Nonlinear scattering terms in a seismic context

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ABSTRACT

The scattering theory can be used as a powerful theoretic approach to understand and process seismic data. Exploring inverse scattering series, which have been used to remove multiples from seismic data, depends on understanding how these series generate primaries and multiples. We are engaged in a study of time lapse seismic problem in the context of scattering theory. To begin this project, we here review the key theoretical antecedent of our near term project, the work of Matson (1996). Analytical calculation of the Born series shows that the higher order terms in the series alter the amplitude and adjust the velocity of the scattered wavefield as well as describe internal multiple reflections. Primary reflections are described by all of the terms in the series whereas a multiple which contains n reflections is described by the n^{th} and all higher order terms.

INTRODUCTION

This paper recapitulates the work reported by Matson (1996), in which the meaning and influence of individual terms in the scattering series construction of a wavefield are determined. We have begun a systematic study of scattering series terms arising from posing the time lapse seismic problem as a perturbation problem, and much of those future insights take the work of Matson as a starting point.

In conventional wave theory, seismic waves propagate through an inhomogeneous earth with different velocities in different layers. These waves undergo reflection and transmission when they incident on a boundary between two layers in earth.

In forward scattering theory, earth is viewed as a homogeneous reference medium and a perturbation about that reference medium rather than a complicated wavefield in an inhomogeneous earth. This provides a powerful theoretical method to describe seismic waves. Using scattering theory, the wavefield in the complex medium (the earth) is treated as a reference wavefield plus a perturbation which is nonlinearly related to the earth properties. This nonlinear relationship can be determined precisely using Lippmann-Schwinger equation which in turn is expanded into an infinite series called the Neumann or Born series. Born series is an infinite terms of wave propagation (reflection and transmission) in the reference medium separated by different orders of perturbation or scattering interaction with a point scatterer earth model. Born series simply constructs a desired wavefield form based on a reference wavefield and a perturbation model (Matson, 1996).

When the difference between the actual and reference medium is not large meaning that values of perturbation are small, the linear term in Born series can be derived and used to predict the earth model properties. This is referred as linearized or Born approximation (Cohen and Bleistein, 1977). For larger values of perturbation larger subsets of Born series terms with higher order of perturbation are required (DeWolf, 1971, 1985).

In seismology, indeed, the wavefield data are recorded and methods are investigated to determine the model earth properties from these data. This method which is inverse of forward scattering mentioned above is referred as inversion scattering problems. There is a relationship between the inverse series terms and primaries and multiples. Therefore inverse scattering can also be applied to predicts and remove multiples (Carvalho and et al., 1991; Weglein and et al., 2003; Innanen, 2009).

In this project Born series terms have been derived for a simple one dimensional model, and then the role of each term has been illustrated. We will start with a brief introduction on the forward scattering Born series followed by deriving the Born series for a one dimensional wave incident on a single planar interface and a single layer embedded between two half spaces. Lastly, the results for general case are summarized.

THEORY: THE BORN SERIES

In exploration seismology, a forward problem is designed to characterize the wavefield emanating from a source and propagating through an earth model. In this model, the earth includes of layers with constant velocities and discontinues velocities at boundaries. The forward problem for this layered model is solved using a boundary value approach in which the solutions are constant velocities for each layer. These separate solutions are matched at the boundary on each consequent layer.

We begin with the one dimensional constant density acoustic wave equation

$$\left[\frac{\partial^2}{\partial x^2} - \left(\frac{1}{c^2(x)}\right)\left(\frac{\partial^2}{\partial t^2}\right)\right] P(x|x_s;t) = \delta(x-x_s)\delta(t) \tag{1}$$

where the scalar $P(x|x_s;t)$ represents the pressure at point x and time t due to a one dimensional planar source at point x_s and time t = 0. The velocity c(x) can be characterized by a constant reference velocity c_0 and a perturbation $\alpha(x)$ so that

$$\frac{1}{c^2(x)} = \left(\frac{1}{c_0^2}\right)[1-\alpha(x)]$$

or

$$\alpha(x) = 1 - \frac{c_0^2}{c^2(x)} \tag{2}$$

Choosing the reference medium as the medium between the source and receiver leads to significant simplifications, and is used throughout this project.

Fourier transforming equation (1) with respect to time and substituting equation (2) gives

$$\left[\frac{\partial^2}{\partial x^2} + \left(\frac{\omega^2}{c_0^2}\right)\right] \widetilde{P}(x|x_s;k) = \delta(x-x_s) + \left(\frac{\omega^2}{c_0^2}\right)\alpha(x)\widetilde{P}(x|x_s;k) \tag{3}$$

where $\widetilde{P}$ is the Fourier transform of P with respect to t, $k = \frac{\omega}{c_0}$ is the spatial wavenumber, and ω is the temporal frequency. To find a solution to equation (3), consider a causal free space Green's function $\widetilde{P}_0(x|x_s;k)$ which satisfies the equation

$$\left[\frac{\partial^2}{\partial x^2}+k^2\right]\widetilde{P}_0(x|x_s;k)=\delta(x-x_s)$$

Using this Green's function as a reference wavefield, an integral equation corresponding to equation (3) and its physical boundary conditions is (Weglein, 1985)

$$\widetilde{P}(x|x_s;k)=\widetilde{P}_0(x|x_s;k)+\int_{-\infty}^{\infty}\widetilde{P}_0(x|x';k)k^2\alpha(x')\widetilde{P}(x'|x_s;k)dx'$$

