Wave propagation and interacting particles continued: plane waves at oblique incidence

Kris Innanen

ABSTRACT

We continue the development of a model of seismic data based on the idea of particles, or groups of particles, undergoing collision and disintegration interactions—rather than seismic wave events reflecting and transmitting. Here we admit plane waves which propagate obliquely with respect to the spatial axis along which they are observed (which we have thus far fixed to be the depth axis, e.g., the well in a VSP experiment). We consider both harmonic and transient waves. Acoustic and multi parameter problems demand the inclusion of additional particles in order that boundary conditions are honoured.

INTRODUCTION

In the last two CREWES reports (Innanen, 2010, 2011) we have been incrementally assembling a theory of seismic wave propagation in which the mechanisms of propagation, reflection, transmission and attenuation have been cast in terms of particles with well defined masses and momenta, moving freely or in potential fields, and colliding inelastically. A range of important phenomena in elastic seismology remain however, which a well formed model must reproduce. These include waves in multiple dimensions, interacting with media described by multiple elastic parameters. Thus far the theory has been discussed for 1D wave propagation only.

The 1D picture involved, implicitly, two space coordinate axes: the depth axis, along which waves propagated, and an orthogonal axis along which the interfaces causing the reflections were oriented. Thereafter, interchanging the depth and time dimensions, the properties of waves in these environments were correctly predicted using the particle model. In this paper we extend our model to incorporate plane waves which are incident obliquely upon this coordinate system. Although this appears to be a simple step, it is important since it opens all multidimensional wave behaviour up to analysis within the particle model. Since all 2D and 3D wave behaviour can be analyzed into harmonic plane waves, if the model admits harmonic plane waves, the model can reproduce all 2D and 3D wave behaviour. The extension to oblique incidence we make in this paper is illustrated in Figure 1a–b.

The key conceptual hurdle when extending to multiple spatial dimensions lies in the problem of switching the roles of space and time. When there was only one space axis, doing so did not involve any real "decisions". Now, if in addition to the z axis there is an x, or even an x and a y axis also, which one do we switch with t? We will deal with this by continuing to examine the wave along a single, preferred space axis, in spite of the fact that it will now vary along several. This observation axis, which will usually be the z axis in this paper, will be the one that is switched with the time axis.

The paper is organized as follows. First we will establish several preliminary results,

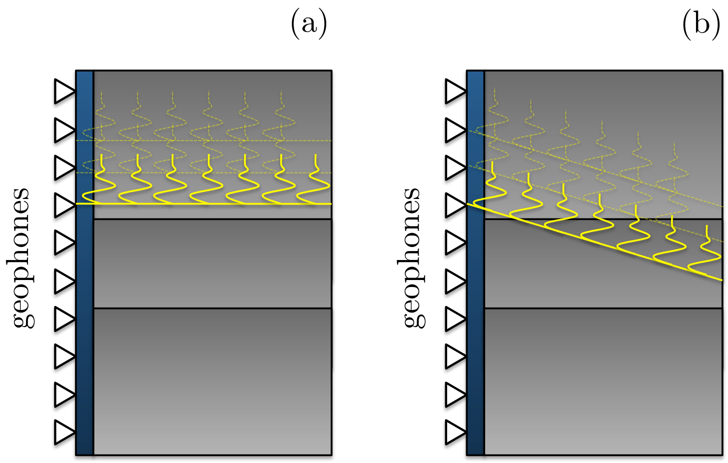

FIG. 1. The collision model is extended from normal (a) to oblique (b) incidence.

expressions from standard wave theory (i.e., the definition of reflection coefficients etc.), which we will attempt to reproduce with the particle model. Then, we will demonstrate that the elementary interactions of a wave reflecting, transmitting and propagating in scalar media are correctly predicted by switching the time and depth axes, and considering the wave events to be individual particles, or groups of particles. Finally we will discuss some of the issues that will come with extending to multi-parameter (e.g., two-parameter acoustic) problems.

