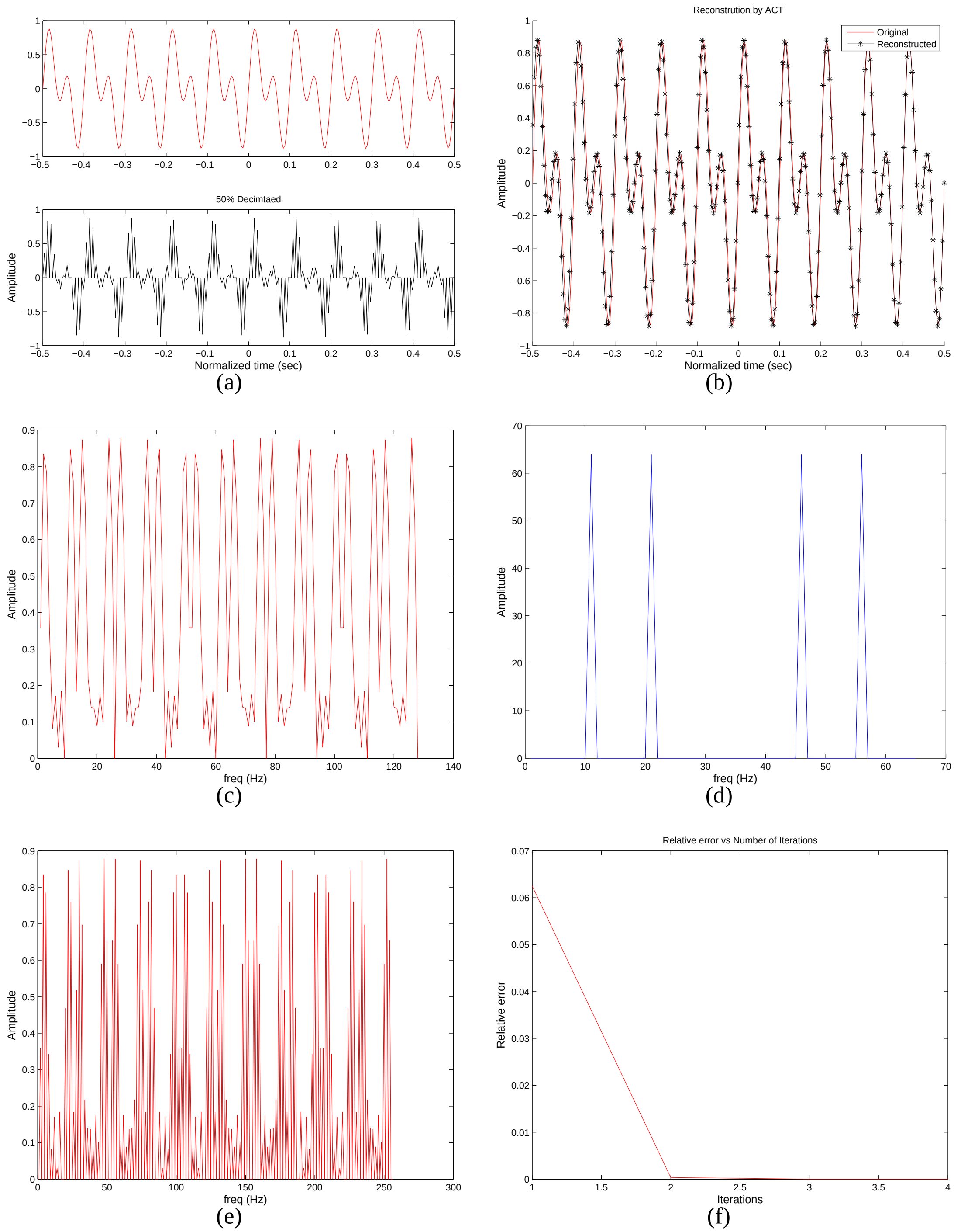


FIG. 1. (a) A simple harmonic and a uniform 50% decimation of that harmonic. (b) The original signal and the ACT reconstruction. (c) The Fourier spectrum of the decimated signal. (d) The Fourier spectrum of the reconstructed signal. (e) The Fourier spectrum of the decimated signal with zeros in place of the unknown samples. (f) The relative error of the ACT inversion after each iteration.