This is called the Lippmann-Schwinger equation and plays a pivotal role in scattering theory. Based on this equation the wavefield in an inhomogeneous medium is the sum of the wavefield in a reference medium and an integral that represents the scattered field due to perturbation. Iterating the Lippmann-Schwinger equation back into itself generates the Born series

$$\begin{aligned}\widetilde{P}(x|x_s;k)&=\widetilde{P}_0(x|x_s;k)+\int_{-\infty}^{\infty}\widetilde{P}_0(x|x';k)k^2\alpha(x')\widetilde{P}_0(x'|x_s;k)dx'\\&+\int_{-\infty}^{\infty}\widetilde{P}_0(x|x';k)k^2\alpha(x')\left[\int_{-\infty}^{\infty}\widetilde{P}_0(x'|x'';k)k^2\alpha(x'')\widetilde{P}_0(x''|x_s;k)dx''\right]dx'\\&+...=\widetilde{P}_0+\widetilde{P}_1+\widetilde{P}_2+...\end{aligned}\tag{4}$$

The terms in this series are referred here either according to their order in α , or their place in the sequence. For example, the zero-th order term is $\widetilde{P}_0$ which is the first term in the series. Similarly, the first order term $\widetilde{P}_1$, is the second term in the series and so on for the other terms.

The first term in the born series which is simply the Green's function, represents a direct wave propagating in the reference medium from the source at x_s , to the measurement point at x. The second term contains $k^2\alpha(x')$ sandwiched between two Green's functions. The Green's function on the right represents a wave which propagates from source at x_s to a point scatterer at x'. The wave interacts with the scatterer at this point and propagate back to x via Green's function on the left. Therefore, the first order term is the integral over all possible single scattering interactions. Similarly, the third term represents a sum over waves which propagates in the reference medium and undergo two scattering interactions. Following this interpretation, each term in Born series involves a series of propagation and interactions with points within the scattering region.

Truncating the Born series after the second term leads to the Born approximation. In this approximation, the data measured is linear in the model. When the perturbation is small, the higher order terms in the Born series become less important and the Born approximation is valid. Higher order terms in Born series play an important role when the perturbation value is lager. In he following sections, contribution of the each term in the Born series to data construction is demonstrated.

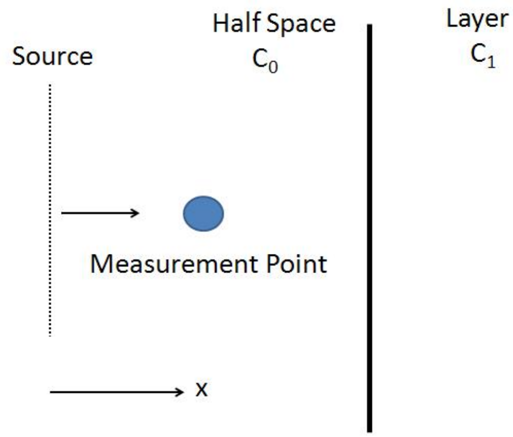

FIG. 1. Plane source of acoustic waves incident on a planer interface.

Scattering by a single interface

Here, I consider two semi-infinite half-spaces with wave velocities c_0 and c_1 are shown in Figure 1. The background medium is a homogeneous medium with velocity c_0. The velocity perturbation is then given as $\alpha(x) = \alpha_0 H(x-x_1)$ where $H(x-x_1)$ is the Heaviside step function, x_1 is the interface location, and $\alpha_0 = 1 - (\frac{c_0}{c_1})^2$. For simplicity, the source location is at $x_s = 0$.

The reference wavefield for a homogeneous whole space and a plane source is the one-dimensional free space Green's function

$$\widetilde{P}_0(x|x_s = 0; k) = \frac{e^{ik|x|}}{2ik}$$

This constitutes the first term in the Born series for this model.

From equation (4), the second term in the Born series is

$$\begin{aligned} \widetilde{P}_1(x|x_s = 0; k) &= \int_{-\infty}^{\infty} \widetilde{P}_0(x|x'; k) k^2 \alpha(x') \widetilde{P}_0(x'|0; k) dx' \\ &= \int_{-\infty}^{\infty} \left(\frac{e^{ik|x-x'|}}{2ik} \right) \left(\frac{e^{ik|x'|}}{2ik} \right) \alpha_0 k^2 H(x' - x_1) dx' \end{aligned} \tag{5}$$

Here, the exponential terms can be combined and expressed as $exp[ik(2x' - x)]$ for $x < x'$ and $exp(ikx)$ for $x > x'$. When $x < x_1$ the measurement position is on the source side of the interface and equation (5) becomes

$$\begin{aligned}\widetilde{P}_1(x < x_1|0;k) &= \int_{x_1}^{\infty}\left[\frac{e^{ik(2x'-x)}}{-4}\right]\alpha_0 dx' \\ &= \left(\frac{-\alpha_0}{4}\right)e^{-ikx}\int_{x_1}^{\infty}e^{ik2x'}dx' \\ &= \frac{-\alpha_0}{4}e^{-ikx}\left[\frac{e^{ik2x'}}{2ik}\right]_{x_1}^{\infty} \\ &= \frac{-\alpha_0}{4}e^{-ikx}\left(\frac{-e^{ik2x_1}}{2ik}\right) \\ &= \left(\frac{\alpha_0}{8ik}e^{ik(2x_1-x)}\right)\end{aligned} \tag{6}$$