SOME PRELIMINARY RESULTS

The relationship between R, θ and θ_1

If a scalar plane wave at velocity c_0 is incident with angle θ on a boundary, below which the velocity is c_0, the reflection coefficient, assuming continuity of the field and its derivative, is

$$R(\theta) = \frac{\frac{c_1}{\cos\theta_1} - \frac{c_0}{\cos\theta}}{\frac{c_1}{\cos\theta_1} + \frac{c_0}{\cos\theta}}, \tag{1}$$

where θ_1 is such that Snell's law $c_0 \sin\theta_1 = c_1 \sin\theta$ holds. By adding and/or subtracting equation (1) from unity in the form of

$$1 = \frac{\frac{c_1}{\cos\theta_1} + \frac{c_0}{\cos\theta}}{\frac{c_1}{\cos\theta_1} + \frac{c_0}{\cos\theta}}, \tag{2}$$

we obtain

$$1 - R(\theta) = \frac{\frac{2c_0}{\cos\theta}}{\frac{c_1}{\cos\theta_1} + \frac{c_0}{\cos\theta}}, \tag{3}$$

and/or

$$1 + R(\theta) = \frac{\frac{2c_1}{\cos\theta_1}}{\frac{c_1}{\cos\theta_1} + \frac{c_0}{\cos\theta}}, \tag{4}$$

by which we find that

$$\frac{c_0}{c_1}\frac{\cos\theta_1}{\cos\theta} = \frac{1 - R(\theta)}{1 + R(\theta)} \tag{5}$$

must hold.

The space and time behaviour of plane waves at oblique incidence

Before switching the roles of space and time and thus invoking a particle view, let us review what the behaviour of obliquely incident plane waves on a vertical observation axis looks like under normal circumstances. In Figure 2a–c three snapshots of such a plane wave are illustrated. An incident wave of amplitude 1 moves down and to the left towards a boundary. At the boundary it is partitioned into a reflected wave of amplitude R and a transmitted wave of amplitude T. Since for illustration purposes we have chosen c_1 to be less than c_0, the transmitted wave's path steepens in comparison to the incident wave.

As the wave propagates the three-armed wave front appears to translate itself from right to left (a–c). How does the wave appear to an observer with access to measurements along the vertical well?

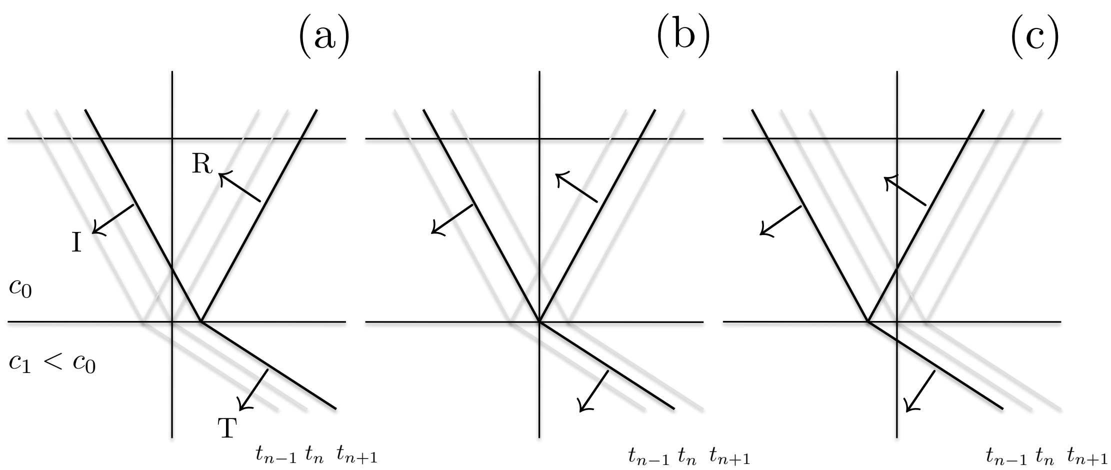

FIG. 2. Snapshots of a plane wave incident obliquely upon a horizontal boundary. Our interest will be to track the progress of the wave from the point of view of the vertical line depicted in the middle of each panel (a)–(c), as time progresses from t_{n-1} to t_n and thence to t_{n+1}.