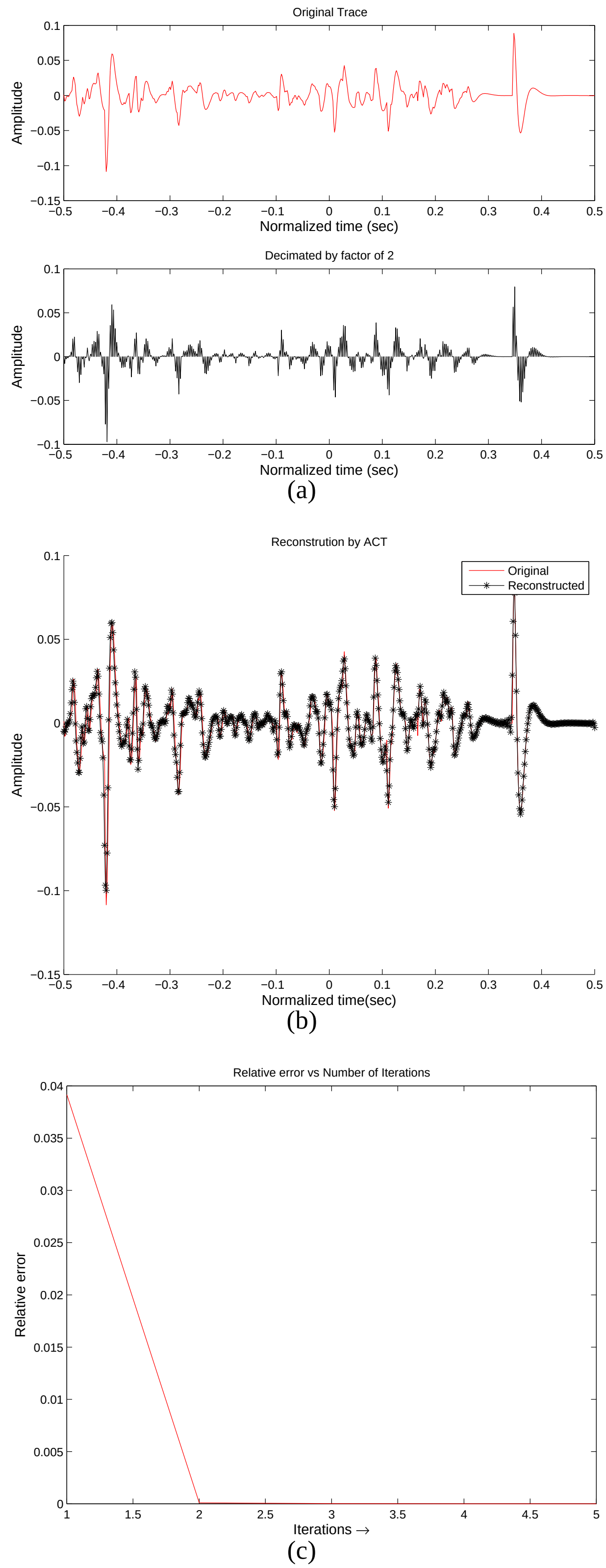


FIG. 2. (a) A stationary seismic trace and a uniform 50% decimation of that trace. (b) The original signal and the ACT reconstruction. (c) The relative error of the ACT inversion after each iteration.