This integral is solved by considering small amount of dissipation in the wave propagation such that waves infinitely far from their source will have negligible amplitude. This is satisfied by adding a small imaginary part to the wave slowness such that $\frac{1}{c_0} = \frac{1}{c_0}(1 + i\epsilon)$ where $\epsilon > 0$ for $\omega > 0$ (Aki and Richard, 1980). This result can also be obtained by a contour integration in the complex x' plane.
Adding $\widetilde{P}_0$ and $\widetilde{P}_1$ gives the Born approximation

$$\widetilde{P}_{Born}(x < x_1|0;k) = \left(\frac{1}{2ik}\right)e^{ikx} + \left(\frac{\alpha_0}{8ik}\right)e^{ik(2x_1-x)}$$

In this expression, the first term is a wave that propagates outward from the source directly to x. The second term is a reflected wave since it travels in the negative direction. The reflection coefficient, which is the ration of the amplitude of the incident and reflected wave, for the Born approximation is

$$R = \frac{|P_1|}{|P_0|} = \frac{\alpha_0}{4} = (\frac{1}{4})\left[\frac{(c_1^2 - c_0^2)}{c_1^2}\right] = (\frac{1}{4})\left[\frac{(c_1 - c_0)(c_1 + c_0)}{c_1^2}\right] \tag{7}$$

This result was also obtained by Weglein and Gray (Weglein and Gray, 1983) in a study on the sensitivity of Born inversion to the choice of reference velocity. The reflection coefficient can be obtained for this example by equating the pressure and the normal component of velocity across the boundary (DeSanto, 1992). This coefficient is $\frac{(c_1-c_0)}{(c_1+c_0)}$ and is clearly not in agreement with equation (7). Figure 2 shows, however, that the Born approximate reflection coefficient does approach the proper value for small values of α_0 ,i.e., when $\frac{c_0}{c_1} \approx 1$.

When $x > x_1$, the second term is expressed as

$$\begin{aligned}\widetilde{P}_1(x > x_1|0;k) &= \left(\frac{-\alpha_0}{4}\right)\left[\int_{x_1}^{x}e^{ikx}dx' + \int_{x}^{\infty}e^{ik(2x'-x)}dx'\right] \\ &= \left(\frac{-\alpha_0}{4}\right)\left[(x - x_1)e^{ikx}\right] + \left(\frac{\alpha_0}{8ik}\right)\left[e^{ik(2x-x)}\right] \\ &= \frac{-\alpha_0}{4}(x - x_1)e^{ikx} + \frac{\alpha_0}{8ik}e^{ikx}\end{aligned}$$

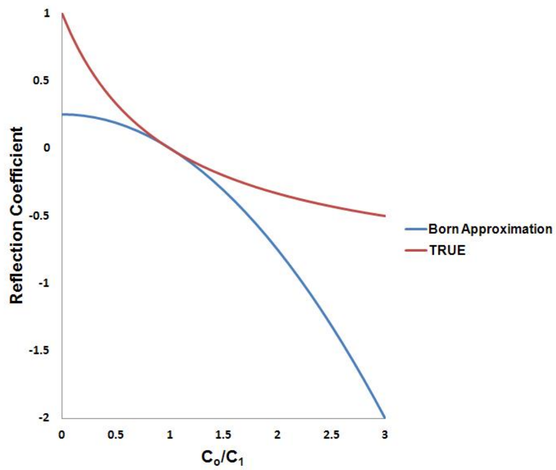

FIG. 2. A comparison of the true reflection coefficient to the Born approximate coefficient for a plane wave incident on a single interface.

Here, the second integral is exactly equation (6) with $x_1 = x$ in the lower limit of the integral. Combining these results with the first term gives the Born approximate transmitted field

$$\begin{aligned}
\widetilde{P}_{Born}(x > x_1|0;k) &= \frac{e^{ikx}}{2ik} + \frac{\alpha_0}{8ik}e^{ikx} - \frac{\alpha_0}{4}(x-x_1)e^{ikx} \\
&= -\frac{e^{ikx}}{2ik}\left[-1 - \frac{\alpha_0}{4} + \frac{\alpha_0}{4}(x-x_1)2ik\right] \\
&= -\left(\frac{e^{ik}}{2ik}\right)\left[\left(\frac{\alpha_0}{4}\right)(2ik(x-x_1)-1)-1\right]
\end{aligned}$$

This solution has the form of a forward traveling transmitted wave. This wave propagates with the velocity of the first medium and not of the second, and the transmission coefficient is not in agreement with the expected $T = \frac{2c_0}{(c_0+c_1)}$ (DeSanto, 1992). By adding more and more terms to the forward Born series, the solutions for the reflected and transmitted waves converge to the classic expressions.

$$\begin{aligned}
T &= \frac{\frac{-1}{2ik}\left[\left(\frac{\alpha_0}{4}\right)(2ik(x-x_1)-1)-1\right]}{\frac{1}{2ik}} \\
&= -\left[\frac{\alpha_0}{4}\left(2ik(x-x_1)-1\right)-1\right]
\end{aligned}$$

The third term in the Born series is written as

$$\widetilde{P}_2(x|0;k) = \int_{x_1}^{\infty} \widetilde{P}_0(x|x';k)k^2\alpha(x')\widetilde{P}_1(x' > x_1|0;k);k)dx'$$