In Figure 3a–d we focus on the incident, reflected, and transmitted waves one at a time, with the waves depicted all together in (a). First, the incident wave: from the point of view of the well, as depicted in Figure 3b, the incident wave moves in the positive z direction at a rate dependent on the velocity of the wave in the direction of propagation and the angle that direction makes with the well. The reflected wave, as it moves from early to late times (Figure 3c) propagates the same manner but in the negative z direction. The transmitted wave (Figure 3d) moves in the positive z direction with a different apparent velocity dictated in accordance with Snell's Law by the new real velocity and the new angle.

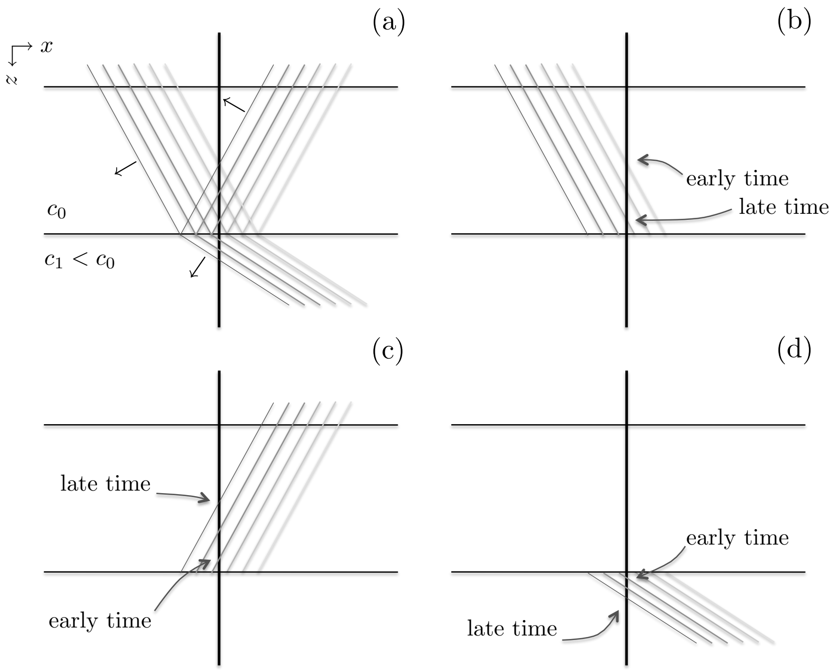

FIG. 3. Apparent plane wave behaviour. (a) The incident, transmitted and reflected plane waves moving from early to late times. (b)–(d) The incident, reflected and transmitted plane waves considered one at a time.

PLANE WAVES AT A BOUNDARY IN TERMS OF INTERACTING PARTICLES

Scalar case

Now let us take our oblique-incidence plane waves, as seen propagating along an observation axis as time increases, and switch those roles. Let us consider the full time-history of the wave, at a fixed point along the observation axis (i.e., depth z), to be one frame of a movie. And, let the movie consist of many such frames being viewed one after another as our position along the observation axis (z) increases.

For z values above the single reflector, which we will place at depth z_I, the movie would look something like what is illustrated in Figure 4. Beginning with Figure 4a, i.e., $z = 0$, we see two events, which from now on we will interpret, the incident wave at an early time (or zero time) and the reflected wave at a relatively late time. The incident wave has by assumption an amplitude of 1 and the reflected wave has an amplitude of $R(\theta)$. By depth z_1 (Figure 4b) the two events have drifted towards each other, and even further towards each other by depth z_2 (Figure 4c).

We are interested in the rate at which this happens—the "velocities" of the particles along the time axis as depth increases. Since for the direct wave we have an arrival time given by τ_D where

$$\tau_D(z) = \frac{z \cos\theta}{c_0} \tag{6}$$

and for the reflected wave

$$\tau_R(z) = -\frac{(2z_I - z)\cos\theta}{c_0}, \tag{7}$$

these rates must be the vertical slownesses

$$v_I = \frac{d\tau_I}{dz} = \frac{\cos\theta}{c_0} \tag{8}$$

and

$$v_R = \frac{d\tau_R}{dz} = -\frac{\cos\theta}{c_0}. \tag{9}$$

As before (Innanen, 2010) we set the notional masses of these two particles to be equal to their plane wave amplitudes, thus:

$$\begin{aligned} m_I &= 1 \\ m_R &= R(\theta). \end{aligned} \tag{10}$$

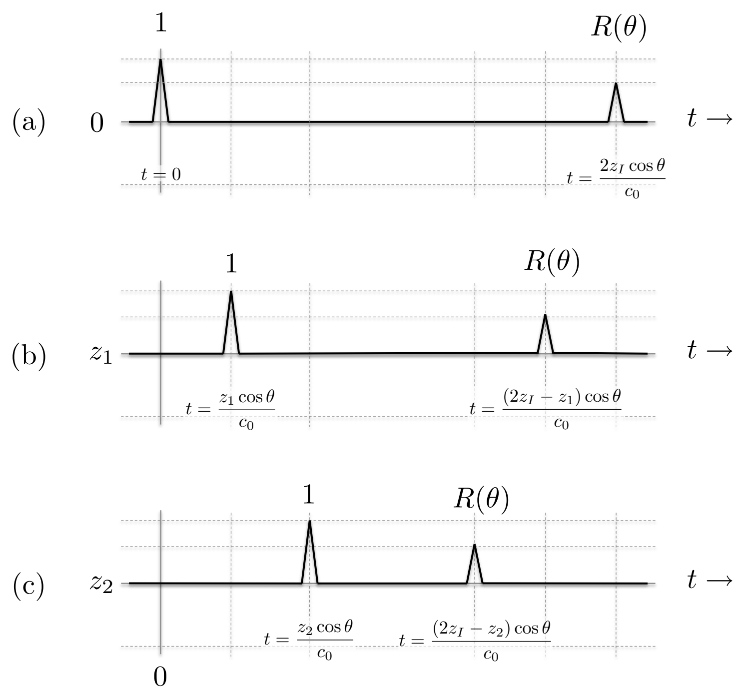

FIG. 4. Events in a simple realization of the particle model, prior to collision. (a)–(c) As we move from low to high z values (the proxy for time in the model), the incident wave of amplitude 1 and the reflected wave of amplitude $R(\theta)$ appear to approach each other at rates determined by the reciprocal wave velocities and the angle of incidence of the wave with respect to the observation axis.

For z values below the reflector, i.e., $z > z_I$, the wave consists of the single transmitted event, moving to greater times as depth increases (Figure 4)a–c. It has an amplitude of

$T(\theta)$. Its travel-time as a function of depth is

$$\tau_T(z) = \frac{z_I \cos\theta}{c_0} + \frac{(z - z_I)\cos\theta_1}{c_1}, \tag{11}$$

where θ_1 is the transmission angle and c_1 is the velocity of the wave along the direction of propagation in the medium below z_I. Therefore, the rate at which the transmitted event traverses the time-axis as depth increases is

$$v_T = \frac{d\tau_T}{dz} = \frac{\cos\theta_1}{c_1}. \tag{12}$$

We choose to view this event as a drifting particle, with a mass defined to be equal to its amplitude:

$$m_T = T(\theta). \tag{13}$$

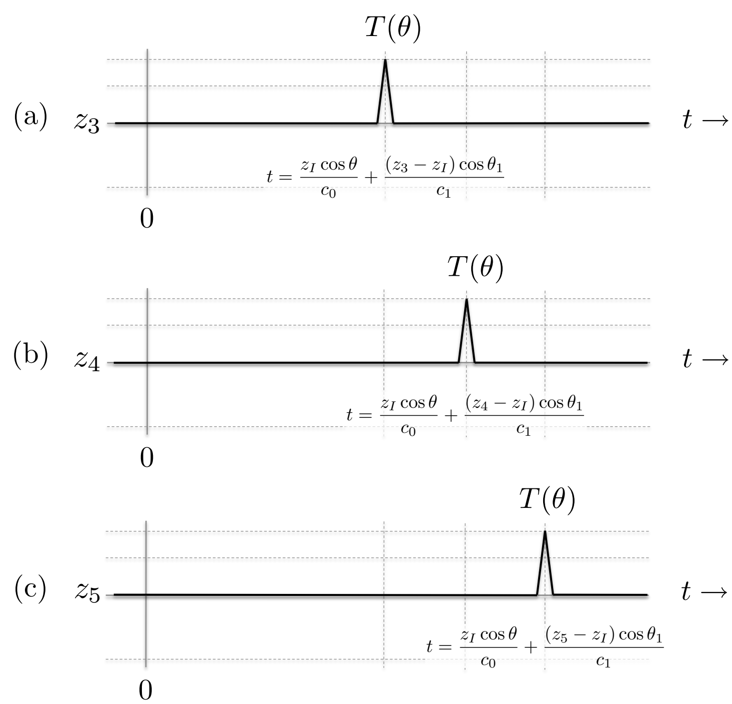

FIG. 5. Events in a simple realization of the particle model, posterior to collision. (a)–(c) As we move from low to high z values (the proxy for time in the model), the drifts to the right at a rate determined by the reciprocal wave velocity and the angle of transmission of the wave with respect to the observation axis.

The particles 1 and $R(\theta)$ meet at $z = z_I$, and thereafter are replaced by the single transmitted event. So, in the particle model, a reflection/transmission interaction is replaced with an inelastic collision. Before the collision, the total mass and momenta of the particles are

$$\begin{aligned} m_I + m_R &= 1 + R(\theta), \\ m_I \times v_I + m_R \times v_R &= 1 \times \frac{\cos\theta}{c_0} + R(\theta) \times \left(-\frac{\cos\theta}{c_0}\right). \end{aligned} \tag{14}$$

After the collision, the same totals are

$$m_T = T(\theta),$$
$$m_T \times v_T = T(\theta) \times \frac{\cos\theta_1}{c_1}. \tag{15}$$

Hence, if it was true that our particle/collision model, in obeying the laws of conservation of mass and momentum, correctly captured the effects of boundary conditions on plane waves, then it would have to be true that

$$1 + R(\theta) = T(\theta), \tag{16}$$

and

$$[1 - R(\theta)]\frac{\cos\theta}{c_0} = T(\theta)\frac{\cos\theta_1}{c_1}, \tag{17}$$

or, using equation (16),

$$\frac{c_0}{c_1}\frac{\cos\theta_1}{\cos\theta} = \frac{1 - R(\theta)}{1 + R(\theta)}. \tag{18}$$

That equation (16) holds is a well-known result of scalar wave theory. That equation (18) holds we established in our preliminary results (equation 5).

This results generalizes to sequences of reflections and transmissions through stacks of layers, with the only difference being additional coefficients wrapped around either side (Innanen, 2010). Upgoing incident waves appear in the collision model as "spontaneous" disintegrations of one particle into two.

Harmonic plane waves and more general wave forms

Replacing the plane wave which is localized in the direction of propagation with a harmonic plane wave requires no particular innovation to be incorporated into the particle model. From the point of view of visualization, the idea of colliding particles is retained, but a continuous stream of particles, to coincide with the quasi steady state idea of incident, transmitted and reflected harmonic plane waves, is invoked (Figures 6a–b). Put another way, a plane harmonic wave at position x and z moving at angle θ above a given reflector,

$$P^+ = \exp\left[i\frac{\omega}{c_0}\left(x\sin\theta + z\cos\theta\right)\right]e^{-i\omega t} + R(\theta)\exp\left[i\frac{\omega}{c_0}\left(x\sin\theta - z\cos\theta\right)\right]e^{-i\omega t} \tag{19}$$

and below the reflector,

$$P^- = T(\theta)\exp\left[i\frac{\omega}{c_1}\left(x\sin\theta_1 + z\cos\theta_1\right)\right]e^{-i\omega t}, \tag{20}$$

have amplitudes so that, should they be interpreted as masses, the results of the previous section are recovered, and the required phase that should they be interpreted as masses with momenta the results of the previous section are recovered, with the only difference being the process of colliding inelastically is a continuous one.

If the harmonic plane wave in question is to propagate in a dissipative medium (as discussed by Innanen, 2011) , we instead view the particle stream as a stream of particle packets, whose number density may be reduced and distributed according to a dispersion law (Figure 6c) in the same way as was the transient wave at normal incidence.