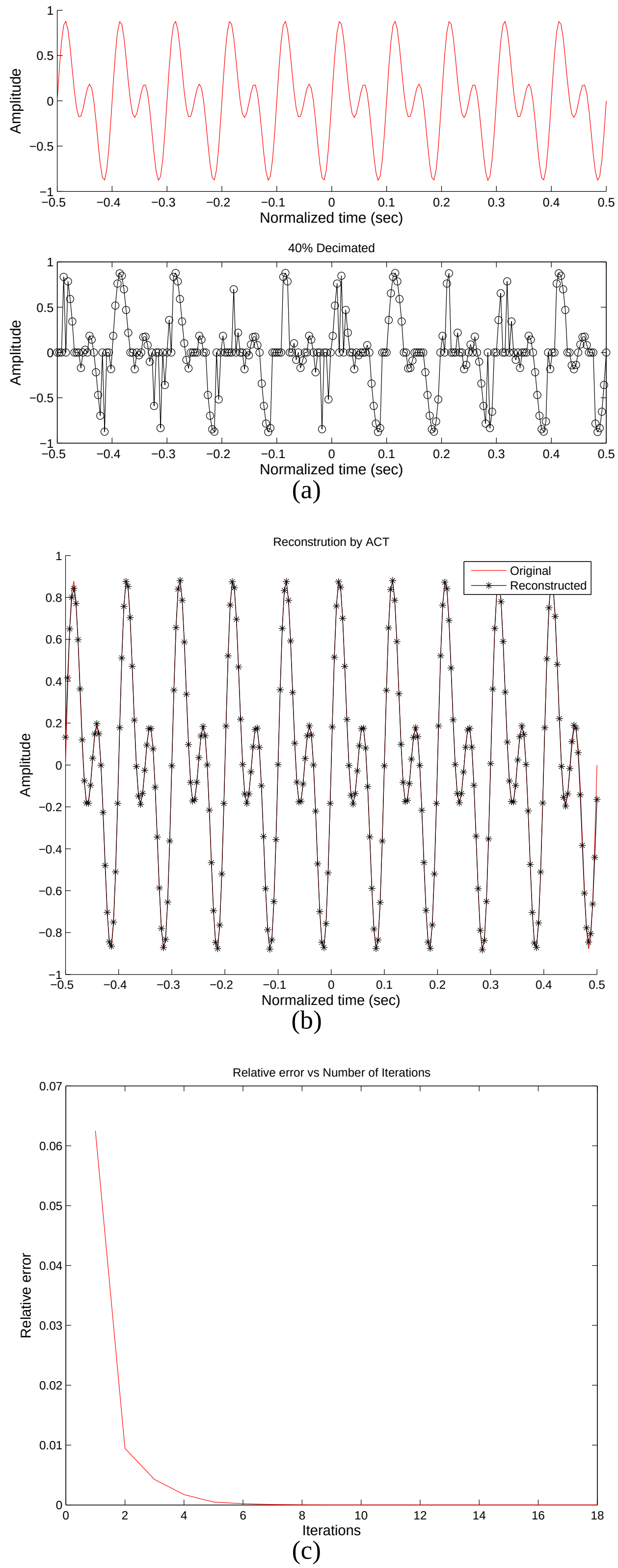


FIG. 3. (a) A simple harmonic and a random 40% decimation of that harmonic. (b) The original signal the ACT reconstruction. (c) The relative error of the ACT inversion after each iteration.