Notice that this wavefield is dependent on the transmitted wavefield for the second term. When $x < x_1$ the measurement position is on the source side of the interface and this equation becomes

$$\begin{aligned}
\widetilde{P}_2(x < x_1|0;k) &= \int_{x_1}^{\infty} \frac{k^2\alpha_0}{2ik} e^{ik(x'-x)} \widetilde{P}_1(x' > x_1|0;k);k)dx' \\
&= \frac{k^2\alpha_0}{2ik} \int_{x_1}^{\infty} e^{ik(x'-x)} \left[\frac{-\alpha_0}{4}(x'-x_1)e^{ikx'} + \frac{\alpha_0}{8ik}e^{ikx'} \right] dx' \\
&= -\frac{k^2\alpha_0^2}{8ik} \int_{x_1}^{\infty} (x'-x_1)e^{ik(2x'-x)}dx' + \left(-\frac{k^2\alpha_0^2}{16k^2}\right) \int_{x_1}^{\infty} e^{ik(2x'-x)}dx' \\
&= \left(-\frac{k^2\alpha_0^2}{8ik}\right)\left(-\frac{e^{ik(2x_1-x)}}{4k^2}\right) + \left(-\frac{k^2\alpha_0^2}{16k^2}\right)\left(-\frac{e^{ik(2x_1-x)}}{2ik}\right) \\
&= \left[\frac{\alpha_0^2 e^{ik(2x_1-x)}}{32ik}\right] + \left[\frac{\alpha_0^2 e^{ik(2x_1-x)}}{32ik}\right] \\
&= \frac{\alpha_0^2}{16ik} e^{ik(2x_1-x)}
\end{aligned}$$

The third term Born series for the reflected wave is

$$\widetilde{P}_2(x < x_1|0;k) = \frac{e^{ik(2x_1-x)}}{ik}\left(\frac{\alpha_0^2}{16}\right)$$

When $x > x_1$, the third term is expressed as

$$\begin{aligned}
\widetilde{P}_2(x > x_1|0;k) &= \left(\frac{k^2\alpha_0}{2ik}\right) \int_{x_1}^{x} e^{ik(x-x')} \widetilde{P}_1(x' > x_1|0;k)dx' \\
&+ \left(\frac{k^2\alpha_0}{2ik}\right) \int_{x}^{\infty} e^{ik(x'-x)} \widetilde{P}_1(x' > x_1|0;k)dx' \\
&= \left(\frac{k^2\alpha_0}{2ik}\right) \int_{x_1}^{x} \left(-\frac{\alpha_0}{4}\right)(x'-x_1)e^{ikx'}e^{ik(x-x')}dx' \\
&+ \left(\frac{k^2\alpha_0}{2ik}\right)\left(\frac{\alpha_0}{8ik}\right) \int_{x_1}^{x} e^{ikx'}e^{ik(x-x')}dx' + \frac{e^{ikx}}{ik}\frac{\alpha_0^2}{16} - \frac{k^2\alpha_0^2}{16ik}e^{ikx}(x-x_1) \\
&= \left(\frac{k^2\alpha_0}{2ik}\right) \int_{x_1}^{x} \left(-\frac{\alpha_0}{4}\right)(x'-x_1)e^{ikx}dx' + \left(\frac{k^2\alpha_0}{2ik}\right)\left(\frac{\alpha_0}{8ik}\right) \int_{x_1}^{x} e^{ikx}dx' + ... \\
&= -\frac{k^2\alpha_0^2}{8ik}e^{ikx} \int_{x_1}^{x} (x'-x_1)dx' + \frac{-k^2\alpha_0^2}{16k^2} \int_{x_1}^{x} e^{ikx}dx' + ... \\
&= -\frac{k^2\alpha_0^2}{8ik}e^{ikx} \left[\frac{(x'-x_1)^2}{2}\right]_{x_1}^{x} - \frac{k^2\alpha_0^2}{16k^2}e^{ikx}(x-x_1) + ... \\
&= -\frac{k^2\alpha_0^2}{16ik}e^{ikx}(x-x_1)^2 - \frac{\alpha_0^2}{16}\left(\frac{ik}{ik}\right)e^{ikx}(x-x_1) + \frac{e^{ikx}}{ik}\frac{\alpha_0^2}{16} - \frac{k^2\alpha_0^2}{16ik}e^{ikx}(x-x_1) \\
&= \left[-k^2(x-x_1)^2\left(\frac{\alpha_0^2}{16}\right) - \left(2ik(x-x_1)\frac{\alpha_0^2}{16}\right) + \left(\frac{\alpha_0^2}{16}\right)\right]\frac{e^{ikx}}{ik}
\end{aligned}$$

The third term Born series for the transmitted wave is

$$\widetilde{P}_2(x > x_1|0;k) = \left[-k^2(x-x_1)^2\left(\frac{\alpha_0^2}{16}\right) - \left(2ik(x-x_1)\frac{\alpha_0^2}{16}\right) + \left(\frac{\alpha_0^2}{16}\right)\right]\frac{e^{ikx}}{ik}$$