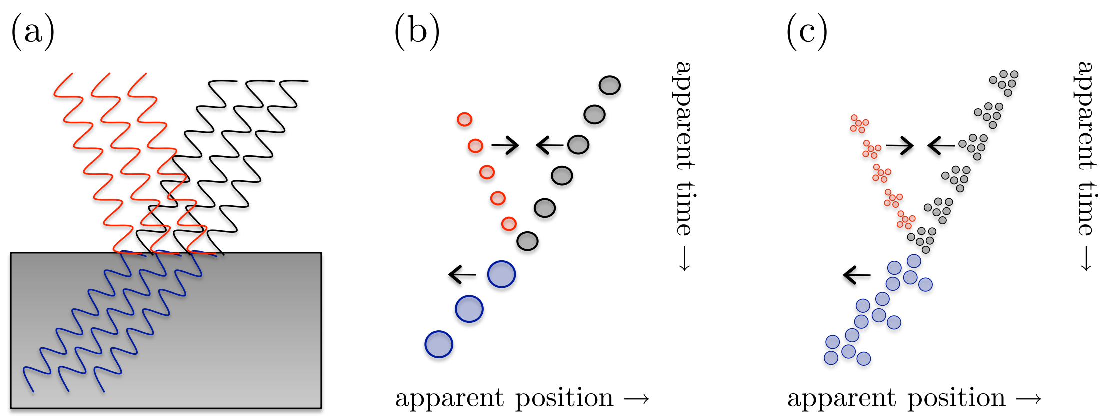

FIG. 6. Harmonic plane waves in the context of the particle model. (a) Harmonic plane waves incident on, reflected from and transmitted through a horizontal boundary. (b) Nondispersive picture: quasi steady state stream of individual particles colliding inelastically and drifting thereafter with mass and momentum conserved. (c) Dispersive or nondispersive picture: each particle is replaced with a large group of particles of equal mass, each drifting with a velocity drawn from a suitable distribution.

The significance of the extension of the model to harmonic plane waves is that it admits to the model any wave problem that can be decomposed into harmonic plane waves, which includes much of multidimensional scalar wave theory.

Nonscalar waves

Tied as they are to reflection and transmission coefficients, we expect the tenets of a particle, space/time-reversed model of seismic waves to be connected to the boundary conditions assumed to hold at the reflecting interfaces. For wave problems of the kind we have been treating, the explicit expressions for R and T are traceable to the following conditions just above (-) and just below (+) a planar boundary Ω:

$$\begin{aligned} &P|_{\Omega}^{+} = P|_{\Omega}^{-}, \quad \text{i.e., continuity of pressure} \\ &\rho_0^{-1} P'|_{\Omega}^{+} = \rho_0^{-1} P'|_{\Omega}^{-}, \quad \text{i.e., continuity of particle velocity.} \end{aligned} \tag{21}$$

We have deliberately left in the density terms ρ_0 required to map the pressure to the velocity, even though (because in scalar problems the density is everywhere constant) they drop out of the final result.

This fact—that in a scalar problem the R and T are determined only by the wave amplitudes and derivatives of the phase terms—makes the particle model simple. Consider equations 15 through 20 in Innanen (2010). In those equations, the derivative operation "brings down" reciprocal velocities, which are then the only coefficients alongside R and T. The particle momenta, therefore, consist only of masses (i.e., R and T) in products with apparent velocities (c_0^{-1} and c_1^{-1}).

When we move to a multi-parameter problem (e.g., an acoustic problem in which both wave velocity and density vary), the boundary conditions are altered to

$$\begin{aligned} P|_\Omega^+ &= P|_\Omega^-, \\ \rho_0^{-1}P'|_\Omega^+ &= \rho_1^{-1}P'|_\Omega^-, \end{aligned} \tag{22}$$

i.e., with the variation in density accounted for in the continuity of particle velocity statement. The previous simplifications disappear, since the densities above (ρ_0) and below (ρ_1) the reflector are now involved.