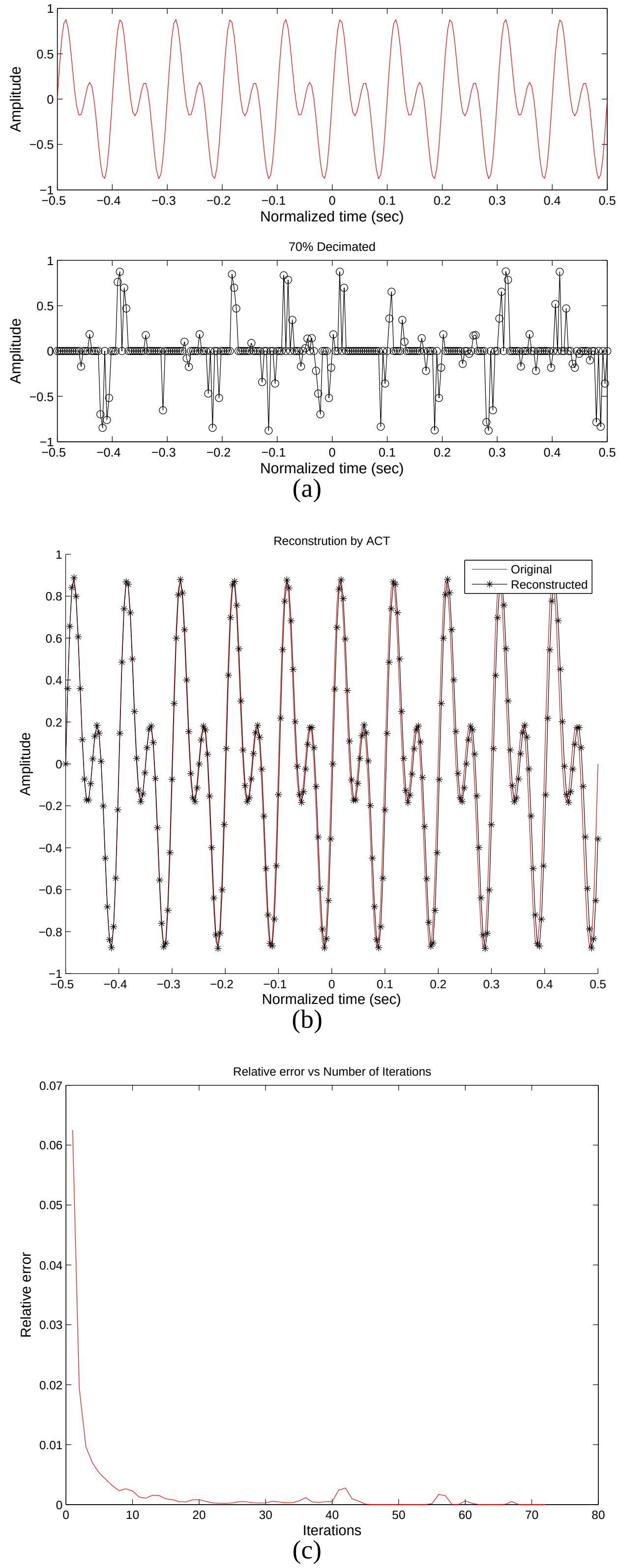


FIG. 4. (a) A simple harmonic and a random 70% decimation of that harmonic. (b) The original signal and a successful ACT reconstruction. (c) The relative error of the ACT inversion after each iteration.

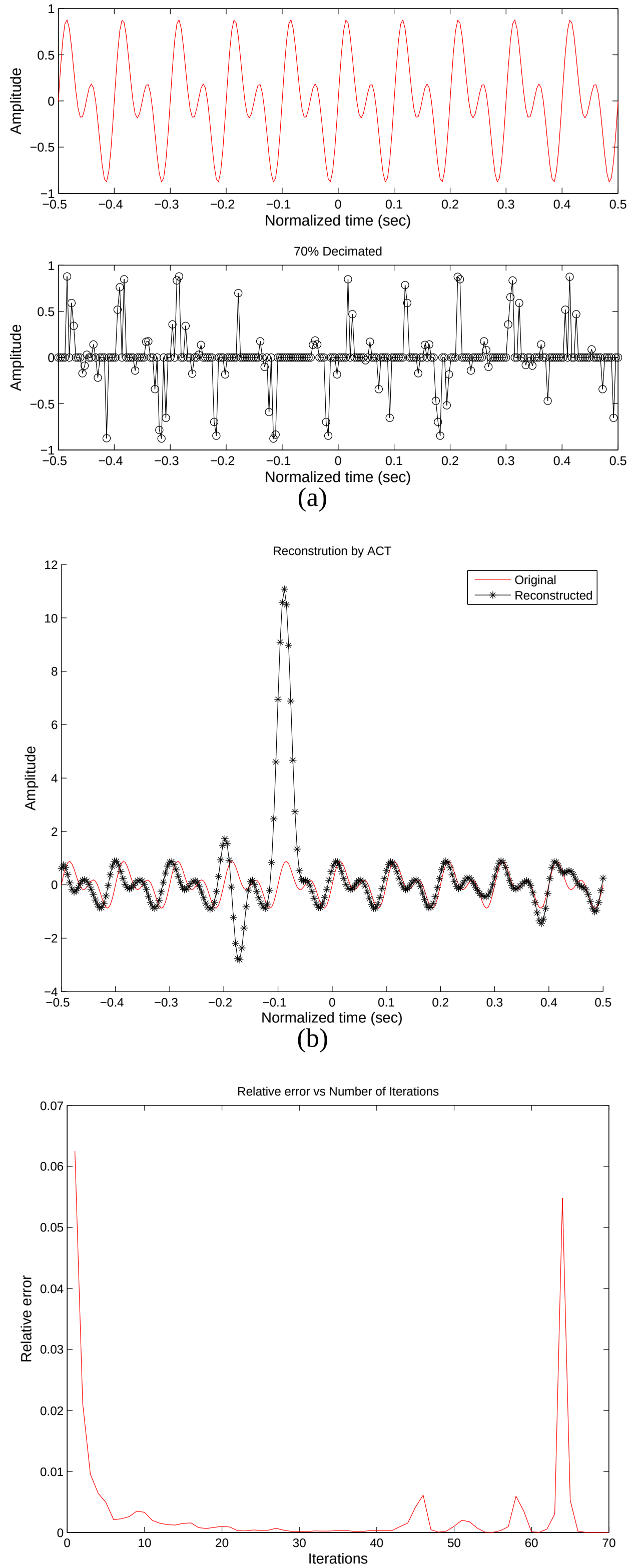


FIG. 5. (a) A simple harmonic and a random 70% decimation of that harmonic. (b) The original signal and an unsuccessful ACT reconstruction. (c) The relative error of the ACT inversion after each iteration.