Each term in the Born series, as it is seen in the third term, depends on the transmitted solution for the previous term. This leads to the following recursive relation which generates all of the terms in the series

$$\begin{aligned}
\widetilde{P}_n(x|0;k) &= \int_{-\infty}^{\infty} \widetilde{P}_0(x|0;k)k^2\alpha(x')\widetilde{P}_{n-1}(x > x_1|0;k)dx' \\
&= \widetilde{P}_n(x < x_1|0;k) + \widetilde{P}_n(x > x_1|0;k) \\
&= \int_{x_1}^{\infty} \left(\frac{k^2\alpha_0}{2ik}\right) e^{ik(x'-x)}\widetilde{P}_{n-1}(x' > x_1|0;k)dx' \\
&+ \int_{x_1}^{x} \left(\frac{k^2\alpha_0}{2ik}\right) e^{ik(x-x')}\widetilde{P}_{n-1}(x' > x_1|0;k)dx' \\
&+ \int_{x}^{\infty} \left(\frac{k^2\alpha_0}{2ik}\right) e^{ik(x'-x)}\widetilde{P}_{n-1}(x' > x_1|0;k)dx'
\end{aligned} \tag{8}$$

where $\widetilde{P}_n$ is the n^{th} term in the series.

Summing a number of terms yields the reflected wavefield

$$\begin{aligned}
&\widetilde{P}_R(x < x_1|0;k) = \left(\frac{e^{ik2x_1}e^{-ikx}}{ik}\right) \\
&\left[\left(\frac{1}{8}\right)\alpha_0 + \left(\frac{1}{16}\right)\alpha_0^2 + \left(\frac{5}{128}\right)\alpha_0^3 + \left(\frac{7}{256}\right)\alpha_0^4 + \left(\frac{21}{1024}\right)\alpha_0^5 + ...\right] \\
&= \widetilde{P}_1 + \widetilde{P}_2 + \widetilde{P}_3 + ...
\end{aligned} \tag{9}$$

where the subscript R denotes the reflected wavefield. Noting that

$$(1-\alpha_0)^{\frac{1}{2}} = 1 - \left(\frac{\alpha_0}{2}\right) - \left(\frac{1}{8}\right)\alpha_0^2 - \left(\frac{1}{16}\right)\alpha_0^3 - \left(\frac{5}{128}\right)\alpha_0^4 - \left(\frac{7}{256}\right)\alpha_0^5 - \left(\frac{21}{1024}\right)\alpha_0^6 + ... \tag{10}$$

equation (9) becomes

$$\widetilde{P}_R(x < x_1|0;k) = \left(\frac{e^{ik2x_1}e^{-ikx}}{ik}\right)\left[\left(\frac{1}{\alpha_0}\right)\left(1 - \left(\frac{\alpha_0}{2}\right) - (1-\alpha_0)^{\frac{1}{2}}\right)\right]$$

This yields the proper reflection coefficient

$$R = \frac{|\widetilde{P}_R|}{|\widetilde{P}_0|} = 2\frac{1-\frac{\alpha_0}{2}-(1-\alpha_0)^{\frac{1}{2}}}{\alpha_0}$$
$$= \left(\frac{2}{\alpha_0}\right)\left(1-\left(\frac{\alpha_0}{2}\right)-(1-\alpha_0)^{\frac{1}{2}}\right)$$
$$= 2\frac{1-\frac{\left(\frac{c_1^2-c_0^2}{c_1^2}\right)}{2}-\left(1-\frac{c_1^2-c_0^2}{c_1^2}\right)^{\frac{1}{2}}}{\left(\frac{c_1^2-c_0^2}{c_1^2}\right)}$$
$$= 2\frac{1-\frac{c_1^2-c_0^2}{2c_1^2}-(\frac{c_0^2}{c_1^2})^{\frac{1}{2}}}{\frac{c_1^2-c_0^2}{c_1^2}}$$
$$= 2\frac{\frac{c_1^2+c_0^2}{2c_1^2}-\frac{c_0}{c_1}}{\frac{c_1^2-c_0^2}{c_1^2}}$$
$$= 2\frac{c_1^2+c_0^2-2c_0c_1}{2c_1^2}\times\frac{c_1^2}{c_1^2-c_0^2}$$
$$= \frac{(c_1-c_0)^2}{c_1^2-c_0^2}$$
$$= \frac{(c_1-c_0)}{(c_1+c_0)}$$

For the transmitted wavefield, equation (8) leads to

$$\widetilde{P}_T(x>x_1|0;k) = \frac{e^{ikx}}{2ik}+\left[\frac{e^{ikx}}{2ik}-2ik(x-x_1)\right]\left(\frac{\alpha_0}{8}\right)$$
$$+\left[\frac{e^{ik}}{2ik}-2ik(x-x_1)-k^2(x-x_1)^2\right]\left(\frac{\alpha_0^2}{16}\right)+\left(\frac{e^{ik}}{2ik}+-2ik(x-x_1)\right)\left(\frac{5\alpha_0^3}{128}\right)$$
$$+\left(-k^2(x-x_1)^2\right)\left(\frac{3\alpha_0^3}{64}\right)+\left(\frac{ik^3}{2}\right)(x-x_1)^3\left(\frac{\alpha_0^3}{32}\right)+...$$
$$=\widetilde{P}_0+\widetilde{P}_1+\widetilde{P}_2+\widetilde{P}_3+...$$