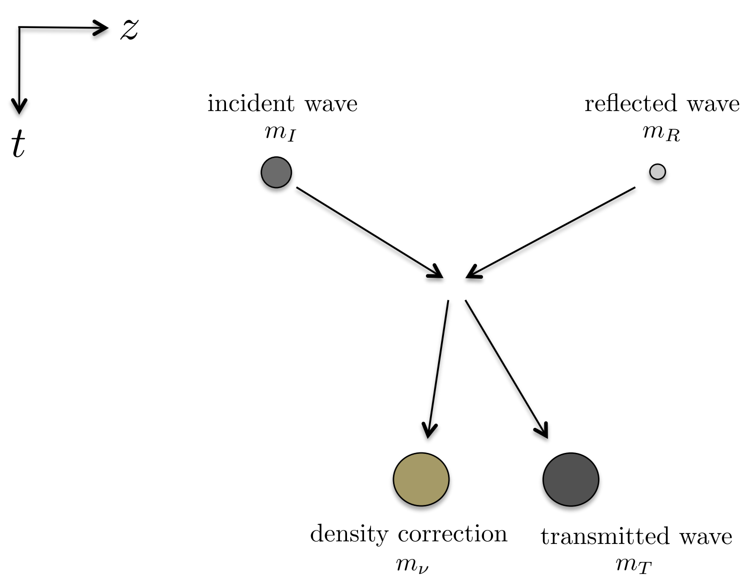

FIG. 7. Multiparameter problems. The simple direct relationship between conservation of momentum and rate of motion along the time axis is a consequence of scalar boundary conditions. To maintain the particle model in a velocity/density problem we must introduce an unmeasured ghost particle ν to carry off a portion of the apparent momentum.

Let us characterize the density influence on the acoustic boundary conditions by forming a density perturbation

$$a_\rho = 1 - \frac{\rho_0}{\rho_1}, \tag{23}$$

in which case

$$\begin{aligned} P'|_\Omega^+ &= \frac{\rho_0}{\rho_1}P'|_\Omega^- \\ &= P'|_\Omega^- - a_\rho P'|_\Omega^-. \end{aligned} \tag{24}$$

This leads to the equality

$$(1)\frac{\cos\theta}{c_0} - R(\theta)\frac{\cos\theta}{c_0} = T(\theta)\frac{\cos\theta_1}{c_1} - a_\rho T(\theta)\frac{\cos\theta_1}{c_1}, \tag{25}$$

in which all of the terms are once again interpretable entirely in the framework of reciprocal velocities, i.e., in terms meaningful to a particle model. The cost is that a new term has had to be incorporated, ν, i.e.,

$$m_I v_I + m_R v_R = m_T v_T + m_\nu v_\nu \tag{26}$$

where $m_\nu = T(\theta)$ and $v_\nu = -a_\rho c_1^{-1} \cos\theta_1$. If the momenta are to be accounted for using reciprocal velocities, then there must a *new particle* to carry off the remainder of the momentum, leftover from the acoustic boundary condition which wants to equate more than just reciprocal apparent velocities. This is phantom particle, which does not appear as an observable event in the data, nevertheless has properties easily describable in terms of the "true" particles and their properties.

CONCLUSIONS

When transient or harmonic scalar plane waves, propagating at arbitrary angles with respect to a chosen observation axis (e.g., the depth axis in the case of VSP data), are treated within the particle model, no new concepts are required beyond those devised for the normal-incidence case. This increases the scope of the particle model significantly, now admitting any multidimensional wave that can be analyzed into plane harmonic waves.

New concepts are, however, needed to model multi-parameter problems. In the variable density acoustic media, continuity of force (or traction, stress, pressure) boundary conditions specifically influence reflected and transmitted amplitudes such that the simple addition of momenta advocated by the scalar particle model leads to imbalances.

In particle physics, the apparent violation of a trusted conservation principle has tended historically to lead to the postulation of new, as-yet undetected particles. In this spirit, we suggest that the particle/collision model of seismic data be imbued with unobserved ν particles, which (though unobserved) are straightforwardly characterized, and which have masses such that momenta are properly conserved and all interactions occur as they would in inelastic collision experiments.

It seems likely that in adding the further complications needed to describe elastic (or anelastic, anisotropic) waves, additional deviations of the boundary conditions, away from straight derivatives of the field, will require the addition of further particles to account for the momenta. Whether this is a flaw or a useful theoretical concept is not clear.

ACKNOWLEDGMENTS

This work was funded by the CREWES project. CREWES sponsors and research personnel are thanked.

REFERENCES

Innanen, K. A., 2010, A particle/collision model of seismic data: CREWES Research Report, **22**.

Innanen, K. A., 2011, A particle view of dispersive wave propagation: CREWES Research Report, **23**.