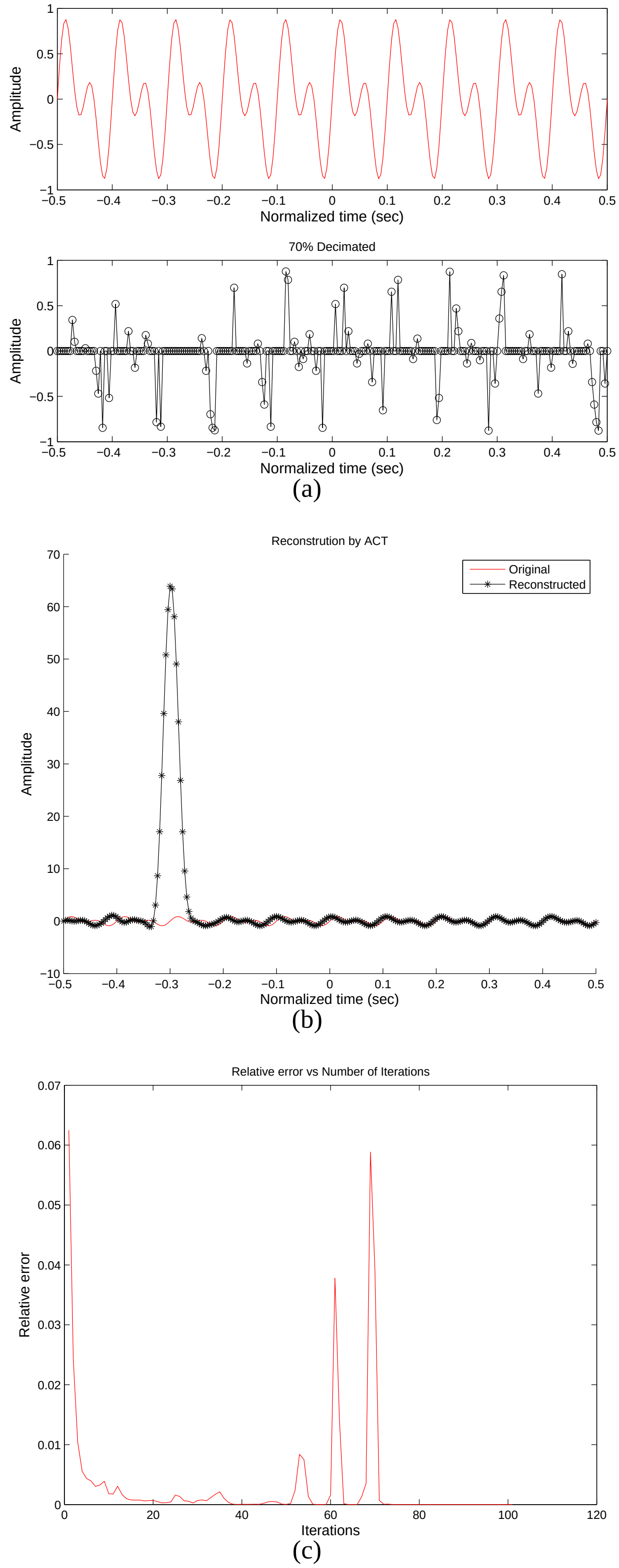


FIG. 6. (a) A simple harmonic and a random 70% decimation of that harmonic. (b) The original signal and an unsuccessful ACT reconstruction. (c) The relative error of the ACT inversion after each iteration.