Rearranging this equation gives

$$\widetilde{P}_T(x>x_1|0;k) = \frac{e^{ikx}}{2ik}+\left(\frac{e^{ikx}}{ik}\right)[(\frac{1}{8}\alpha_0+\frac{1}{16}\alpha_0^2+\frac{5}{128}\alpha_0^3+\frac{7}{256}\alpha_0^4+\frac{21}{1024}\alpha_0^5+...)$$
$$-2ik(x-x_1)(\frac{1}{8}\alpha_0+\frac{1}{16}\alpha_0^2+\frac{5}{128}\alpha_0^3+\frac{7}{256}\alpha_0^4+\frac{21}{1024}\alpha_0^5+...)$$
$$-k^2(x-x_1)^2(\frac{1}{16}\alpha_0^2+\frac{3}{64}\alpha_0^3+\frac{9}{256}\alpha_0^4+\frac{7}{256}\alpha_0^5+...)$$
$$+\left(\frac{ik^3}{3}\right)(x-x_1)^3(\frac{1}{32}\alpha_0^3+\frac{1}{32}\alpha_0^4+\frac{7}{256}\alpha_0^5+...)+...]+...$$
$$=\widetilde{P}_0+\widetilde{P}_1+\widetilde{P}_2+\widetilde{P}_3+... \tag{11}$$

Using equation (10) and defining $\gamma = (1-\alpha_0)^{\frac{1}{2}} = \frac{c_0}{c_1}$ equation (11) written as

$$\widetilde{P}_T(x > x_1|0;k) = \frac{e^{ikx}}{2ik} + \left(\frac{e^{ikx}}{ik}\right)\left(\frac{R}{(1-\gamma)}\right)$$
$$\left[\frac{(1-\gamma)}{2} - ik(x-x_1)(1-\gamma) - \left(\frac{k^2(x-x_1)^2}{2}\right)(1-\gamma)^2 + ... + \left(\frac{(-ik(x-x_1)^n)}{n!}\right)(1-\gamma)^n\right]$$
$$= \left(\frac{e^{ik}}{2ik}\right) + \left(\frac{e^{ik}}{ik}\right)\left(\frac{R}{(1-\gamma)}\right)\left[e^{-ik(x-x_1)(1-\gamma)} - \frac{(1+\gamma)}{2}\right]$$

Knowing that

$$e^A = 1 + A + \frac{A^2}{2!} + \frac{A^3}{3!} + ...$$
$$A = -ik(x-x_1)(1-\gamma)$$

And reducing the terms

$$\frac{R}{(1-\gamma)} = \frac{c_1}{(c_1+c_0)}$$

and

$$\frac{R(\gamma+1)}{(\gamma+1)} = -1$$

results in

$$\widetilde{P}(x > x_1|0;k) = \widetilde{P}_T(x > x_1|0;k) = \left[\frac{c_1}{(c_1+c_0)}\right]\left[\frac{e^{ik_1(x-x_1)}e^{ikx_1}}{ik}\right]$$

where the subscript T denotes the transmitted field and $k_1 = k\gamma = \frac{\omega}{c_1}$ is the wavenumber in the second medium. This gives the transmitted coefficient

$$T = \frac{|\widetilde{P}_R|}{|\widetilde{P}_0|} = \left[\frac{2c_1}{(c_1+c_0)}\right] = 1 + R$$

as expected.

This is an amazing result since not only $\widetilde{P}_T$ has the proper amplitude and propagation direction, it propagates with the velocity c_1 . Therefore the various scattered wave propagating in the reference medium superpose to create a wave which travels at the proper medium velocity. In addition, these waves also superpose to cancel the direct wave $\widetilde{P}_0$.

This simple example illustrate how the higher order terms in the Born series alter the amplitude and adjust the propagation velocity of the scattered wavefield. The next example demonstrates that the higher order terms also describe waves which are multiply reflected between interfaces.

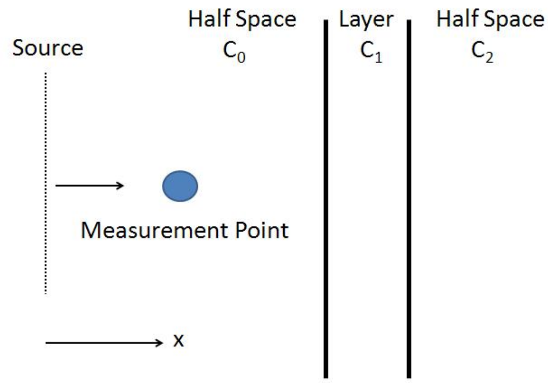

FIG. 3. Plane source of acoustic wave incident on a layer between two half-space.

Scattering by a layer between two half-spaces

In this section, the case is a single layer with a velocity c_1 between two semi-infinite half spaces having velocities c_0 and c_1.In this model,shown in Figure 3, internal multiple reflections will be generated in addition to primary reflections.Using the same constant reference velocity c_0, the perturbation part of the velocity is written as

$$\alpha(x) = \alpha_1 H(x - x_1) + (\alpha_2 - \alpha_1)H(x - x_2)$$
$$\alpha_1 = 1 - \frac{c_{0^2}}{c_1^2}$$
$$\alpha_2 = 1 - \frac{c_{0^2}}{c_2^2}$$

where x_1 and x_2 are the location of the first and second interfaces, As before, the source is located at x_s .

When the measurement point is in the same region of the source, the measured data are reflection data and includes both the direct and scattered wavefields. The first term in the Born series for this region, i.e. $x < x_1$ is

$$\begin{aligned}\widetilde{P}_1(x < x_1|0; k) &= \int_{-\infty}^{\infty} \widetilde{P}_0(x|x'; k)k^2\alpha(x')\widetilde{P}_0(x'|0; k)dx' \\ &= -\int_{x_1}^{x_2} \left(\frac{\alpha_1}{4}\right) e^{ik(2x'-x)}dx' - \int_{x_2}^{\infty} \left(\frac{\alpha_2}{4}\right) e^{ik(2x'-x)}dx'\end{aligned}$$

As before, a small amount of dissipation allows direct evaluation of the indefinite integral. This gives

$$\widetilde{P}_1(x < x_1|0; k) = \left(\frac{e^{-ikx}}{8ik}\right) \left[\alpha_1 e^{ik2x_1} + (\alpha_2 - \alpha_1)e^{ik2x_2}\right] \tag{12}$$

which describes waves reflected from the first and second boundaries respectively. For the interval $x_1 < x < x_2$ $\widetilde{P}_1$ is expressed as

$$\begin{aligned}
&\widetilde{P}_1(x_1 < x < x_2|0;k) \\
&= -\int_{x_1}^{x} \left(\frac{\alpha_1}{4}\right) e^{ikx} dx' - \int_{x}^{x_2} \left(\frac{\alpha_1}{4}\right) e^{ik(2x'-x)} dx' - \int_{x_2}^{\infty} \left(\frac{\alpha_2}{4}\right) e^{ik(2x'-x)} dx' \\
&= \left(\frac{1}{8ik}\right) \left[-\alpha_1 \left(2ik(x - x_1) - 1\right) e^{ik} + (\alpha_2 - \alpha_2) e^{ik(2x_2 - x)}\right]
\end{aligned} \tag{13}$$

This solution represents the transmitted wave through the first interface and the reflected wave from the second interface. For $x > x_2$,

$$\begin{aligned}
&\widetilde{P}_1(x > x_2|0;k) \\
&= -\int_{x_1}^{x_2} \left(\frac{\alpha_1}{4}\right) e^{ikx} dx' - \int_{x_2}^{x} \left(\frac{\alpha_2}{4}\right) e^{ikx} dx' - \int_{x}^{\infty} \left(\frac{\alpha_2}{4}\right) e^{ik(2x'-x)} dx' \\
&= \left(\frac{e^{ikx}}{8ik}\right) \left[\alpha_1 2ik(x_1 - x_2) - \alpha_2 \left(2ik(x - x_2) - 1\right)\right]
\end{aligned} \tag{14}$$

which is transmitted wave only.

The Born approximate solutions to first order for this model predict the expected types of waves. However, they display errors in both amplitude and phase and also they do not generate the interlayer multiple reflections. Starting equation (12) - (14), the following recursive relations enable the calculation of higher order terms in the Born series

$$\begin{aligned}
\widetilde{P}_n(x < x_1|0;k) &= \int_{x_1}^{x_2} \left(\frac{k^2\alpha_1}{2ik}\right) e^{ik(x'-x)} \widetilde{P}_{n-1}(x_1 < x' < x_2|0;k) dx' \\
&+ \int_{x_2}^{\infty} \left(\frac{k^2\alpha_2}{2ik}\right) e^{ik(x'-x)} \widetilde{P}_{n-1}(x' > x_2|0;k) dx' \\
\widetilde{P}_n(x_1 < x < x_2|0;k) &= \int_{x_1}^{x} \left(\frac{k^2\alpha_1}{2ik}\right) e^{ik(x-x')} \widetilde{P}_{n-1}(x_1 < x' < x_2|0;k) dx' \\
&+ \int_{x}^{x_2} \left(\frac{k^2\alpha_1}{2ik}\right) e^{ik(x'-x)} \widetilde{P}_{n-1}(x_1 < x' < x_2|0;k) dx' \\
&+ \int_{x_2}^{\infty} \left(\frac{k^2\alpha_2}{2ik}\right) e^{ik(x'-x)} \widetilde{P}_{n-1}(x' > x_2|0;k) dx' \\
\widetilde{P}_n(x > x_2|0;k) &= \int_{x_1}^{x_2} \left(\frac{k^2\alpha_1}{2ik}\right) e^{ik(x-x')} \widetilde{P}_{n-1}(x_1 < x' < x_2|0;k) dx' \\
&+ \int_{x_2}^{x} \left(\frac{k^2\alpha_2}{2ik}\right) e^{ik(x-x')} \widetilde{P}_{n-1}(x' > x_2|0;k) dx' \\
&+ \int_{x}^{\infty} \left(\frac{k^2\alpha_2}{2ik}\right) e^{ik(x'-x)} \widetilde{P}_{n-1}(x' > x_2|0;k) dx'
\end{aligned}$$

Summing a number of terms gives an expression in which the various terms can be grouped according to the type of wave they describe. For reflection data this is written as

$$\widetilde{P}(x < x_1|0;k) = \widetilde{P}_0(x < x_1|0;k) + \widetilde{P}^{pr1}(x < x_1|0;k) + \widetilde{P}^{pr2}(x < x_1|0;k) + \widetilde{P}^{mlt1}(x < x_1|0;k) + \widetilde{P}^{mlt2}(x < x_1|0;k) + higher - oreder - multiples,$$

where the subscripts pr1, pr2, mlt1, mlt2 represent the first and second primary reflections, and the first and second order multiple reflections, respectively.

As each term in the Born series represents a different order of scattering interaction, there is a relationship between these terms and internal multiples reflections. This relationship for reflection data is such that the different order multiples do not appear until a certain odd number of scattering interactions are summed. For example, a first order multiple involves three reflections: two from the second interface and one from the first. In the Born series, this type of wave path is first described by the forth term which contains waves with three scattering interactions as they propagate from the source to to the measurement point. For a given multiple, its first term and all of the following terms contains information about that multiple. So not only do the higher order terms contribute to the primary reflections and transmissions, they describe higher and higher order multiple reflections.