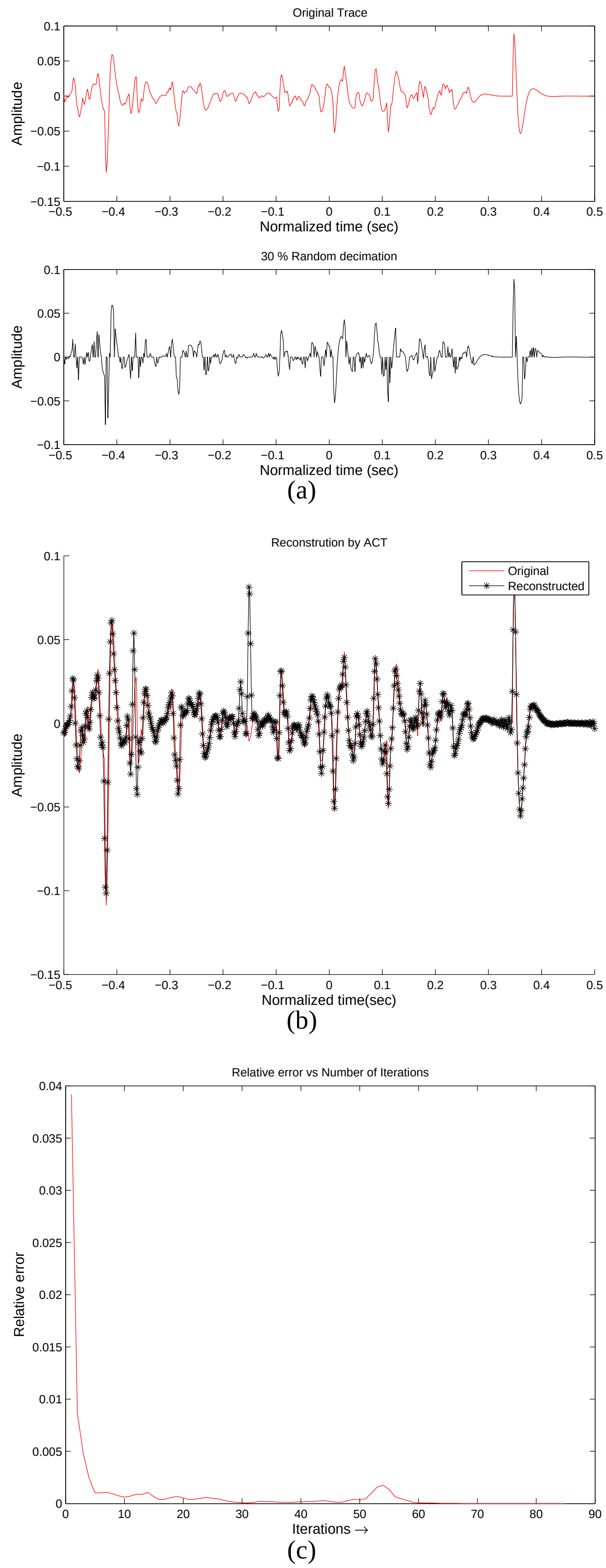


FIG. 7. (a) A stationary seismic trace and a random 30% decimation of that trace. (b) The original signal and a successful ACT reconstruction. (c) The relative error of the ACT inversion after each iteration.