The first primary reflection is

$$\widetilde{P}^{pr1}(x < x_1|0;k) = \left[\frac{e^{-ikx}e^{ik2x_1}}{ik}\right] \times \left[\left(\frac{1}{8}\right)\alpha_1 + \left(\frac{1}{16}\right)\alpha_1^2 + \left(\frac{5}{128}\right)\alpha_1^3 + \left(\frac{7}{256}\right)\alpha_1^4 + \left(\frac{21}{1024}\right)\alpha_1^5 + ...\right]$$

As in the previous example, using equation (10) leads to the solution

$$\widetilde{P}^{pr1}(x < x_1;k) = R_1\left[\frac{e^{-ikx}e^{ik2x_1}}{2ik}\right]$$

where

$$R_1 = \frac{\left[2 - 2(a-\alpha_1)^{\frac{1}{2}} - \alpha_1\right]}{\alpha_1} = \frac{(c_1 - c_0)}{(c_1 + c_0)}$$

is the reflection coefficient corresponding to the first interface.

The second primary reflection is represented by a more complicated expression involving terms in k_1, x_2 , x_1 , α_1 and α_2 . When certain series expansions are used, these terms reduce to

$$\widetilde{P}^{pr2}(x;k) = T_1T_1^{'}R_2\left[\frac{e^{-ikx}e^{ik2x_1}e^{ik2(x_2-x_1)}}{2ik}\right] \tag{15}$$

where k_1 is the wavenumber in the second medium. The transmission coefficients T_1 and T_1' are given by $1 + R_1$ and $1 - R_1$ respectively, and correspond to wave transmitted through the first interface in the + and - directions. The reflection coefficient R_2 is related o the second interface and can be written as

$$R_2 = \frac{\left[2 - \alpha_1 - \alpha_2 - 2(1-\alpha_1)^{\frac{1}{2}}(1-\alpha_2)^{\frac{1}{2}}\right]}{(\alpha_2 - \alpha_1)} = \frac{(c_2 - c_1)}{(c_2 + c_1)} \tag{16}$$

The Born series is obtained from equation (15) by the first expanding $exp\,[ik_1 2(x_2 - x_1)]$ as a power series in $ik_1 2(x_2 - x_1)$ and then substituting the expansion for $(1-\alpha_1)^{\frac{1}{2}}$ into k_1, T_1 , and T_1' . Next, the expansions for $(1-\alpha_1)^{\frac{1}{2}}$, $(1-\alpha_2)^{\frac{1}{2}}$ and $\frac{1}{(\alpha_2-\alpha_1)}$ are substituted into equation (16) which is in turn substituted into equation (15).

Like the second primary, the first order multiple $\widetilde{P}^{mlt1}$ is also sum of terms in k , x_2 , x_1 , α_1 and α_2 . By employing the same series expansions as for the reflected waves, this portion of the series becomes

$$\begin{aligned}
\widetilde{P}^{mlt1}(x;k) &= -R_1(1-R_1^2)R_2^2\left[\frac{e^{-ikx}e^{ik2(2x_2-x_1)}}{2ik}\right][1 + 4(x_2-x_1)(\gamma_1-1)(ik) \\
&- \frac{\left(4(x_2-x_1)(\gamma_1-1)\right)^2 k^2}{2!} + ... + \frac{\left(ik4(x_2-x_1)(\gamma_1-1)\right)^n}{n!}] \\
&= -R_1(1-R_1^2)R_2^2\left[\frac{e^{-ikx}e^{ik2(2x_2-x_1)}}{2ik}\right]e^{ik4(x_2-x_1)(\gamma_1-1)} \\
&= -R_1(1-R_1^2)R_2^2\left[\frac{e^{-ikx}e^{ik2x_1}e^{ik_1 4(x_2-x_1)}}{2ik}\right]
\end{aligned}$$

where $\gamma_1 = (1-\alpha_1)^{\frac{1}{2}}$. Note that this properly describes the multiple with the respect to both the amplitude and the phase.
For the second multiple, the terms reduce to

$$\widetilde{P}^{mlt2}(x;k) = -R_1^2(1-R_1^2)R_2^3\left[\frac{e^{-ikx}e^{ik2x_1}e^{ik_1 6(x_2-x_1)}}{2ik}\right]$$

As before, the corresponding portion of he Born series is obtained by expanding out R_1 , R_2, and $exp\,[ik_1 6(x_2 - x_1)]$.

CONCLUSIONS

The scattering theory is applied to investigate a mapping method between the earth model and seismic data. The Born series is established and full series terms are derived. Theses series were able to predict and interpret seismic reflection data including primary and multiple events.Scattering theory could simplify a complex wave field in a complex medium into a reference wavefield, Green's function, which is perturbed by medium. The perturbation function is defined base on the medium property or slowness. This work enhances our understanding of inverse scattering which is a step by step process for inverting seismic data into earth parameters by understanding how earth properties are mapped into primary and multiples reflections in the forward problem. This can form the basis for multiple attenuation. The calculation in this project have not only confirmed that the role of the higher order terms is to alter the amplitude of reflected and transmitted wave, adjust the propagation velocity of transmitted wave, and describe inter layer multiples, but have also shown the details of how this occurs.

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