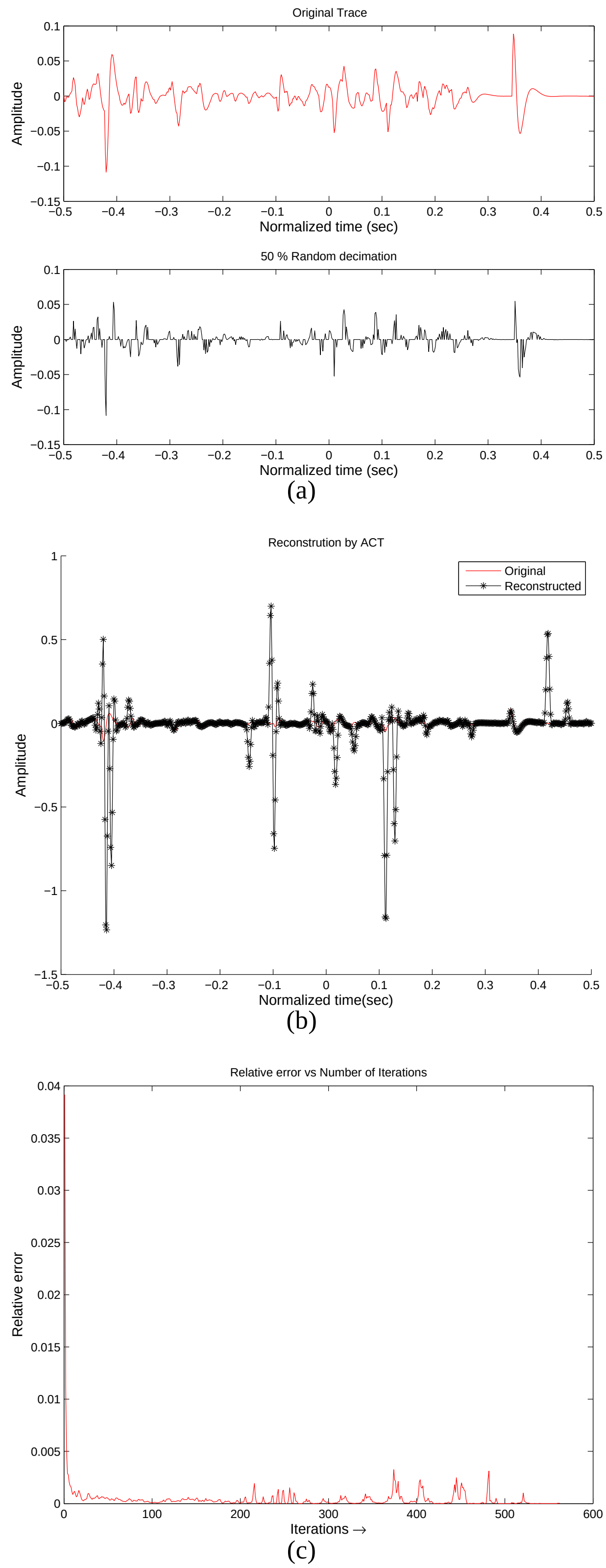
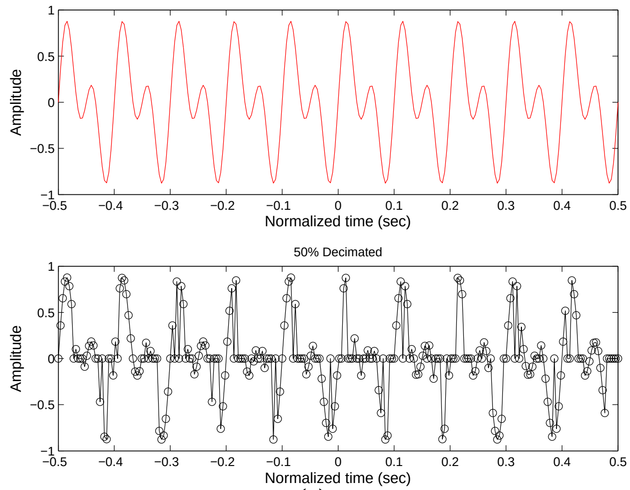
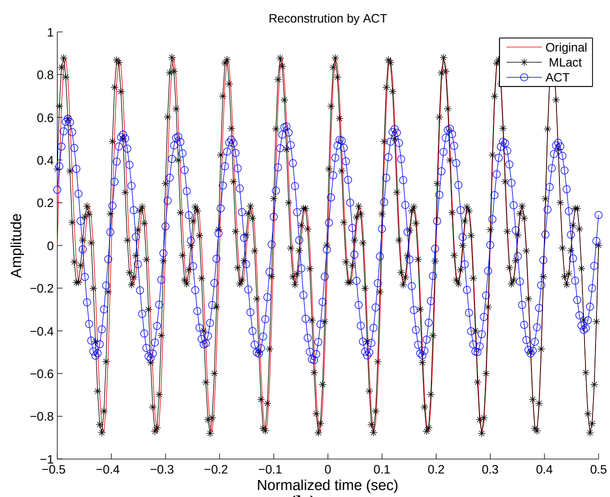


FIG. 8. (a) A stationary seismic trace and a random 50% decimation of that trace. (b) The original signal and an unsuccessful ACT reconstruction. (c) The relative error of the ACT inversion after each iteration.



(a)



(b)

FIG. 9. (a) A simple harmonic and a random 50% decimation of that harmonic. (b) The original signal, the ACT reconstruction with underestimated bandwidth, and the MLACT automatic bandwidth reconstruction.